

contradicting (1). This proves that we must have $q \equiv 1 \pmod{3}$, as was to be shown.

- (786) (Sierpinski [39], Problem #176) There is only one, namely $x = 3$. Setting $x = t + 3$, (*) is reduced to

$$(**) \quad 2t(t^2 + 9t + 21) = 0.$$

Since the quadratic polynomial $t^2 + 9t + 21$ has no real roots, the only solution of (**) is $t = 0$. It follows that the only solution of (*) is $x = 3$.

- (787) Assume that such a solution $\{x, y\}$ exists. Since $9|117$, we must have that $9|x^3 + 5$, which is impossible, since $x^3 + 5 \equiv 4, 5, 6 \pmod{9}$.

REMARK: In his book [32], Joe Roberts makes the following interesting observation:

In the chapter "Diophantine Equations: p -adic Methods" in Studies in Number Theory, [22] D.J. Lewis states on page 26 that "The equation $x^3 - 117y^3 = 5$ is known to have at most 18 integral solutions but the exact number is not known." Finkelstein and London (1971) [12] made use of the field $\mathbf{Q}(\sqrt[3]{117})$, where the cube root is real, to show that, in fact, the equation has no solutions in integers. Halter-Koch (1973) [17] and Udrescu (1973) [33] independently observed that by considering the equation modulo 9 we get $x^3 \equiv 5 \pmod{9}$ and this congruence clearly has no solutions. Consequently we immediately see that the equation has no solutions.

- (788) It is important to make sure that each of the terms $(\dots)^3$ is positive. To do so, if we take $a \geq 3$ and $b = a + 1$, it is easy to see that each of the four expressions $(\dots)^3$ is positive. Since the Ramanujan identity holds for each integer $a \geq 3$, the first result is proved. On the other hand, to find the "double" representation of 1729, we first set $a = 3$ and $b = 4$ in (*), in which case we obtain

$$7^3 + 84^3 = 63^3 + 70^3.$$

Dividing each of the four terms of this last identity by 7^3 , we obtain the double representation of 1729 noticed by Ramanujan.

- (789) If $ax + by = b + c$ is solvable, then $d = (a, b)|(b + c)$. Since $d|b$, it follows that $d|c$, which implies that $ax + by = c$ is solvable. The other implication can be handled in a similar manner.
- (790) We know that $ax + by = c$ is solvable if and only if $d = (a, b)|c$, which is equivalent to $d|(a, b, c)$. This shows that $(a, b) = (a, b, c)$.
- (791) Since $(a, b) = 1$, there exist integers x^* and y^* such that $ax^* + by^* = 1$. The solutions of $ax + by = n$ are then given by $x = nx^* - bk$ and $y = ny^* + ak$, where $ax^* + by^* = 1$. Hence, the equation has positive solutions if $nx^* - bk > 0$ and $ny^* + ak > 0$, that is if

$$-\frac{y^*n}{a} < k < \frac{x^*n}{b}.$$

To show that there exists at least one such a value of k , we only need to show that

$$-\frac{y^*n}{a} + 1 < \frac{x^*n}{b},$$

an inequality which is equivalent to $n(ax^* + by^*) > ab$, that is $n > ab$.

Finally, if $n = ab$, then $-y^*b < k < x^*a$; and since $ax^* + by^* = 1$, we obtain $ax^* - 1 < k < ax^*$, which is impossible.

- (792) The solution is $x = 19$, $y = 11$, $z = 70$. Indeed, if we multiply the first equation by 2 and subtract this new equation from the second one, we obtain

$$(*) \quad 30y - 19z = -1000.$$

Reducing modulo 19, we obtain $11y \equiv 7 \pmod{19}$ and therefore (multiplying by 7) we have

$$y \equiv 11 \pmod{19} \quad \text{that is} \quad y = 11 + 19k, \quad k \in \mathbb{Z}.$$

Substituting this value in (*), we find $z = 70 + 30k$ and finally $x = 19 - 49k$. For these solutions to be positive, we must choose $k = 0$, which gives the solution stated above.

- (793) (Marco Carmosini, Queen's University, Canadian Congress of Students in Mathematics, May 1999). If P and A stand respectively for the perimeter and the area of such a triangle, then using Heron's formula $A = \sqrt{\frac{P}{2}(\frac{P}{2} - a)(\frac{P}{2} - b)(\frac{P}{2} - c)}$ where $P = a + b + c$, we are led to the equation

$$a + b + c = \sqrt{\frac{a + b + c}{2} \cdot \frac{-a + b + c}{2} \cdot \frac{a - b + c}{2} \cdot \frac{a + b - c}{2}},$$

that is

$$(*) \quad 16(a + b + c) = (-a + b + c)(a - b + c)(a + b - c).$$

Since the left-hand side of (*) is even, it follows that $(-a + b + c)$ or $(a - b + c)$ or $(a + b - c)$ must be even. It is easy to see that if any of these three quantities is even, each of the other two will also be even. It follows that there exist three integers $m \leq n \leq k$ such that

$$-a + b + c = 2m, \quad a - b + c = 2n, \quad a + b - c = 2k,$$

so that

$$a = n + k, \quad b = m + k, \quad c = m + n.$$

Substituting these values in (*), we obtain that

$$mnk = 4(m + n + k).$$

We will treat separately the following four cases:

$$m = 1, \quad m = 2, \quad m = 3, \quad m \geq 4.$$

If $m = 1$, we obtain successively $nk = 4(1 + n + k)$, $nk - 4n - 4k = 4$, $nk - 4n - 4k + 16 = 4 + 16$ and $(n - 4)(k - 4) = 20$, in which case the only possible values of (n, k) are $(n, k) = (5, 24)$, $(6, 14)$ and $(8, 9)$. To these values correspond the three triangles whose sides a, b, c are $(a, b, c) = (20, 15, 7)$, $(17, 10, 9)$ and $(29, 25, 6)$.

In the case $m = 2$, we have $2nk = 4(2 + n + k)$; that is $(n - 2)(k - 2) = 8$, which gives the only possible values $(n, k) = (3, 10)$ and $(4, 6)$, thus yielding the two triangles of lengths a, b, c given by $(a, b, c) = (13, 12, 5)$ and $(10, 8, 6)$.

In the case $m = 3$, we obtain successively $3nk = 4(3 + n + k)$, $(3n - 4)(3k - 4) = 52$, and hence the pair $(n, k) = (2, 10)$, which must be rejected since $n = 2 < 4 = m$.

It remains to consider the case $m \geq 4$. Let us assume that there exists a solution (m, n, k) , with $m \geq 4$, to the equation $mnk = 4(m + n + k)$. We would then successively have

$$m = \frac{4n + 4k}{nk - 4} \geq 4, \quad n + k \geq nk - 4, \quad k \leq \frac{5}{n-1} + 1 \quad \text{for all } n \geq 4.$$

In particular, we would have

$$k \leq \frac{5}{4-1} + 1 = \frac{8}{3} < 3, \quad \text{and therefore } 4 \leq m \leq n \leq k < 3,$$

a contradiction.

To sum up, the only solutions (a, b, c) are given by the five triples

$$(13, 12, 5), \quad (10, 8, 6), \quad (29, 25, 6), \quad (20, 15, 7), \quad (17, 10, 9).$$

- (794) There are none. Indeed, if x, y is a solution, since x is not a multiple of 3, then $x^2 \equiv 1 \pmod{3}$, in which case $x^2 + 3y \equiv 1 \pmod{3}$, while $5 \equiv 2 \pmod{3}$.
- (795) We may assume that $x^2 + y^2 = z^2$. We proceed by contradiction by assuming that $xyz \not\equiv 0 \pmod{5}$, in which case $x^2, y^2, z^2 \equiv 1, 4 \pmod{5}$. The only three possible values modulo 5 of $x^2 + y^2$ are therefore $1 + 1$, $1 + 4$ and $4 + 4$, that is 2, 0 and 3 modulo 5, while we should have 1 or 4.
- (796) If $x > 1$, then using the first equation, we have that $y < 1$, which in turn implies that $z > 1$. But " $x > 1, z > 1$ " contradicts the third equation. Hence, $x \leq 1$. By a similar argument, one can show that $x \geq 1$. Hence, $x = 1$. We can then do the same reasoning with each unknown, allowing us to conclude that $x = y = z = 1$.
- (797) (*TYCM, Vol. 13, 1982, p. 263*). Assume that there exist nonnegative integers x and y such that $ax + by = ab - a - b$. In this case, we have $a(x + 1) = b(a - y - 1)$. Since a and b are relatively prime, it is clear that

$$b|x + 1 \quad \text{and} \quad a|a - y - 1,$$

which implies that $a|y + 1$. Hence, $y + 1 \geq a$, $x + 1 \geq b$ and therefore $ab = (x + 1)a + (y + 1)b \geq 2ab$, which is impossible, since a and b are positive.

