Solutions

(1) (a) If $n = 1$, the result is true. Assume that the result is true for n and let us prove it for $n+1$. Since

$$1 + 2 + \cdots + n + (n+1) = \frac{n(n+1)}{2} + (n+1) = \frac{(n+1)(n+2)}{2},$$

the result follows.

(b) If $n = 1$, the result is true. Assume that the result is true for n and let us prove it for $n+1$. Since

$$\begin{aligned} 1^2 + 2^2 + \cdots + n^2 + (n+1)^2 &= \frac{n(n+1)(2n+1)}{6} + (n+1)^2 \\ &= \frac{(n+1)(n+2)(2n+3)}{6}, \end{aligned}$$

the result follows.

(c) If $n = 1$, the result is true. Assume that the result is true for n and let us prove it for $n+1$. Since

$$\begin{aligned} 1^3 + 2^3 + \cdots + n^3 + (n+1)^3 &= \frac{n^2(n+1)^2}{4} + (n+1)^3 \\ &= \frac{(n+1)^2(n+2)^2}{4}, \end{aligned}$$

the result follows.

(2) We only need to observe that

$$\begin{aligned} n^3 &= (1^3 + 2^3 + \cdots + n^3) - (1^3 + 2^3 + \cdots + (n-1)^3) \\ &= \left(\frac{n(n+1)}{2}\right)^2 - \left(\frac{(n-1)n}{2}\right)^2, \end{aligned}$$

where we used the identity of Problem 1 (c).

(3) The given expression can be written as $\frac{1}{2}\left(\frac{3}{2} - \frac{2n+1}{n(n+1)}\right)$ for each positive integer n.

(4) By using the formula of Problem 1 (a), we obtain

$$2 + 4 + \cdots + 2n = 2(1 + 2 + \cdots + n) = 2 \cdot \frac{n(n+1)}{2} = n(n+1).$$

(5) Since $\sum_{j=1}^{n} j = \frac{n(n+1)}{2}$, it is enough to check that

$$(*) \qquad \sum_{j=1}^{n} (-1)^j j^2 = (-1)^n \frac{n(n+1)}{2}.$$

We use induction. First of all, $(*)$ is true for $n = 1$. Assume that $(*)$ is true for $n = k$; we will show that this implies that it is true for $n = k+1$.

Indeed, this implies successively

$$\begin{aligned}\sum_{j=1}^{k+1}(-1)^j j^2 &= \sum_{j=1}^{k}(-1)^j j^2 + (-1)^{k+1}(k+1)^2\\ &= (-1)^k\frac{k(k+1)}{2} + (-1)^{k+1}(k+1)^2\\ &= (-1)^{k+1}\left((k+1)^2 - \frac{k(k+1)}{2}\right)\\ &= (-1)^{k+1}\frac{(k+1)(k+2)}{2},\end{aligned}$$

as required.

(6) This result can be proved by induction by observing that

$$\begin{aligned}a^{2n+1}+b^{2n+1} &= a^{2n+1}+a^2b^{2n-1}-a^2b^{2n-1}+b^{2n+1}\\ &= a^2(a^{2n-1}+b^{2n-1})-b^{2n-1}(a^2-b^2).\end{aligned}$$

(7) This result can be proved by induction by observing that

$$\begin{aligned}a^{4n+4}-b^{4n+4} &= a^{4n+4}-a^4b^{4n}+a^4b^{4n}-b^{4n+4}\\ &= a^4(a^{4n}-b^{4n})+b^{4n}(a^4-b^4).\end{aligned}$$

(8) This result can be proved using induction by observing that

$$a^{n+1}-b^{n+1} = a^{n+1}-a^nb+a^nb-b^{n+1} = a^n(a-b)+b(a^n-b^n).$$

(9) We use induction as well as the relation

$$\sum_{j=1}^{n+1} j\cdot j! = \sum_{j=1}^{n} j\cdot j! + (n+1)(n+1)!.$$

(10) Multiplying the given inequality $(2n)! < 2^{2n}(n!)^2$ by the trivial inequality $(2n+1) < 2^2(n+1)^2$, then using induction, one easily proves the inequality.

(11) Multiplying the relation $(n+1)^3/n^3 < (n+1)$ (valid for $n \geq 3$) by the given inequality $n^3 < n!$ allows one to use induction and thereby obtain the result.

(12) This follows from

$$\begin{aligned}(1+\theta)^{n+1} = (1+\theta)^n(1+\theta) \geq (1+n\theta)(1+\theta) &= 1+\theta+n\theta+n\theta^2\\ &> 1+(n+1)\theta.\end{aligned}$$

REMARK: This inequality is often called the *Bernoulli inequality*, being attributed to Jacques Bernoulli (1654–1705).