- (798) We know that the solutions of $n = ax + by$ are of the form $x = x_0 + bt$, $y = y_0 - at$, where $ax_0 + by_0 = n$, $t \in \mathbb{Z}$. We must choose t so that $y_0 - at \geq 0$ and $x_0 + bt \geq 0$, which is equivalent to $-(x_0/b) \leq t \leq (y_0/a)$. The number of solutions is therefore $[y_0/a] - [-x_0/b]$, and since $[a] - [b] = [a - b]$ or $[a - b] + 1$, we obtain the result.
- (799) Say Peter has paid \$1.04. The only way this can happen is if Peter has bought x apples and y oranges, with x and y such that $5x + 7y = 104$. We can express this situation as $(x, y) = (x_0, y_0) = (4, 12)$. All the integer solutions of $5x + 7y = 104$ are given by $x = x_0 + 7t = 4 + 7t$ and $y = y_0 - 5t = 12 - 5t$. Since we must have $4 + 7t > 0$ (that is $t > -4/7$) and $12 - 5t > 0$ (that is $t < 12/5$), it follows that $0 \leq t \leq 2$. The only three suitable values of t are therefore 0, 1 and 2. Since Peter's purchase corresponds to the value $t = 0$, Paul's purchase must necessarily

correspond to $t = 1$ or to $t = 2$, that is $x = 11$ and $y = 7$ or $x = 18$ and $y = 2$. Since by hypothesis $y \geq 3$, we may conclude that Paul has bought 11 apples and 7 oranges.

- (800) First of all, since $(3, 7) = 1 \mid 11$, the given Diophantine equation has integer solutions. We easily establish that $(x_0, y_0) = (6, -1)$ is a particular solution of this Diophantine equation. The set of all the solutions is therefore given by

$$x = 6 + 7t, \quad y = -1 - 3t, \quad \text{where } t \in \mathbb{Z}.$$

The solutions located in the second quadrant are those corresponding to the points (x, y) satisfying

$$x = 6 + 7t < 0 \quad \text{and} \quad y = -1 - 3t > 0,$$

that is when $t < -\frac{6}{7}$ and $t < -\frac{1}{3}$. This means that we must have $t \leq -1$. The set A of integer points (x, y) which are solutions of $3x + 7y = 11$ and which are located in the second quadrant is therefore given by

$$A = \{(x, y) : x = 6 + 7t \text{ and } y = -1 - 3t, \text{ where } t = -1, -2, -3, \dots\}.$$

- (801) We first establish that $(x_0, y_0) = (5, -2)$ is a particular solution of this equation. This point generates the solutions

$$x = 5 + 7t, \quad y = -2 - 5t, \quad \text{where } t \in \mathbb{Z}.$$

Since we are interested in the points (x, y) such that $y > x$, we need to establish the integer values of t for which

$$-2 - 5t > 5 + 7t, \quad \text{that is} \quad t < -7/12,$$

which is only possible if $t \leq -1$. The required set E is therefore

$$E = \{(x, y) : x = 5 + 7t \text{ and } y = -2 - 5t, \text{ where } t = -1, -2, -3, \dots\}.$$

- (802) Taking $t = 0$, we obtain that $(x, y) = (5, 1)$ is a solution of $(*)$. Choosing $t = 1$, we obtain the solution $(x, y) = (1, -2)$. These two solutions give rise to the system

$$\begin{cases} 5a + b = 11, \\ a - 2b = 11, \end{cases}$$

a solution of which is $a = 3$ and $b = -4$, which produces the required numbers a and b .

- (803) Since $(x, y, z) = 1$, we have $(x, y) = (x, z) = (y, z) = 1$, and therefore only one of the terms x , y and z can be even. If x is even, then $x^2 \equiv 0 \pmod{4}$. The fact that x is even implies that y and z are both odd and $z^2 - 3y^2 \equiv 2 \pmod{4}$. It follows that x must be odd and that y or z is even.

I) If y is even, then $(z + x, z - x) = 2$ and therefore $z + x = 2u$ and $z - x = 2v$, where $(u, v) = 1$. We then have $3y^2 = (z - x)(z + x) = 4uv$. Hence, $(y/2)^2 = uv/3$, and since $(u, v) = 1$, we may assume that $3 \mid u$, so that there exists a positive integer m such that $u = 3m$. It follows that there exist positive integers r and s such that $v = r^2$ and $m = s^2$, in which case

$$\frac{z + x}{2} = u = 3m = 3s^2 \quad \text{and} \quad \frac{z - x}{2} = v = r^2.$$

We easily see that in this case, we must have

$$(r, s) = 1, \quad s > r, 3 \nmid r, \quad y = 2rs, \quad x = 3s^2 - r^2, \quad z = 3s^2 + r^2.$$

II) If z is even, then $(z+x, z-x) = 1$, in which case for r and s odd and $(r, s) = 1, \quad s > r, 3 \nmid r$, we have

$$y = rs, \quad x = \frac{3s^2 - r^2}{2}, \quad z = \frac{3s^2 + r^2}{2}.$$

(804) (Sierpinski [39], Problem #170). We begin with the identity

$$(*) \quad (x+y+z)^3 - (x^3 + y^3 + z^3) = 3(x+y)(x+z)(y+z).$$

It follows that if x, y and z are integers such that $x+y+z = 3$ and $x^3 + y^3 + z^3 = 3$, then, by $(*)$, we have

$$(**) \quad 8 = (x+y)(x+z)(y+z) = (3-x)(3-y)(3-z),$$

so that, in light of $x+y+z = 3$, we have

$$(***) \quad 8 = (3-x)(3-y)(3-z).$$

Relation $(***)$ implies that either the three numbers $3-x, 3-y, 3-z$ are even or else only one of the three is even. In the first case, in light of $(**)$, they are all in absolute value equal to 2; therefore, by $(***)$, they are equal to 2, in which case $x = y = z = 1$. In the second case, in light of $(**)$, one of the numbers $3-x, 3-y, 3-z$ is in absolute value equal to 8, while the others are in absolute value equal to 1; thus, by $(***)$, one of the two is equal to 8, while the others are equal to -1 . This finally yields $x = -5$ and $y = z = 4$, or $x = y = 4$ and $z = -5$ or $x = 4, y = -5$ and $z = 4$. We can therefore conclude that the system of equations has only four integer solutions, namely $(1, 1, 1), (-5, 4, 4), (4, -5, 4)$ and $(4, 4, -5)$.

(805) We only need to consider, for each positive integer n , the triples $\{x, y, z\}$, where

$$\begin{aligned} x &= n^{10}(n+1)^8, \\ y &= n^7(n+1)^5, \\ z &= n^4(n+1)^3. \end{aligned}$$

(806) Consider the equation $5x + 7y = 136$. Reducing this equation modulo 5, we obtain $y = 3 + 5k$. Substituting in the equation, we obtain $x = 23 - 7k$. The condition " $x > 0$ and $y > 0$ " allows us to conclude that solutions are possible for $k = 0, 1, 2, 3$, that is $136 = 115 + 21 = 80 + 56 = 45 + 91 = 10 + 126$.

(807) Setting $x = a - r, y = a$ and $z = a + r$, we find the equation $x^2 + y^2 = z^2$ becomes $a(a - 4r) = 0$. Hence, $a = 4r$ and therefore $x = 3r, y = 4r$ and $z = 5r$, where $r \in \mathbb{N}$.

(808) Setting $r = 16$ and $s = 5$ in Theorem 34, we obtain $z = 281, y = 160$ and $x = 231$.

(809) Since $3|xy$ and $4|xy$, we have that $12|xyz$. Hence, we only need to show that $5|xyz$. We first observe that if $5 \nmid m$, then $m = 5k \pm 1$ or $m = 5k \pm 2$, for a certain integer k . In the first case, $m^2 = 5(5k^2 \pm 2k) + 1$ and in the second case, $m^2 = 5(5k^2 \pm 4k) + 4$. Using this observation, we see that if none of the numbers x, y, z are divisible by 5, then $x^2 + y^2$ gives, after dividing by 5, the remainders 2, 3 or 0. Since $x^2 + y^2 = z^2$, the first

two cases are clearly impossible. The only possibility is the third one, in which case z^2 is divisible by 5, so that z is divisible by 5.

- (810) Since $x = r^2 - s^2$, $y = 2rs$ and $z = r^2 + s^2$, we have $s(r - s) = 6$. Solving for s , we find

$$s = \frac{r \pm \sqrt{r^2 - 24}}{2}.$$

Hence, in order for s to exist, we must have $r^2 \geq 24$, in which case $\sqrt{r^2 - 24}$ is an integer. Therefore, there exists an integer u such that $r^2 - 24 = u^2$, in which case

$$(r - u)(r + u) = 24 = 1 \cdot 24 = 2 \cdot 12 = 3 \cdot 8 = 4 \cdot 6.$$

From this, we derive the values $r = 7$ and $r = 5$. We thus obtain the Pythagorean triples $(20, 21, 29)$, $(16, 30, 34)$, $(13, 84, 85)$ and $(48, 14, 50)$.

- (811) The equations $x^2 + y^2 = z^2$ and $x + y + z = xy$ allow us to obtain the equation $(x - 2)(y - 2) = 2$. Hence, $x = 3$, $y = 4$ and $z = 5$ are the dimensions of the required triangle.
- (812) The relation $(n - 1)^2 + n^2 = (n + 1)^2$ implies that $n^2 = 4n$; that is $n = 4$.
- (813) Whatever the parity of n , the left-hand side is always odd, while the right-hand side is always even, a contradiction.
- (814) Since $x^2, y^2 \equiv 0, 1 \pmod{4}$, we have $x^2 + y^2 \not\equiv 3 \pmod{4}$, while $4x + 7 \equiv 3 \pmod{4}$.
- (815) First observe that the primitive solutions of $x^2 + y^2 = (z^2)^2$ are given by $x = r^2 - s^2$, $y = 2rs$ and $z^2 = r^2 + s^2$, with $r > s > 0$, $(r, s) = 1$, r, s of opposite parity. Since the primitive solutions of $z^2 = r^2 + s^2$ are in turn given by $r = m^2 - n^2$, $s = 2mn$ and $z = m^2 + n^2$, with $m > n > 0$, $(m, n) = 1$, m, n of opposite parity, we may conclude that all primitive solutions of $x^2 + y^2 = z^4$ are given by $y = 4mn(m^2 - n^2)$, $x = m^4 + n^4 - 6m^2n^2$ and $z = m^2 + n^2$, with $m > n > 0$, $(m, n) = 1$, m, n of opposite parity.
- (816) First of all, it is clear that $(x_0, y_0) = (2, -1)$ is a solution of the Diophantine equation $x + y = 1$. All the integer solutions (x, y) of the equation are therefore given by

$$x = 2 + t, \quad y = -1 - t \quad (t \in \mathbb{Z}).$$

Hence, we are looking for the values of x and y such that $x^2 + y^2 \leq 9$. But

$$x^2 + y^2 = (2 + t)^2 + (-1 - t)^2 = 2t^2 + 6t + 5.$$

This means that we must have

$$2t^2 + 6t + 5 \leq 9,$$

an inequality which is verified for the integers $t = -3, -2, -1, 0$, yielding the integer solutions

$$(-1, 2), \quad (0, 1), \quad (1, 0), \quad (2, -1).$$

- (817) First observe that $3136 = 56^2$. We are therefore looking for the primitive solutions of

$$(*) \quad x^2 + 56^2 = z^2.$$

Since all the primitive solutions of $X^2 + Y^2 = Z^2$ are given by

$$X = r^2 - s^2, Y = 2rs, Z = r^2 + s^2,$$

where $r > s > 0$, $(r, s) = 1$, r, s of opposite parity, we must look for integers r and s such that

(**) $Y = 56 = 2rs$, where $r > s > 0$, $(r, s) = 1$, r, s of opposite parity.

Hence, we only need to search for the solutions of (**). There are two of them, namely $(r, s) = (28, 1)$ and $(r, s) = (7, 4)$, these in turn giving rise to the solutions $(X, Y, Z) = (x, 56, z) = (783, 56, 785)$ and $(X, Y, Z) = (x, 56, z) = (33, 56, 65)$.

(818) Comparing the geometric mean with the arithmetic mean (see Theorem 5) we have, for any positive real numbers x and y ,

$$\frac{x^2 + y^2}{2} \geq (x^2 y^2)^{1/2} = xy, \text{ and therefore } x^2 + y^2 \geq 2xy.$$

Hence, we cannot have $x^2 + y^2 = xy$ unless $x = y = 0$. This is why the only integer solution of $x^2 + y^2 = xy$ is $(x, y) = (0, 0)$.

(819) It is obvious that x must be even. Setting $x = 2u$, we have

$$2u^2 + y^2 = 2z^2.$$

It is then clear that y must be even, in which case setting $y = 2v$, we obtain

$$u^2 + 2v^2 = z^2.$$

Reducing modulo 4, we see that v must be even. Setting $v = 2w$, we then have

$$u^2 + 8w^2 = z^2.$$

This equation has infinitely many solutions for each fixed value of u . Indeed, $w = u$ and $z = 3u$ is a solution for each positive integer u . Moreover, for each solution of (*) we can write

$$8w^2 + u^2 = (3w - a)^2,$$

for some integer a . Thus, we have

$$w = 3a \pm \sqrt{8a^2 + u^2}.$$

Since $w_1 = u$ is a solution, it follows that

$$w_2 = 3u + \sqrt{8u^2 + u^2} = 6u$$

is also a solution. Other solutions are given by

$$w_3 = 35u,$$

$$w_4 = 204u,$$

and more generally by

$$w_n = 6w_{n-1} - w_{n-2}.$$

To find all the solutions for a fixed u , we only need to search for the solutions such that w is between 0 and u inclusively and then to iterate from these solutions.

- (820) (*Contribution of John Brillhart, Arizona*). Let a, b, c be the lengths of the three sides of the required triangle and let α and 2α be the angles opposite to the sides a and b . Calling upon the law of sines and thereafter to the law of cosines, we obtain successively

$$\frac{b}{a} = \frac{\sin 2\alpha}{\sin \alpha} = \frac{2 \sin \alpha \cos \alpha}{\sin \alpha} = 2 \cos \alpha = \frac{b^2 + c^2 - a^2}{bc},$$

so that

$$\begin{aligned} b^2 c &= ab^2 + ac^2 - a^3, \\ b^2 c - b^2 a &= a(c^2 - a^2), \\ b^2(c - a) &= a(c^2 - a^2), \\ b^2 &= a(c + a), \quad \text{since } c \neq a. \end{aligned}$$

It is clear that the choice $a = 4, c = 5, b = 6$ serves our purpose.

- (821) It is easy to check that the only solution is $x = 1$ and $y = 3$. As for the other equation, it has no integer solution.

- (822) Since $X^2 + Y^2 = Z^2$ implies $\left(\frac{X}{Z}\right)^2 + \left(\frac{Y}{Z}\right)^2 = 1$, we have

$$x = \frac{X}{Z} = \frac{2rs}{r^2 + s^2}, \quad y = \frac{Y}{Z} = \frac{r^2 - s^2}{r^2 + s^2}.$$

Dividing the numerator and the denominator by r^2 , we obtain by setting $t = s/r$,

$$x = \frac{2t}{1 + t^2}, \quad y = \frac{1 - t^2}{1 + t^2}, \quad 0 \leq t \leq 1.$$

- (823) Let $y = 2rs = 24$, so that $rs = 12 = 12 \cdot 1 = 6 \cdot 2 = 4 \cdot 3$. We thus find $(r, s) = (12, 1) = (6, 2) = (4, 3)$, and this is why the primitive Pythagorean triangles are obtained when $(r, s) = (12, 1)$ and $(r, s) = (4, 3)$.
- (824) Assume that x, y and z is a primitive solution of $x^2 + y^2 = z^2$. Hence, $x = t^2 - s^2, y = 2ts$ and $z = t^2 + s^2$, so that letting A be the area of the triangle and letting r be the radius of the inscribed circle, we have

$$A = \frac{xy}{2} = \frac{rx}{2} + \frac{ry}{2} + \frac{rz}{2},$$

and therefore

$$r = \frac{xy}{x + y + z} = \frac{2ts(t^2 - s^2)}{2ts + (t^2 - s^2) + (t^2 + s^2)} = s(t - s),$$

an integer.