(13) By using the induction hypothesis and by observing that

$$\begin{aligned}(1+\theta)^{n+1} = (1+\theta)(1+\theta)^n &\geq (1+\theta)(1+n\theta+\frac{n(n-1)}{2}\theta^2)\\ &\geq 1+(n+1)\theta+\frac{(n+1)n}{2}\theta^2,\end{aligned}$$

the result follows.

(14) We prove this result by induction. For $n = 1$, the result is true. Assume that it is true for n and let us prove it for $n+1$. Since

$$\frac{1}{3}\Big((n+1)^3 + 2(n+1)\Big) = \frac{1}{3}(n^3 + 3n^2 + 3n + 1 + 2n + 2) = \frac{1}{3}(n^3 + 2n) + n^2 + n + 1$$

is an integer because of the induction hypothesis, the result follows.

(15) Let $f(n) = 10^n + 3 \cdot 4^{n+2} + 5$. Since $f(0) = 54$ is divisible by 9 and since for each integer $n \geq 0$, $\dfrac{f(n+1) - f(n)}{9} = 10^n + 4^{n+2}$ is an integer, the result follows.

(16) The first equation is equivalent (after simplification) to

$$\frac{1}{n-k} = \frac{1}{k+1},$$

which in turn is equivalent to $n = 2k + 1$, as was to be shown.

(17) (a) By adding the relations

$$2^{2n} = (1+1)^{2n} = \sum_{k=0}^{2n} \binom{2n}{k} \text{ and } 0 = (1-1)^{2n} = \sum_{k=0}^{2n} (-1)^k \binom{2n}{k},$$

we obtain

$$2^{2n} = \sum_{k=0}^{2n} (1 + (-1)^k) \binom{2n}{k} = 2 \sum_{k=0}^{n} \binom{2n}{2k},$$

which yields the result.

(b) This follows essentially from part (a) and the fact that $\sum_{k=0}^{2n} \binom{2n}{k} = 2^{2n}$.

(18) Comparing the geometric mean with the arithmetic mean, we obtain

$$(n!)^{1/n} \leq \frac{1 + 2 + \cdots + n}{n} = \frac{n(n+1)}{2n} = \frac{n+1}{2},$$

so that the result follows by raising each side to the power n.

(19) (Gelfand [**13**]) For $n = 8$ the result is true. Assume that k can be written as a sum of 3's and 5's. Then, this sum contains one 5 (possibly many) or none at all. In the first case, we replace a 5 by two 3's. The new number $k+1$ then contains 3's or 5's. In the second case, there is at least three 3's, and we can replace them by two 5's. The new number $k+1$ then contains 3's or 5's. This proves the result.

(20) Let P_n be the following proposition: the number of lines thus created by n points for which no combination of three of these points are on a straight line is $n(n-1)/2$. Since two points determine a straight line and since $2(2-1)/2 = 1$, P_2 is true. Assume now that P_n is true for an integer $n \geq 2$. If a new point is added to the collection of n points in such a way that it cannot be on a straight line created by two of the points, then n additional lines will thus be added and the new collection of $n+1$ points will determine $n(n-1)/2 + n = \frac{n(n+1)}{2}$ lines. The result then follows by induction.

(21) The proof is done by induction. If

$$1+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\cdots+\frac{1}{\sqrt{k}}>\sqrt{k}$$

then, since $\sqrt{k+1}-\sqrt{k}<1/\sqrt{k+1}$, the sum of these two inequalities gives the result.

(22) Let

$$R_{2k-1}:=1^3+3^3+5^3+\cdots+(2k-1)^3.$$

In light of Problem 1 (c), we know that

$$S_n:=1^3+2^3+3^3+\cdots+n^3=\left(\frac{n(n+1)}{2}\right)^2.$$

For $n=2k-1$, this last sum can be written as

$$\begin{aligned} S_{2k-1} &= \left(1^3+3^3+\cdots+(2k-1)^3\right)+\left(2^3+4^3+\cdots+(2k-2)^3\right)\\ &= \left(1^3+3^3+\cdots+(2k-1)^3\right)+2^3\left(1^3+2^3+3^3+\cdots+(k-1)^3\right)\\ &= R_{2k-1}+2^3S_{k-1}.\end{aligned}$$

It follows that

$$\begin{aligned} R_{2k-1} &= S_{2k-1}-2^3S_{k-1}=\left(\frac{(2k-1)(2k)}{2}\right)^2-8\left(\frac{(k-1)k}{2}\right)^2\\ &= k^2(2k-1)^2-2k^2(k-1)^2=k^2(2k^2-1),\end{aligned}$$

as was to be shown.