- (825) One only needs to reduce the equation modulo 8, thereby obtaining a contradiction.
- (826) We will show that at least two of the numbers x, y, z must be even. Assume the contrary, that is that the three numbers x, y, z are odd. Then t^2 is a number of the form $8k + 3$ and therefore must be odd, which contradicts the fact that t is even. If only one of the numbers x, y, z is even, the sum $x^2 + y^2 + z^2 = t^2$ is of the form $4k + 2$, which is impossible since the square of an even number must be of the form $4k$.

- (827) If $(x, y) = (y, z) = (x, z) = 1$, then x and z are odd and y is even. Set $z - x = 2u$, $z + x = 2v$, where $(u, v) = 1$ and u and v are of opposite parity. Substituting in the equation, we find that $y^2 = 2uv$. Assuming that u is even, set $u = 2M$, that is $y^2 = 4Mv$, in which case we must have that $M = r^2$ and $v = s^2$, $(r, s) = 1$. By substitution, we obtain $y = 2rs$, $z = s^2 + 2r^2$ and $x = s^2 - 2r^2$, with $(r, s) = 1$.
- (828) (*AMM*, Vol. 65, 1958, p. 43). The first equation becomes

$$(a - b - c)(a^2 + (b - c)^2 + ab + bc + ca) = 0.$$

Since $a^2 + (b - c)^2 + ab + bc + ca \neq 0$ (because $a, b, c > 0$), we have $a - b - c = 0$, which implies that $a = b + c = a^2/2$ and allows us to conclude that $a = 2$ and $b = c = 1$.

- (829) (*AMM*, Vol. 73, 1966, p. 895). Multiplying the equation of the statement by 4 and adding 1, we obtain

$$4x^4 + 4x^3 + 4x^2 + 4x + 1 = (2y + 1)^2.$$

For $x = -1$, we find $y = -1$ or 0 ; for $x = 0$, we find $y = -1$ or 0 ; for $x = 2$, we find $y = -6$ or 5 ; finally, for $x = 1$, y is not an integer. On the other hand, for $x < -1$ or $x > 2$, the left-hand side of the above equation is larger than $(2x^2 + x)^2$ but smaller than $(2x^2 + x + 1)^2$ and therefore cannot be the square of an integer for integer values of x , while the right-hand side is the square of an integer for all integer values of y . It follows that the six solutions listed above are the only integer solutions of the given equation.

- (830) (*AMM*, Vol. 75, 1968, p. 193). If $x^2 + ry^2 = p$, then $x^2 \equiv -ry^2 \pmod{p}$ and therefore $-ry^2$; that is $-r$ is a quadratic residue of p . Hence, the required prime number must satisfy $\left(\frac{-r}{p}\right) = 1$ for $1 \leq r \leq 10$. This will be satisfied if $-1, 2, 3, 5$ and 7 are quadratic residues of p . It follows that the congruences $p \equiv 1 \pmod{8}$, $p \equiv 1 \pmod{3}$, $p \equiv 1$ or $-1 \pmod{5}$, and $p \equiv 1, 2$, or $4 \pmod{7}$. The required prime number p must therefore satisfy

$$p \equiv 1, 121, 169, 289, 361 \text{ or } 529 \pmod{840},$$

and this is why the smallest prime number satisfying the conditions is $p = 1009$. We have therefore obtained

$$\begin{aligned} 1009 &= 15^2 + 28^2 = 19^2 + 2 \cdot 18^2 = 31^2 + 3 \cdot 4^2 = 15^2 + 4 \cdot 14^2 \\ &= 17^2 + 5 \cdot 12^2 = 25^2 + 6 \cdot 8^2 = 1^2 + 7 \cdot 12^2 = 19^2 + 8 \cdot 9^2 \\ &= 28^2 + 9 \cdot 5^2 = 3^2 + 10 \cdot 10^2. \end{aligned}$$

- (831) (*AMM*, Vol. 75, 1968, p. 685). Setting $x = a + 3d$, $y = a + 4d$, $z = a + 5d$ and $w = a + 6d$, we find the given equation becomes $a(a^2 + 9ad + 21d^2) = 0$. The only integer solution a of this equation is $a = 0$, and therefore the only solution $\{x, y, z, w\}$ of the given equation is $\{3d, 4d, 5d, 6d\}$.
- (832) We have $x_{n+1} = x_1 + 6n(n+2)$, $n \geq 1$, and therefore $x_{n+1} = (n+2)^3 - n^3$. Hence, if x_{n+1} is a cube, say A^3 , then we have $x_{n+1} = A^3 = (n+2)^3 - n^3$. Since no integer satisfies such an equation, the result is proved.

- (833) (*AMM*, Vol. 85, 1978, p. 118). Assume that there exists a solution $\{x, y, n\}$. Since

$$x^n = y^{n+1} - 1 = (y-1)(y^n + y^{n-1} + \cdots + 1),$$

it is clear that any prime divisor p of $y-1$ divides x , and since $(x, n+1) = 1$, we have $p \nmid (n+1)$. Since $y \equiv 1 \pmod{y-1}$, it follows that

$$1 + y + y^2 + \cdots + y^n \equiv n+1 \pmod{y-1}$$

and therefore that the numbers $y-1$ and $1 + y + \cdots + y^n$ are relatively prime. Hence, we may write

$$x^n = (y-1)(1 + y + \cdots + y^n),$$

which implies that $1 + y + \cdots + y^n$ is an n -th power of an integer. But this is impossible since

$$y^n < 1 + y + \cdots + y^n < (y+1)^n.$$

- (834) (*AMM*, Vol. 87, 1980, p. 138). If the exponent of 3 is not zero, it is easy to see that none of these equations are solvable modulo 3.

- (835) (*AMM*, Vol. 76, 1969, p. 308). If $x \leq y \leq z$, then $4^x + 4^y + 4^z$ is a perfect square under the condition that there exists a positive integer m and a positive odd integer t such that

$$1 + 4^{y-x} + 4^{z-x} = (1 + 2^m t)^2.$$

Therefore,

$$(*) \quad 4^{y-x}(1 + 4^{z-y}) = 2^{m+1}t(1 + 2^{m-1}t),$$

so that we must have $m = 2y - 2x - 1$. Substituting this value in $(*)$, we obtain

$$\begin{aligned} t - 1 &= 4^{y-x-1}(4^{z-2y+x+1} - t^2) \\ &= 4^{y-x-1}(2^{z-2y+x+1} + t)(2^{z-2y+x+1} - t). \end{aligned}$$

Since t is odd, this last equation is possible when $t = 1$, and consequently $z = 2y - x - 1$. Therefore, the only integer solutions are $\{x, y, 2y - x - 1\}$, with arbitrary x and y . Finally, these values produce the square $(2^x + 2^{2y-x-1})^2$.

- (836) (*AMM*, Vol. 76, 1969, p. 84). Setting $a = 3d$, $c = 2b - 3d$, we obtain $x + y = 3b$, and the second equation boils down to

$$(x - y)^2 = (b - 8d)^2 - 40d^2,$$

of which a solution is given by

$$x - y = m^2 - 10n^2, \quad b - 8d = m^2 + 10n^2, \quad d = mn,$$

where $m, n \in \mathbb{N}$. Hence, the solutions are: $a = 3d$, $b = 8d + m^2 + 10n^2$, $c = 2b - 3d$, $x = 12d + 2m^2 + 10n^2$, $y = 12d + m^2 + 20n^2$. To obtain infinitely many solutions when a, b, c are in arithmetic progression, it is enough to choose

$$a = 3mn, \quad b = m^2 + 8mn + 10n^2, \quad c = 2m^2 + 13mn + 20n^2$$

and

$$x = 2m^2 + 12mn + 10n^2, \quad y = m^2 + 12mn + 20n^2.$$

- (837) (*AMM*, Vol. 83, 1976, p. 569). First consider the equation (*), $x^m(x^2 + y) = y^{m+1}$. If $m = 0$, the solutions are $x = 0$ and y arbitrary. If $m \geq 1$, the solutions are given by $x = b(b^m - 1)$, $y = b^2(b^m - 1)$, where $b \in \mathbb{Z}$. It is easy to verify that these are indeed solutions. Let (x, y) be a nontrivial solution; that is $xy \neq 0$. Then, we can write $x = ac$, $y = bc$ where a and b are relatively prime and $a \geq 1$. From (*), we derive

$$a^m(a^2c + b) = b^{m+1}.$$

This implies $a = 1$ and $c = b(b^m - 1)$, hence the solutions x and y . The only solutions are therefore $(m, x, y) = (0, 0, y)$ where y is arbitrary, and $(m, x, y) = (m, b(b^m - 1), b^2(b^m - 1))$, where $m \geq 1$ and $b \in \mathbb{Z}$.

Let us now examine the equation (**), $x^m(x^2 + y^2) = y^{m+1}$. If $m = 0$, we have the only solution $x = 0$, $y = 1$. If $m \geq 1$, we only have the trivial solution $x = y = 0$. Indeed, assume that there exists a nontrivial solution (x, y) . Then $xy \neq 0$, and we write again $x = ac$, $y = bc$, where a and b are relatively prime and $a \geq 1$. From (**) we derive

$$a^m c(a^2 + b^2) = b^{m+1}.$$

This implies $a = 1$, so that $1 + b^2$ divides b^{m+1} . Since $b \neq 0$, we obtain a contradiction. The only solutions are therefore $(m, x, y) = (0, 0, 1)$ and $(m, 0, 0)$ where $m \geq 1$.

- (838) (*AMM*, Vol. 95, 1988, p. 141). These equations cannot be satisfied by integers. Indeed, for each integer h , we have

$$h^2 \equiv \begin{cases} 0 \pmod{8} & \text{if } h \equiv 0 \pmod{4}, \\ 1 \pmod{8} & \text{if } h \equiv 1 \text{ or } 3 \pmod{4}, \\ 4 \pmod{8} & \text{if } h \equiv 2 \pmod{4}. \end{cases}$$

We therefore have

$$h^2 + k^2 \equiv \begin{cases} 0, 1 \text{ or } 4 \pmod{8} & \text{if } h \equiv 0 \pmod{4}, \\ 1, 2 \text{ or } 5 \pmod{8} & \text{if } h \equiv 1 \text{ or } 3 \pmod{4}, \\ 0, 4 \text{ or } 5 \pmod{8} & \text{if } h \equiv 2 \pmod{4}, \end{cases}$$

for each integer h and k . Hence, since $\{x + 1, x + 2, x + 3, x + 4\}$ forms a complete residue system modulo 4, the congruences

$$\begin{aligned} (x + 1)^2 + a^2 &\equiv (x + 2)^2 + b^2 \equiv (x + 3)^2 + c^2 \equiv (x + 4)^2 + d^2 \\ &\equiv n \pmod{8} \end{aligned}$$

are satisfied only if $n \in \{0, 1, 4\} \cap \{1, 2, 5\} \cap \{0, 4, 5\}$, which is impossible.

- (839) Verifying the parity, we easily notice that two of the three integers are even, while the other is odd. Setting $x = 2m$, $y = 2n$ and $z = 2r + 1$, the equation becomes

$$4m^2 + 4n^2 + 4r^2 + 4r + 2 = 4mn(2r + 1),$$

which would mean that $4|2$, which is nonsense. Hence, there are no solutions.

- (840) If $x = 0$, then from (*), $y = \pm 1$; hence, because of (**), we have $y = 1$. It follows that $(x, y) = (0, 1)$ is a solution of the system. Similarly, $(x, y) = (1, 0)$ is a solution of the system. Assume that $x \neq 0$ and $y \neq 0$. If $x > 1$, then $2x^3 - x^2 + y^2 = x^2(2x - 1) + y^2 > x^2 + y^2 > 1 + y^2 > 1$, which contradicts (*); hence $x \leq 1$. Similarly, $y \leq 1$. By adding (*) and (**),

we derive that $x^3 + y^3 = 1$. If $x < 0$, then $y > 1$, which contradicts $y \leq 1$; hence $x > 0$. Similarly, $y > 0$. We then have $0 < x < 1$ and $0 < y < 1$. By hypothesis, $2x^3 - x^2 + y^2 = 2y^3 - y^2 + x^2$, so that $x^3 - x^2 = y^3 - y^2$. Let $u = y/x$. It follows that $x^3 - x^2 = u^3x^3 - u^2x^2$ and therefore that

$$(1) \quad x - 1 = u^2(ux - 1).$$

If $u > 1$, then $u^2(ux - 1) > ux - 1$, which contradicts (1). Similarly, we cannot have $u < 1$. It follows that $u = 1$ and therefore that $y = x$. It then follows from (*) that $x^3 = 1/2$ and therefore that $x = y = (1/2)^{1/3}$. The only solutions of the system are therefore

$$(0, 1), \quad (1, 0), \quad \left(\frac{1}{2^{1/3}}, \frac{1}{2^{1/3}} \right).$$

- (841) (*MMAG*, Vol. 52, 1979, p. 47). If one of the numbers is 1, the other must also be equal to 1. Assume that (x, y) is a solution with $x \geq 2$ and $y \geq 2$. Then, $x^y = y^{x-y} > 1$ and therefore $x > y$. Dividing both sides of the equation by y^y , we obtain $(x/y)^y = y^{x-2y}$. Since $x/y > 1$, we have $(x/y)^y = y^{x-2y} > 1$. It follows that $x - 2y$ is a positive integer and thus $x/y > 2$, so that $(x/y)^y$ is a positive integer. This implies that x/y is a positive integer. Since the function $f(x) = 2^x - 4^2$ is strictly increasing for $x \geq 5$, it follows that $2^x > 4x$ and therefore for $x/y \geq 5$, we have

$$\frac{x}{y} = y^{(x/y)-2} \geq 2^{(x/y)-2} > \frac{x}{y},$$

a contradiction. On the other hand, when $2 < x/y < 5$ we obtain that x/y must be equal to 3 or 4. Since $x/y = y^{(x/y)-2}$, it follows that by choosing $x/y = 3$, we have $y = 3$ and $x = 9$, and choosing $x/y = 4$, we have $y = 2$ and $x = 8$. Therefore, the only solutions are $(1, 1)$, $(9, 3)$ and $(8, 2)$.

- (842) (*MMAG*, Vol. 63, 1990, p. 190). Since $1 + x + x^2 > 0$, $1 + y + y^2 > 0$ and $1 + z + z^2 > 0$, it follows that x, y, z are positive integers. Without any loss in generality, we may assume that $x \geq y \geq z$. Then, $2x(1 + x + x^2) \geq 3(1 + x^4)$, so that $(x - 1)^2(3x^2 + 4x + 3) \leq 0$. Therefore, $x = 1$, which yields the only real solution $x = y = z = 1$.
- (843) (*MMAG*, Vol. 63, 1990, p. 190). This follows from the fact that any integer $n > 2$ satisfies the identity $(n^2 + n)!(n - 1)! = (n^2 + n - 1)!(n + 1)!$ and the chain of inequalities $n^2 + n > n^2 + n - 1 > n + 1 > n - 1$.
- (844) Since $m^3 \equiv 0, 1$ or $8 \pmod{9}$, it follows that

$$x^3 + y^3 + z^3 \equiv 0, 1, 2, 3, 6, 7 \text{ or } 8 \pmod{9}.$$

We conclude that neither of these two equations is solvable in integers.

- (845) The answer is YES. Indeed, this Diophantine equation can be written successively as

$$\begin{aligned} x^4 &= 4y^2 + 4y + 1 - 81, \\ x^4 + 81 &= 4y^2 + 4y + 1, \\ x^4 + 3^4 &= (2y + 1)^2, \end{aligned}$$

this last equation having integer solutions only if $x = 0$, in which case we obtain $2y + 1 = \pm 9$, that is $y = 4$ or -5 . We then have only two solutions, namely $(x, y) = (0, 4)$ and $(x, y) = (0, -5)$.

- (846) The answer is NO. Indeed, given an arbitrary integer a , we always have $a^4 \equiv 0$ or $1 \pmod{5}$. Therefore, the only possible values of $x^4 + y^4 + z^4$ modulo 5 are 0, 1, 2 or 3. Since $363932239 \equiv 4 \pmod{5}$, there is no hope for a solution.
- (847) The answer is NO. The reason is that $(303, 57) = 3$, while 3 never divides $a^2 + 1$.
- (848) The answer is NO. Indeed, this Diophantine equation can be written as

$$x^4 + 2^4 = (2y + 1)^2.$$

But we know that the Diophantine equation $X^4 + Y^4 = Z^2$ has integer solutions only if $X = 0$ or $Y = 0$. Here, $Y = 2$, and therefore $X = 0$. It then follows that $16 = (2y + 1)^2$, which makes no sense.

- (849) The answer is YES. It is enough to take $x = 0$, $y = 1$ and $z = 9$, in which case we do have

$$x^4 + (2y + 1)^4 = 0^4 + 3^4 = 9^2.$$

And this is the only solution in nonnegative integers.