(23) Using the Binomial Theorem, we have

$$\begin{aligned} (n+1)^{k+1}-1 &= \left((n+1)^{k+1}-n^{k+1}\right)+\left(n^{k+1}-(n-1)^{k+1}\right)+\cdots\\ &\qquad\qquad +\left(2^{k+1}-1^{k+1}\right)\\ &= \sum_{j=1}^{n}\left\{(j+1)^{k+1}-j^{k+1}\right\}\\ &= \sum_{j=1}^{n}\left\{j^{k+1}+\binom{k+1}{1}j^k+\binom{k+1}{2}j^{k-1}+\cdots\right.\\ &\qquad\qquad \left.+\binom{k+1}{k+1}j^0-j^{k+1}\right\}\\ &= \sum_{j=1}^{n}\left\{\binom{k+1}{1}j^k+\binom{k+1}{2}j^{k-1}+\cdots+\binom{k+1}{k+1}j^0\right\}\\ &= \sum_{j=1}^{n}\left\{\binom{k+1}{k}j^k+\binom{k+1}{k-1}j^{k-1}+\cdots+\binom{k+1}{0}j^0\right\}\\ &= \sum_{j=1}^{n}\sum_{r=0}^{k}\binom{k+1}{r}j^r\\ &= \sum_{r=0}^{k}\binom{k+1}{r}\sum_{j=1}^{n}j^r=\sum_{r=0}^{k}\binom{k+1}{r}S_r(n).\end{aligned}$$

Therefore, if $S_r(n)$ is a polynomial of degree $r+1$ for each positive integer $r \le k-1$, we may conclude, using induction, that $S_k(n)$ is a polynomial of degree $k+1$.

(24) (a) The required formula is $\displaystyle\prod_{i=2}^{n}\left(1-\frac{1}{i}\right)=\frac{1}{n}$, since

$$\prod_{i=2}^{n}\left(1-\frac{1}{i}\right)=\frac{1}{2}\cdot\frac{2}{3}\cdot\frac{3}{4}\cdots\frac{n-2}{n-1}\cdot\frac{n-1}{n}=\frac{1}{n}.$$

(b) The required formula is $\displaystyle\prod_{i=2}^{n}\left(1-\frac{1}{i^2}\right)=\frac{n+1}{2n}$, since

$$\prod_{i=2}^{n}\left(1-\frac{1}{i^2}\right)=\frac{3}{2^2}\cdot\frac{(3-1)(3+1)}{3^2}\cdot\frac{(4-1)(4+1)}{4^2}\cdots$$
$$\cdot\frac{(5-1)(5+1)}{5^2}\cdot\frac{(n-2)n}{(n-1)^2}\cdot\frac{(n-1)(n+1)}{n^2}=\frac{n+1}{2n}.$$

(25) Let S_n be the given sum. Since

$$i^4+i^2+1=(i^2+1)^2-i^2=(i^2+i+1)(i^2-i+1)$$

and since

$$\frac{i}{(i^2+i+1)(i^2-i+1)}=\frac{1}{2}\left(\frac{1}{i^2-i+1}-\frac{1}{i^2+i+1}\right),$$

we have

$$\begin{aligned}S_n &= \frac{1}{2}\left(\sum_{i=1}^{n}\frac{1}{i^2-i+1}-\sum_{i=1}^{n}\frac{1}{i^2+i+1}\right)\\ &= \frac{1}{2}\left(\sum_{i=1}^{n}\frac{1}{(i-1)i+1}-\sum_{i=1}^{n}\frac{1}{i(i+1)+1}\right)\\ &= \frac{1}{2}\left(1-\frac{1}{n(n+1)+1}\right),\end{aligned}$$

and the result follows.

(26) We will show that the choice $(m,n,r)=(2,3,7)$, for which $S=41/42$, maximizes the sum S in the sense that for any other choice (m,n,r), with $S<1$, we must have $S<\frac{41}{42}$. So let us consider such a triple (m,n,r). Without any loss in generality, we may assume that $2\le m\le n\le r$. We shall first show that $m=2$. Indeed, if $m\ge 3$, then

$$S\le\frac{1}{3}+\frac{1}{3}+\frac{1}{4}=\frac{11}{12}<\frac{41}{42}.$$

Hence, $m=2$. Let us now show that $n=3$. If $n\ge 4$, we have

$$S\le\frac{1}{2}+\frac{1}{4}+\frac{1}{5}=\frac{19}{20}<\frac{41}{42}.$$

Hence, $n=3$. It remains to show that $r=7$. Two situations need to be considered: $3\le r\le 6$ and $r\ge 8$. In the first case, we have

$$S=\frac{1}{2}+\frac{1}{3}+\frac{1}{r}\ge\frac{1}{2}+\frac{1}{3}+\frac{1}{6}=1,$$

a contradiction. In the second case, we have

$$S = \frac{1}{2} + \frac{1}{3} + \frac{1}{r} \leq \frac{1}{2} + \frac{1}{3} + \frac{1}{8} = \frac{23}{24} < \frac{41}{42}.$$

We may therefore conclude that $r = 7$, thus completing the proof.

(27) The problem is equivalent to the combinatorial problem which consists in distributing k balls in ℓ urns with the restriction that there must be at least one ball in the first urn. We shall call upon the combinatorial theorem according to which there are $\binom{k-1}{\ell-1}$ distinct vectors with (positive) integer components satisfying the relation

$$x_1 + x_2 + \cdots + x_\ell = k.$$

We then place the first ball in the first urn and distribute the $k-1$ remaining balls in the ℓ urns. The above result then yields

$$A_k(\ell) = \binom{k+\ell-2}{\ell-1},$$

as required.

(28) We prove the first formula; the proof of the second formula is similar. First of all, it is true for $n = 1$. Assume that it is true for each odd number $n \leq k$, k odd. Since

$$1^2 + 3^2 + 5^2 + \cdots + k^2 + (k+2)^2 = \binom{k+2}{3} + (k+2)^2,$$

the result will follow if we manage to show that

$$\binom{k+2}{3} + (k+2)^2 = \binom{k+4}{3}$$

or similarly that

$$\binom{k+4}{3} - \binom{k+2}{3} = (k+2)^2.$$

But this relation is true, since

$$\begin{aligned}\binom{k+4}{3} - \binom{k+2}{3} &= \frac{(k+4)!}{3!(k+1)!} - \frac{(k+2)!}{3!(k-1)!}\\ &= \frac{(k+2)!}{3!(k-1)!}\left(\frac{(k+3)(k+4)}{k(k+1)} - 1\right)\\ &= \frac{(k+2)!}{3!(k-1)!}\frac{(6k+12)}{k(k+1)}\\ &= \frac{k+2}{6}(6k+12) = (k+2)^2,\end{aligned}$$

which completes the proof.

(29) (*CRUX, 1975*). Since the sum of the elements of any subset cannot exceed $90 + 91 + \cdots + 99 = 945$, the sum of the elements of the subsets of S can be found amongst the numbers $1, 2, \ldots, 945$. Since the set S contains 10 elements, we can form $2^{10} - 1 = 1023$ nonempty different subsets. The Pigeonhole Principle then allows us to conclude that there exist (at least) two subsets having the same sum. By removing the elements which are common to these two subsets, we obtain two disjoint subsets with the same sum.

(30) There are exactly 50 possible remainders when we divide the numbers by 50, and these remainders are the numbers: 0,1,2,..., 49. Since we have 51 integers and only 50 possible remainders, it follows that using the Pigeonhole Principle, there are at least two numbers amongst these 51 integers having the same remainder. Then, the difference of these two integers has 0 as a remainder and is therefore divisible by 50.

(31) For each n-th day of the year, let a_n be the total number of solved problems between the first day and the n-th day inclusively. Then $a_1, a_2, \ldots$ is a strictly increasing sequence of positive integers. Consider another sequence $b_1, b_2, \ldots$ obtained by adding 20 to each element of the preceding sequence, that is $b_n = a_n + 20$, $n = 1, 2, \ldots$. The b_n's are strictly increasing and are also all distinct. But for a period of eight consecutive weeks (one needs to consider at least seven consecutive weeks) during the year, the student cannot solve more than $11 \cdot 8 = 88$ problems. Then, the numbers a_n are located between 1 and 88 inclusively, while the b_n's are between 21 and 108 inclusively. Since there are 56 days in eight weeks, the concatenation of the two sequences gives

$$a_1, a_2, \ldots, a_{56}, a_1 + 20, a_2 + 20, \ldots, a_{56} + 20,$$

which yields a total of 112 distinct integers all located between 1 and 108 inclusively. By the Pigeonhole Principle, at least two elements of the concatenated sequence must be equal. One of the two must be in the first half of the sequence and the other in the second part. Let a_j and $a_k + 20$ be these two integers. We then have $a_k - a_j = 20$, which implies that the student must solve exactly 20 problems between the $(j+1)$-th day and the k-th day of the year.