- (850) Since $x^2 - y^4 = 8$, we have $(x - y^2)(x + y^2) = 8$, which means that the only two possible cases are

$$\begin{cases} x - y^2 = 1, \\ x + y^2 = 8 \end{cases} \quad \text{and} \quad \begin{cases} x - y^2 = 2, \\ x + y^2 = 4. \end{cases}$$

The first of these two systems has no solutions, while the second implies that $x = 3$ and $y = 1$. The only positive solution of this Diophantine equation is therefore $(x, y) = (3, 1)$.

- (851) Since $x^2 - y^4 = pq$, we have $(x - y^2)(x + y^2) = pq$, which gives rise to the systems of equations

$$\begin{cases} x - y^2 = 1, \\ x + y^2 = pq \end{cases} \quad \text{and} \quad \begin{cases} x - y^2 = p, \\ x + y^2 = q. \end{cases}$$

The second system has the solution

$$x = \frac{p+q}{2}, \quad y^2 = \frac{q-p}{2} = \frac{8}{2} = 4,$$

which provides the following solutions to the Diophantine equation: $x = \frac{p+q}{2}$, $y = \pm 2$.

The first system implies that $2y^2 + 1 = pq = p(p+8)$; that is $2y^2 + 17 = (p+4)^2$ with $p+8$ prime. This last equation may have solutions depending on the value of p . When $p = 3$ with $q = p+8 = 11$, we obtain the solutions $x = \pm 7$ and $y = \pm 2$, and therefore we have found in this case more than one solution.

- (852) The answer is NO. Indeed, this Diophantine equation can be written as

$$(x + 1)^2 + (y + 2)^2 = -4z + 3.$$

Assume that this equation has a solution (x, y, z) . Since the left-hand side of this equation is congruent to 0, 1 or 2 modulo 4, while the right-hand side is congruent to 3 modulo 4, we have reached a contradiction.

- (853) The answer is NO. This follows immediately from the fact that, since the geometric mean is no larger than the arithmetic mean (see Theorem 5),

$$\frac{x^4 + y^4 + z^4 + u^4}{4} \geq (x^4 y^4 z^4 u^4)^{1/4} = |xyz u|.$$

- (854) The answer is NO. To prove it, we use the method of infinite descent of Fermat. Indeed, assume that this Diophantine equation has solutions. Let $x = x_0$ be the value corresponding to the smallest positive value of x for which this equation has a solution, say (x_0, y_0, z_0) . We then have

$$(1) \quad x_0^3 + 2y_0^3 = 4z_0^3.$$

It is clear from (1) that $2|x_0^3$, which implies that $8|x_0^3$; hence $x_0 = 2X$ for a certain positive integer X . The equation can therefore be rewritten as $8X^3 + 2y_0^3 = 4z_0^3$, that is

$$(2) \quad 4X^3 + y_0^3 = 2z_0^3.$$

It follows from (2) that $2|y_0^3$ and therefore that $8|y_0^3$, and equation (2) becomes $4X^3 + 8Y^3 = 2z_0^3$, with $2Z = z_0$, that is

$$(3) \quad 2X^3 + 4Y^3 = z_0^3.$$

It follows from (3) that $2|z_0^3$ and therefore that $8|z_0^3$, and equation (3) becomes $2X^3 + 4Y^3 = 8Z^3$, that is

$$X^3 + 2Y^3 = 4Z^3,$$

which is not possible, since we would have thus obtained a solution (X, Y, Z) to the equation $x^3 + 2y^3 = 4z^3$, with $0 < X < x_0$, thereby contradicting the minimal choice of x_0 .

- (855) If $m = 4k + r$, $0 \leq r \leq 3$, then $m^2 \equiv 0, 1$ or $4 \pmod{8}$. Consequently, $x^2 + y^2 \equiv 0, 1, 2, 4$ or $5 \pmod{8}$ while $8z + 7 \equiv 7 \pmod{8}$. Therefore, we conclude that this Diophantine equation has no integer solutions.

- (856) We may assume that the numbers x and y are not divisible by 7. Consequently, these numbers are of the form $7k \pm 1$, $7k \pm 2$ or $7k \pm 3$. Since $(7k \pm 1)^2 = 7(7k^2 \pm 2k) + 1$, $(7k \pm 2)^2 = 7(7k^2 \pm 4k) + 4$, $(7k \pm 3)^2 = 7(7k^2 \pm 6k + 1) + 2$, it follows that

$$(7k \pm 1)^4 = 7M + 1, \quad (7k \pm 2)^4 = 7N + 2, \quad (7k \pm 3)^4 = 7K + 4$$

and therefore that

$$x^4 + y^4 \equiv 1, 2, 3, 4, 5, 6 \pmod{7},$$

while $7z^2 \equiv 0 \pmod{7}$. Hence, equation $x^4 + y^4 = 7z^2$ has no integral solution.

For the second equation, the answer is again NO. In this case, we only need to reduce the given equation modulo 5.

- (857) We easily see that the left-hand side of the equation is congruent to 0, 2 or 4 modulo 8, while the right-hand side is congruent to 5 or 6 modulo 8. It follows that the equation has no integer solutions.

- (858) We have $z^n = z^2 z^{n-2} = z^{n-2}(x^2 + y^2) > x^n + y^n$, a contradiction.

- (859) We write the initial relation as

$$X^3 + 3Y^3 = 9Z^3.$$

We proceed by contradiction by first assuming that amongst all the solutions with $Z > 0$, the smallest one (in Z) is x, y, z . From the above equation, we derive that x is a multiple of 3, and we write $x = 3n$, which we immediately substitute in the relation. We then obtain that $9n^3 + y^3 = 3z^3$, which implies that y is also a multiple of 3. We then set $y = 3m$, which gives rise to the relation $9n^3 + 27m^3 = 3z^3$, which in turn implies that $3n^3 + 9m^3 = z^3$. Hence, $z = 3k$ and $3n^3 + 9m^3 = 27k^3$, which is equivalent to $n^3 + 3m^3 = 9k^3$. But this relation is of the same form as the initial equation. But $k = z/3$, which contradicts the minimal choice of z .

- (860) The answer is NO. Indeed, since $p|x$, we can set $x = x_0p$ for some positive integer x_0 . Substituting in the original equation, we obtain the equation

$$p^3x_0^4 + y^4 + pz^4 = p^2w^4.$$

This equation implies that $p|y$. As above, we then write $y = y_0p$, which yields the equation

$$p^2x_0^4 + p^3y_0^4 + z^4 = pw^4.$$

It follows that $p|z$. Hence, write $z = z_0p$, giving rise to the equation

$$px_0^4 + p^2y_0^4 + p^3z_0^4 = w^4.$$

This implies that $p|w$. Writing $w = w_0p$, we obtain

$$x_0^4 + py_0^4 + p^2z_0^4 = p^3w_0^4.$$

Now, this equation is of the same type as the original equation, but the integers x_0, y_0, z_0, w_0 are respectively strictly smaller than x, y, z, w . Therefore, the method of infinite descent of Fermat then guarantees that the original equation has no integer solution.

- (861) We proceed by contradiction by assuming that such a solution x, y, z exists, with positive integers x, y, z . First assume that x is odd and that y and z are even. We have

$$x^2 + y^2 + z^2 \equiv 1 \pmod{4} \quad \text{while} \quad 2xyz \equiv 0 \pmod{4},$$

which makes no sense. Similarly, x and y cannot be odd with z even. We can then finally show that x, y, z are odd. We have thus arrived at the conclusion that the three integers x, y, z must be even, say $x = 2x_1$, $y = 2y_1$, $z = 2z_1$, so that

$$x_1^2 + y_1^2 + z_1^2 = 4x_1y_1z_1 \equiv 0 \pmod{4},$$

which again implies that x_1, y_1 and z_1 must in turn be even. Continuing this process, we build triples (x_2, y_2, z_2) , (x_3, y_3, z_3) , ... delivering each time smaller and smaller even integers, which is impossible. This proves that there exist no integer solutions x, y, z .