(32) Divide the surface of the table into squares of 3 inches by 3 inches. We then have a total of 2000 squares. The diagonal of each of these squares is $\sqrt{18}$ inches long, that is approximately 4.25 inches. Therefore, a cylindrical glass of diameter 5 inches will cover entirely any given square. Hence, if we place seven marbles in each square, there will be a total of 14000 marbles on the table. Hence, by the Pigeonhole Principle, since we have a total of 14001 marbles, one of these will contain at least eight marbles.

(33) We easily see that the number N_1 of secants thus drawn is given by $N_1 = \binom{n}{2}$. Let N_2 be the number of points of intersection of these secants. For any group of points taken four by four, there is exactly two secants joining the points that intersect inside the circle, so that $N_2 = \binom{n}{4}$.

We are then ready to count the number of regions in terms of n. At each drawn secant, the circle is divided into an additional region. At each intersection point, the circle is divided into an additional region. The solution is therefore given by

$$1 + N_1 + N_2 = \binom{n}{2} + \binom{n}{4} + 1,$$

which can also be written in the polynomial form $n^4/24 - n^3/4 + 23n^2/24 - 3n/4 + 1$, provided that $n > 3$.

(34) Let $f(n)$ be the number of required moves, that is the number of moves that are necessary to succeed in transferring a tower of n disks. It is easy to see that we must

(a) first move the $n-1$ disks from the top of the first post to the second post (using in the process the third post);
(b) then move the largest disk to the third post;
(c) and finally move the $n-1$ disks from the second post to the third (using if need be the first post).

We then obtain, by setting $f(0)=0$ and $f(1)=1$,

$$f(n)=2f(n-1)+1 \qquad (n\geq 1).$$

We observe that $f(2)=3$, $f(3)=7,\ldots$, and we then conjecture that

$$f(n)=2^n-1,$$

a result which can easily be proved by induction.

(35) Let N be an arbitrary positive integer. Let F_{i_1} be such that $F_{i_1}\leq N< F_{i_1}+1$. Set $\Delta_1=N-F_{i_1}$. If $\Delta_1=0$, we are done, since $N=F_{i_1}$. Otherwise, let F_{i_2} with $i_2<i_1$ be such that $F_{i_2}\leq \Delta_1<F_{i_2}+1$. If $\Delta_2:=\Delta_1-F_{i_2}=0$, we are done, since in this case $N=\Delta_1+F_{i_1}=\Delta_2+F_{i_2}+F_{i_1}=F_{i_1}+F_{i_2}$. Otherwise, we choose F_{i_3} such that $F_{i_3}\leq \Delta_2<F_{i_3}+1$, and so on. The process will end since the sequence of positive integers Δ_i is decreasing, so that eventually we will obtain $\Delta_r=0$ for a certain positive integer r, in which case we have

$$N=F_{i_1}+F_{i_2}+\cdots+F_{i_r}.$$

(36) (Problem #360 in Barbeau, Klamkin & Moser [**3**]) We first observe that

$$\begin{array}{rcl} 1^2-0^2 & = & 1\\ 3^2-2^2 & = & 5\\ 5^2-4^2 & = & 9\\ 7^2-6^2 & = & 13\\ \vdots & & \vdots \end{array}$$

while

$$\begin{array}{rcl} 2^2-1^2 & = & 3\\ 4^2-3^2 & = & 7\\ 6^2-5^2 & = & 11\\ 8^2-7^2 & = & 15\\ \vdots & & \vdots \end{array}$$

Hence, it easily follows that

$$\big((m+3)^2-(m+2)^2\big)-\big((m+1)^2-m^2\big)=4 \qquad (m=0,1,2,3,\ldots);$$

that is

$$4=m^2-(m+1)^2-(m+2)^2+(m+3)^2 \qquad (m=0,1,2,3,\ldots).$$

It follows from this that if n can be written as

$$n=e_1 1^2+e_2 2^2+e_3 3^2+e_4 4^2+\cdots+e_k k^2,$$

the same is true for $n+4$, since we then have

$$n+4 = e_1 1^2 + e_2 2^2 + \cdots + e_k k^2 + (k+1)^2 - (k+2)^2 \\ -(k+3)^2 + (k+4)^2.$$