- (862) Let (x, y) be a solution of

$$(1) \quad x^3 + y^3 = x^2 + y^2.$$

If $x = 0$, we find the two solutions $(x, y) = (0, 0)$ and $(x, y) = (0, 1)$. On the other hand, if $x \neq 0$, set $a = y/x$. Since, from (1), we always have $y \neq -x$, it follows that $a \neq -1$. Substituting $y = ax$ in (1), we find $x = \frac{1+a^2}{1+a^3}$. Similarly, substituting $x = y/a$ in (1), we find $y = \frac{a(1+a^2)}{1+a^3}$. We

have thus established that any solution (x, y) of (1) with $x \neq 0$ is of the form

$$(2) \quad x = \frac{1+a^2}{1+a^3}, \quad y = \frac{a(1+a^2)}{1+a^3} \quad (a \neq -1).$$

Reciprocally, one easily verifies that (2) produces a solution (x, y) of (1).
 (863) (*Problem due to Leo Moser*). We will provide a particular solution. Set $x_1 = 2$ and then, for each r , $2 \leq r \leq n$, let

$$x_r = x_1 x_2 \cdots x_{r-1} + 1.$$

Of course the equation is satisfied for $n = 1$. Hence, assume that the set $\{x_1, x_2, \dots, x_r\}$ satisfies the equation for $n = r$; we will show that the corresponding set with $n = r + 1$ satisfies the equation as well. But for $n = r + 1$, we have

$$\begin{aligned} \frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_{r+1}} + \frac{1}{x_1 x_2 \cdots x_{r+1}} &= 1 - \frac{1}{x_1 x_2 \cdots x_r} \\ &\quad + \frac{1}{x_{r+1}} + \frac{1}{x_1 x_2 \cdots x_{r+1}} \\ &= 1 + \frac{1}{x_{r+1}} - \frac{x_{r+1} - 1}{x_1 x_2 \cdots x_{r+1}} = 1 + \frac{1}{x_{r+1}} - \frac{x_1 \cdots x_r}{x_1 \cdots x_r x_{r+1}} = 1, \end{aligned}$$

as required.

REMARK: The sequence 2, 3, 7, 43, 1807, 3 263 443, ... is also the subject of Problem 479.

(864) We proceed by contradiction by assuming that such a solution x, y, z exists with positive integers x, y, z . First assume that x is odd and that y and z are even. We thus have

$$x^2 + y^2 + z^2 \equiv 1 \pmod{4} \quad \text{while} \quad x^2 y^2 \equiv 0 \pmod{4},$$

which makes no sense. Similarly, one can show x and y cannot be odd while z is even. Finally, one can show that x, y, z cannot be odd. We therefore arrive at the conclusion that all three integers x, y, z must be even, say $x = 2x_1, y = 2y_1, z = 2z_1$, so that we have

$$x_1^2 + y_1^2 + z_1^2 = 4x_1^2 y_1^2 \equiv 0 \pmod{4},$$

which again implies that x_1, y_1 and z_1 must also be even. Continuing, we construct triples $(x_2, y_2, z_2), (x_3, y_3, z_3), \dots$ each time made up of even numbers getting smaller and smaller, an endless process, which makes no sense. This argument implies that there is no integer solution x, y, z .

(865) Assuming that x is even or odd, we reach a contradiction.

(866) (*Problem introduced by Johann Walter*). Assume that such odd integers x, y, z exist. Then

$$(x^2 + 2xy + y^2) + (x^2 + 2xz + z^2) = y^2 + 2yz + z^2$$

and therefore

$$x^2 + xy + xz = yz.$$

By adding yz on each side, we obtain

$$(*) \quad (x+y)(x+z) = 2yz,$$

which is impossible because each of the expressions $x+y$ and $x+z$ is even, so that the left member of (*) is divisible by 4, while its right member is not, y and z being odd.

- (867) One easily checks that $x = (s^2 - pr^2)/2$, $y = rs$, $z = (s^2 + pr^2)/2$ is a solution. Conversely, if x, y, z is a primitive solution, then $y^2 = (z^2 - x^2)/p$ and thus $p|(z \pm x)$. Setting $s^2 = z \mp x$ and $r^2 = (z \pm x)/p$, we obtain the result.
- (868) Let $n, m \in \mathbb{N}$. Setting $x = n(n^2 - 12m^2)$, $y = m(4m^2 - 3n^2)$, we obtain $z = n^2 + 4m^2$, which yields infinitely many solutions.
- (869) The only solutions are $(x, y, n) = (0, 0, n)$ (with arbitrary n) and $(x, y, n) = (2, 2, 1)$. Indeed, first consider the case $n = 1$. The equation $x + y = xy$ becomes $x = (x-1)y$, which means that $x = 0$ (and $y = 0$) or that $x-1|x$, or in other words that $x-1 = 1$ or -1 (since $x-1$ and x are two consecutive integers). If $x-1 = 1$, then $x = 2$ and $y = 2$. If $x-1 = -1$, then $x = 0$ and $y = 0$. The second case is the one where $n \geq 2$. If x and y are positive, assume that $x > y > 0$; we then have $xy = x^n + y^n > x^n \geq x^2 \geq xy$, a contradiction.

Let us now examine the case where at least one of x, y is negative; clearly both cannot be negative. Assume that $x < 0$ and $y > 0$. If n is even, we are back to the above case. On the other hand, if n is odd, $n \geq 3$, then we can write $x = -a$, with $a > 0$, and say $y = b$. Then, the equation $x^n + y^n = xy$ becomes $a^n - b^n = ab$. Setting $a = b + r$, we have

$$\begin{aligned} a^n - b^n &= (b+r)^n - b^n > nb^{n-1}r + \binom{n}{2}b^{n-2}r^2 \geq 3b^2r + 3br^2 \\ &> b^2 + br = ab, \end{aligned}$$

a contradiction.

- (870) Since $z \geq \max(x, y)$, we derive from the equation that $n^x|n^y$ and $n^y|n^x$. Consequently, $x = y$ and it follows that $z = x + 1$ and $n = 2$.
- (871) Assume that $w \geq \max\{x, y\}$. Then, $n^x|n^w$. Since $n^x|n^z$, we have $n^x|n^y$. By the symmetry of the problem, we also have $n^y|n^x$. Therefore, $x = y$ and the equation we need to solve is reduced to (*) $2n^x + n^w = n^z$. In this case, we derive that $n^w|2n^x$, so that $2n^{x-w}$ is an integer for $x \leq w$. If $n > 2$, then $x = w$ and (*) becomes $3n^x = n^z$, which implies that $n = 3$ and the solution is $x = y = w = z - 1$. For $n = 2$, the solution is $x = y = w - 1 = z - 2$.
- (872) (*Contribution of A. Ivić, Belgrade*). Assume that the equation $x^p + y^q = z^r$ has a solution in positive integers x, y, z . Let $A = x^p$, $B = y^q$, $C = z^r$, so that

$$\prod_{p|ABC} p = \prod_{p|x^p y^q z^r} p = \prod_{p|xyz} p \leq xyz.$$

But since $x \leq z^{r/p}$, $y \leq z^{r/q}$, then according to the *abc* conjecture, for all $\varepsilon > 0$, there exists a positive constant $M = M(\varepsilon)$ such that

$$z^r \leq M \cdot \left(\prod_{p|ABC} p \right)^{1+\varepsilon} = M \cdot (xyz)^{1+\varepsilon} \leq M \cdot (z^r)^{(1+\varepsilon)(\frac{1}{p} + \frac{1}{q} + \frac{1}{r})}.$$

If $z \geq z_0$, we obtain

$$1 \leq (1 + \varepsilon) \left(\frac{1}{p} + \frac{1}{q} + \frac{1}{r} \right),$$

which contradicts (*) for ε sufficiently small.

- (873) In order to prove the result, we first observe that if $1 < a < b < c$ are three consecutive integers, then $ac + 1 = b^2$.

Now, since ac , 1 and b^2 are relatively prime, it follows from the *abc* conjecture that, for each $\varepsilon > 0$, there exists a positive constant $M = M(\varepsilon)$ such that

$$(*) \quad b^2 \leq M \cdot (\gamma(abc))^{1+\varepsilon} \leq M \cdot (\sqrt{abc})^{1+\varepsilon} \leq M \cdot b^{3(1+\varepsilon)/2},$$

where $\gamma(n)$ stands for the product of the prime numbers dividing n and where we used the fact that $ac < b^2$. It then follows from (*) that

$$b^{(1-3\varepsilon)/2} \leq M.$$

Choosing ε small enough, we find that b as well as a and c are bounded, which proves the result.