As we mentioned in the statement of the problem, the numbers 1, 2, 3 and 4 can be written in the stated form. We may therefore conclude that all the integers can be written in this form. Our argument therefore establishes the result without however providing the explicit form taken by any given integer. Curiously, it is nevertheless possible to obtain explicitly such a representation. Here it is. Each integer ≥ 5 is of the form $4r+1$, $4r+2$, $4r+3$ or $4r+4$, with $r \geq 1$, and we can establish that

$$\begin{aligned} 4r+1 &= 1^2 + \sum_{i=1}^{2r} (-1)^{i+1}((2i)^2 - (2i+1)^2), \\ 4r+2 &= -1^2 - 2^2 - 3^2 + 4^2 + \sum_{i=2}^{2r+1} (-1)^i ((2i+1)^2 - (2i+2)^2), \\ 4r+3 &= -1^2 + 2^2 + \sum_{i=1}^{2r} (-1)^{i+1}((2i+1)^2 - (2i+2)^2), \\ 4r+4 &= -1^2 - 2^2 + 3^2 + \sum_{i=2}^{2r+1} (-1)^i ((2i)^2 - (2i+1)^2). \end{aligned}$$

(37) We construct a procedure, using MAPLE software (here, we have used version 5), which gives the positive integers $n \leq N$ such that $(!n, n!) \neq 2$.

```
> kurepa:=proc(N)
> local n;
> for n from 1 to N do
> if gcd(sum(k!,k=0..n-1),n!)<>2
> then print(' n'=n) else fi;od; end:
> kurepa(1000);
```

(38) Assume that the integers a and b are increased by n. Then we have

$$a+n = (b+n)652 + 8634 - 651n.$$

Since the remainder must be positive, it follows that $651n \leq 8634$, that is $n \leq 13.26$. Hence, $n = 13$.

REMARK: More generally, with $a = bq + r$ instead of $(*)$, the quantity n is given by $n = \left[\frac{r}{q-1}\right]$.

(39) Let n be the number of "1"s in N. We then have

$$N = 1 + 2^1 + 2^2 + 2^3 + \cdots + 2^{n-1} = 2^n - 1.$$

Therefore,

$$\begin{aligned} N^2 &= 2^{2n} - 2^{n+1} + 1 = 2^{n+1}\left(2^{n-1} - 1\right) + 1 \\ &= 2^{n+1}\left(2^{n-2} + 2^{n-3} + \cdots + 2 + 1\right) + 1 \\ &= 2^{2n-1} + 2^{2n-2} + \cdots + 2^{n+2} + 2^{n+1} + 1, \end{aligned}$$

an expression which can be written as follows in basis 2:

$$\underbrace{11\ldots11}_{n-1}\underbrace{00\ldots00}_{n}1.$$

(40) Let $N = 7^{37}+13^{37}+19^{37}$. We will show that $3|N$ and that $13|N$. Indeed,

$$N = (6+1)^{37} + (12+1)^{37} + (18+1)^{37} = 3A+3$$

for a certain integer A, while

$$N = (13-6)^{37} + 13^{37} + (13+6)^{37} = 13^{37}B + (-6)^{37} + 6^{37} = 13^{37}B$$

for a certain integer B. Since $(3,13) = 1$, it follows that $39|N$.
REMARK: This result remains true when the integer 37 is replaced by an arbitrary odd positive integer. Moreover, note that by using congruences, the proof is almost immediate.

(41) We first observe that

$$49 = 48+1, \quad 2352 = 7^2 \cdot 48 \quad \text{and} \quad 2304 = 48^2.$$

We therefore need to show that

(1) $$48^2|49^n - 49\cdot 48\cdot n - 1.$$

In fact, we will show the more general result

(2) $$(a-1)^2|a^n - a(a-1)n - 1.$$

First of all, we observe that

$$\begin{aligned} a^n - a(a-1)n - 1 &= (a^n-1) - a(a-1)n \\ &= (a-1)(a^{n-1}+a^{n-2}+\cdots+a+1) - a(a-1)n \\ &= (a-1)\left(a^{n-1}+a^{n-2}+\cdots+a+1-an\right). \end{aligned}$$

Since the expression $a^{n-1}+a^{n-2}+\cdots+a+1-an$ vanishes when $a=1$ and is divisible by $a-1$, we have

$$a^n - a(a-1)n - 1 = (a-1)^2\cdot N$$

for a certain positive integer N, which establishes (2) and therefore (1).

(42) Let n be an arbitrary positive integer and let $N = n^4+2n^3+2n^2+2n+1$. It is clear that

$$\begin{aligned} N &= (n+1)^4 - (2n^3+4n^2+2n) \\ &= (n+1)^4 - 2n(n+1)^2 = (n+1)^2(n^2+1). \end{aligned}$$

Therefore, if N is a perfect square, there exists a positive integer a such that $(n+1)^2(n^2+1) = a^2$, in which case there exists another integer b such that $n^2+1 = b^2$. Since two perfect squares cannot be consecutive, the result is proved.