- (874) Assume that the number $m = n^3 + 1$ is powerful. Then, according to the *abc* conjecture, we have that for each $\varepsilon > 0$ there exists a positive constant $M = M(\varepsilon)$ such that

$$m < M \cdot \gamma(mn^3)^{1+\varepsilon},$$

where $\gamma(a)$ is the product of the prime numbers dividing a . Since $(m, n) = 1$, $\gamma(m) \leq \sqrt{m}$ and $n < m^{1/3}$, it follows that

$$m < M \cdot \gamma(nm)^{1+\varepsilon} < M \cdot (m^{1/3}n^{1/2})^{1+\varepsilon} = M \cdot m^{5(1+\varepsilon)/6},$$

so that

$$m^{\frac{1}{6} - \frac{5\varepsilon}{6}} < M.$$

Taking ε sufficiently small, we find that m is bounded, which proves the result.

The numbers $n = 2$ and $n = 23$ are the two smallest numbers (and possibly the only ones) such that the corresponding number $n^3 + 1$ is powerful: $2^3 + 1 = 3^2$ and $23^3 + 1 = 2^3 \cdot 3^2 \cdot 13^2$.

- (875) Assume that x, y, z are three 4-powerful numbers relatively prime and verifying $x + y = z$. We apply the *abc* conjecture to the triple (x, y, z) so that

$$z \leq M \cdot r(xyz)^{1+\varepsilon} \leq M \cdot (xyz)^{(1+\varepsilon)/4} \leq M \cdot z^{3(1+\varepsilon)/4}.$$

It follows that

$$z^{(1-3\varepsilon)/4} \leq M$$

and therefore that z is bounded, and similarly for x and y .

- (876) Let $a + b = c$, where c is 4-powerful and where a and b are 3-powerful, $(a, b) = 1$. For each $\varepsilon > 0$, we have $c < M(\varepsilon) \cdot \gamma(abc)^{1+\varepsilon}$. By hypothesis, we have

$$\gamma(a) \leq a^{1/3}, \quad \gamma(b) \leq b^{1/3}, \quad \gamma(c) \leq c^{1/3},$$

which implies that, using the *abc* conjecture,

$$c < M(\varepsilon) \left(a^{1/3} b^{1/3} c^{1/3} \right)^{1+\varepsilon} < M(\varepsilon) (c^{11/12})^{1+\varepsilon},$$

an inequality which cannot hold if ε is sufficiently small and c large enough. This clearly proves that only a finite number of such a, b, c integers can exist.

- (877) Let $y > 0$ be fixed and let $\varepsilon > 0$ be fixed and sufficiently small. Let also p_1, p_2, \dots, p_r be the list of all prime numbers $\leq y$. If $P(p^2 - 1) \leq y$ for a certain prime number p , then there exist nonnegative integers $\alpha_1, \dots, \alpha_r$ such that

$$p^2 - 1 = p_1^{\alpha_1} \cdots p_r^{\alpha_r}$$

and therefore

$$p^2 = p_1^{\alpha_1} \cdots p_r^{\alpha_r} + 1.$$

It follows from the *abc* conjecture that for all $\varepsilon > 0$, there exists a positive constant $M = M(\varepsilon) > 0$ such that

$$p^2 < M \cdot (p_1 p_2 \cdots p_r p)^{1+\varepsilon},$$

so that

$$p^{1-\varepsilon} < M \cdot (p_1 p_2 \cdots p_r)^{1+\varepsilon} < M \cdot y^{r(1+\varepsilon)}.$$

Since ε is small and $r = \pi(y)$ is fixed (as well as y), it follows that p is bounded, and the result is proved.

REMARK: For each odd prime number $y \leq 19$, here is the conjectured value of the largest element $p_* = p_*(y)$ of the set of prime numbers A_y :

$y =$	3	5	7	11	13	17	19
$p_* =$	17	31	4801	4801	8191	388961	1419263

Let us mention that although $P(p^2 - 1) > 11$ for each prime number $p > 4801$, the largest prime number p such that $P(p^2 - 1) = 11$ is $p = 881$ (in fact $P(4801^2 - 1) = 7$). For the other prime numbers y listed above, we have $P(p_*^2 - 1) = y$.

- (878) Since $p \equiv 1 \pmod{4}$, we have $q \equiv 1 \pmod{8}$, so that $\left(\frac{2}{q}\right) = 1$ and therefore, by Euler's Criterion,

$$2^{p-1} = 2^{\frac{q-1}{2}} \equiv \left(\frac{2}{q}\right) = 1 \pmod{q}.$$

On the other hand, since

$$n - 1 = pq - 1 = 2p^2 - p - 1 = (p - 1)(2p + 1) \quad \text{and} \quad 2^{p-1} \equiv 1 \pmod{p},$$

it follows that

$$\begin{aligned} 2^{n-1} &= (2^{p-1})^{2p+1} \equiv 1 \pmod{p}, \\ 2^{n-1} &= (2^{p-1})^{2p+1} \equiv 1 \pmod{q}, \end{aligned}$$

which implies that $2^{n-1} \equiv 1 \pmod{pq}$, as required.

- (879) First we define the function

$$M_0(\varepsilon) = \max_{\delta \geq \varepsilon} M(\delta),$$

which is decreasing for all $\varepsilon > 0$ and is such that $M(\varepsilon) \leq M_0(\varepsilon)$ for each $\varepsilon > 0$.

It follows from the *abc* conjecture that, for $i = 1, 2, 3$,

$$x_i \leq M(\varepsilon/3) \cdot (\gamma(x_1 x_2 x_3))^{1+\varepsilon/3} \leq M_0(\varepsilon/3) \cdot (\gamma(x_1 x_2 x_3))^{1+\varepsilon/3}$$

and therefore that

$$x_1 x_2 x_3 \leq M_0(\varepsilon/3)^3 \cdot (\gamma(x_1 x_2 x_3))^{3+\varepsilon}.$$

Hence, if the conclusion (*) is false, then

$$x_i > M_0(\varepsilon) \cdot \gamma(x_i)^{3+\varepsilon} \quad (i = 1, 2, 3)$$

and therefore

$$x_1 x_2 x_3 > M_0(\varepsilon)^3 \cdot (\gamma(x_1 x_2 x_3))^{3+\varepsilon},$$

in which case we would have

$$M_0(\varepsilon)^3 \cdot \gamma(x_1 x_2 x_3)^{3+\varepsilon} < x_1 x_2 x_3 \leq M_0(\varepsilon/3)^3 \cdot \gamma(x_1 x_2 x_3)^{3+\varepsilon},$$

and therefore

$$M_0(\varepsilon) < M_0(\varepsilon/3),$$

which is impossible since M_0 is decreasing.

(880) (*Math. Intelligencer* **18** (1996), p. 58). It is false. Indeed, choosing $n = 3$, $x = 10$, $y = 16$, $z = 17$, we obtain a contradiction. This counterexample is due to Roger Apéry, the famous mathematician who proved the irrationality of $\zeta(3) = \sum_{n=1}^{\infty} 1/n^3$.

(881) From Wilson's Theorem, we have $(p-1)! \equiv -1 \pmod{p}$ so that

$$\xi := \frac{(p-1)! + 1}{p} \quad \text{is an integer.}$$

It follows that

$$r^{(p-1)!+1} = \underbrace{r^{(p-1)!} + \dots + r^{(p-1)!}}_r.$$

Hence,

$$(r^\xi)^p = \left(r^{(p-1)!/\alpha_1}\right)^{\alpha_1} + \left(r^{(p-1)!/\alpha_2}\right)^{\alpha_2} + \dots + \left(r^{(p-1)!/\alpha_r}\right)^{\alpha_r}.$$

The result follows by setting

$$n = r^\xi \quad \text{and} \quad x_i = r^{(p-1)!/\alpha_i} \quad \text{for } i = 1, 2, \dots, r.$$

(882) From Wilson's Theorem, $(p-1)! \equiv -1 \pmod{p}$ and this is why

$$\xi := \frac{(p-1)! + 1}{p} \quad \text{is a positive integer. It follows that}$$

$$\begin{aligned} (2^\xi)^p &= 2^{(p-1)!+1} = 2^{(p-1)!} + 2^{(p-1)!} = \left(2^{(p-1)!/(p-1)}\right)^{p-1} \\ &\quad + \left(2^{(p-1)!/(p-1)}\right)^{p-1}. \end{aligned}$$

The result then follows by choosing

$$x = 2^{(p-2)!}, \quad y = 2^{(p-2)!}, \quad z = 2^\xi.$$

(883) It is all the prime numbers p satisfying one of the congruences $p \equiv 1, 5, 7, 9, 19, 25, 35, 37, 39, 43 \pmod{44}$.