(43) We write $N = 10a+b$ where $0 < a \le 9$, $0 \le b \le 9$ and $M = 10b+a$. In this case, $M-N = 9(b-a)$ and the first result is proved. In order to find the integers N such that $|M-N| = 18$, it is enough to choose $|b-a| = 2$. Therefore, $N = 13, 20, 24, 31, 35, 42, 46, 53, 57, 64, 68, 75, 79, 86, 97$.

(44) The answer is YES. Since $n = 3m+r$, $0 \le r \le 2$, it is obvious that $(3, n^2+1) = 1$, and the statement is verified.

(45) The answer is YES. By simply writing $n = 5m + r$, where $0 \leq r \leq 4$, we easily obtain the result. The result is clearly the same when we replace 5 by 7.

(46) Let $n = kp + r$, $0 \leq r \leq p - 1$. Using the Binomial Theorem, we obtain

$$\begin{aligned} N(n) &= a_0 + a_1(kp + r) + \cdots + a_{s-1}(kp + r)^{s-1} + a_s(kp + r)^s \\ &= N(r) + pM, \end{aligned}$$

for a certain integer M. Hence, $p|N(n)$ if and only if $p|N(r)$.

Setting $n = 7k + r$, $0 \leq r \leq 6$, we find that the required integers are those of the form $n = 7k + 1$ as well as those of the form $n = 7k + 4$, where $k \in \mathbb{Z}$.

(47) (*1987 American Invitational Mathematics Examination*). Let N be the number to compute. Since $324 = 18^2$ and since

$$a^4 + 18^2 = (a^2 + 18)^2 - 36a^2 = (a^2 + 18 + 6a)(a^2 + 18 - 6a),$$

the number N can be written as

$$\begin{aligned} N &= \prod_{k=0}^{4} \frac{(10 + 12k)^4 + 18^2}{(4 + 12k)^4 + 18^2} \\ &= \prod_{k=0}^{4} \frac{[(10 + 12k)^2 + 18 + 6(10 + 12k)][(10 + 12k)^2 + 18 - 6(10 + 12k)]}{[(4 + 12k)^2 + 18 + 6(4 + 12k)][(4 + 2k)^2 + 18 - 6(4 + 12k)]} \\ &= \prod_{k=0}^{4} \frac{(144k^2 + 312k + 178)(144k^2 + 168k + 58)}{(144k^2 + 168k + 58)(144k^2 + 24k + 10)} \\ &= \prod_{k=0}^{4} \frac{144k^2 + 312k + 178}{144k^2 + 24k + 10}. \end{aligned}$$

But since $144(k + 1)^2 + 24(k + 1) + 10 = 144k^2 + 312k + 178$, the number N can be written as

$$N = \frac{144 \cdot 4^2 + 312 \cdot 4 + 178}{10} = 373.$$

(48) Consider the number 10101 in basis $b \geq 2$. Then

$$\begin{aligned} 10101 = 1 \cdot b^4 + 0 \cdot b^3 + 1 \cdot b^2 + 0 \cdot b + 1 \cdot b^0 &= b^4 + b^2 + 1 \\ &= (b^2 + b + 1)(b^2 - b + 1), \end{aligned}$$

a product of two integers larger than 1.

(49) The product of four consecutive integers is

$$N := n(n + 1)(n + 2)(n + 3).$$

Since a member of the product is divisible by 4 and another is divisible by 2, this shows that $8|N$. On the other hand, if we write $n = 3k + r$, $0 \leq r \leq 2$, we easily see that $3|N$. Since $(3, 8) = 1$, we conclude that $24|N$.

(50) If n is an odd positive integer, we know that $a + b|a^n + b^n$. Therefore,

$$7 = 1 + 6|1^{47} + 6^{47}, \quad 7 = 2 + 5|2^{47} + 5^{47}, \quad 7 = 3 + 4|3^{47} + 4^{47},$$

and the result follows.

(51) (*CRUX, 1987; solution given by Aage Bondesen*). Let $N = n(n+1)(n+2)(n+3)(n+4)$ be the given product. It is clear that N must contain two or three multiples of 2, one or two multiples of 3, only one multiple of 5 and no more than one multiple of any other prime number. Thus, if the product is a perfect square, each of the integers $n+j$ $(0 \leq j \leq 4)$ can be written as

$$2^r 3^s 5^{2a} 7^{2b} \cdots \qquad (r \geq 2, s \geq 1, a \geq 1, b \geq 0, \ldots).$$

In short, each of the integers $n+j$ $(0 \leq j \leq 4)$ is of one of the following forms:

(i) r even, s even:
$n+j = 2^{2k}3^{2m}5^{2a}7^{2b}\cdots = \left(2^k 3^m 5^a 7^b \cdots\right)^2$, a perfect square;

(ii) r odd, s even:
$n+j = 2^{2k+1}3^{2m}5^{2a}7^{2b}\cdots = 2\left(2^k 3^m 5^a 7^b \cdots\right)^2$, twice a perfect square;

(iii) r even, s odd:
$n+j = 2^{2k}3^{2m+1}5^{2a}7^{2b}\cdots = 3\left(2^k 3^m 5^a 7^b \cdots\right)^2$, that is three times a perfect square;

(iv) r odd, s odd:
$n+j = 2^{2k+1}3^{2m+1}5^{2a}7^{2b}\cdots = 6\left(2^k 3^m 5^a 7^b \cdots\right)^2$, that is six times a perfect square.

But we have five factors in the product N, each being of one of the above four types. Using the Pigeonhole Principle, we may conclude that two of the factors $n+j$ must be of the same type. Let us first examine the possibility that it is one of the types (ii), (iii) or (iv). This is not possible, since if we take for example type (ii), we would have that two amongst five consecutive numbers belong to the sequence 2, 8, 18, 32, 50, ..., that is numbers separated by at least 6. Therefore, two of the factors $n+j$ must be of type (i), that is perfect squares. But the only chain of five consecutive numbers which contains two perfect squares is 1, 2, 3, 4, 5, whose product is equal to 120, which is not a perfect square. This completes the proof.

(52) We only need to observe that $n^5 - n = (n-2)(n-1)n(n+1)(n+2) + 5n(n^2-1)$.

(53) Since n and $(n+1)$ are two consecutive integers, $2|n(n+1)$. To show that $3|n(n+1)(2n+1)$, it is enough to consider the three cases: $n = 3k+r$, $0 \leq r \leq 2$. If $n = 3k$ or $n = 3k+2$, the result is immediate. When $n = 3k+1$, we have that $3|(2n+1)$. The result is therefore true for each $n \geq 1$.

(54) We only need to observe that using the Binomial Theorem, there exists a positive integer M such that

$$(a+1)^{n+1} = 1 + (n+1)a + a^2 M.$$

(55) Observe that

$$n^2 + 1 = (n+1-1)^2 + 1 = (n+1)^2 - 2(n+1) + 2.$$

Hence, for the relation to be true, we must have that $(n+1)|2$, that is $n = 1$.

(56) Since

$$n^6 + 206 = (n^2 + 2 - 2)^3 + 206 = (n^2 + 2)^3 - 6(n^2 + 2)^2 + 12(n^2 + 2) + 198,$$

the relation will be true if $(n^2+2)|198$. The only possibilities are therefore: $n^2 + 2 = 1, 2, 3, 6, 9, 11, 18, 22, 33, 66, 99, 198$, in which case the positive values of the required n are: 1, 2, 3, 4, 8 and 14. For example, with $b = 5$ and $a = 2$, we have $5|2^2 + 1$, $5 \nmid 2 \cdot 2^2$ and $5 \nmid 2^4 + 1 = 17$.

(57) First observe that $n^6 + 216 = (n^3 + 2 - 2)^2 + 216$. Then we proceed as in the preceding problem, and we obtain that the only possible integer n satisfying the given property is $n = 2$.

(58) The answer is NO. Indeed, if $b|a^2+1$, then there exists a positive integer k such that $a^2+1 = kb$. It follows that $a^4+1 = (kb-1)^2+1 = k^2b^2-2kb+2$ and therefore that in order to have $b|a^4 + 1$, we must have that $b|2$. It is easy to choose integers a and b in such a way that $b|a^2 + 1$ and $b \nmid 2a^2$. For instance, with $b = 5$ and $a = 2$, we obtain that $5|2^2 + 1$, $5 \nmid 2 \cdot 2^2$ and $5 \nmid 2^4 + 1 = 17$.

(59) (a) Since

$$n\left(\binom{n}{k}, \binom{n-1}{k-1}\right) = \left(n\binom{n}{k}, n\binom{n-1}{k-1}\right) = \left(n\binom{n}{k}, k\binom{n}{k}\right) = \binom{n}{k}(n, k),$$

we obtain the result.

(b) This follows from the fact that

$$(n+1-k)\left(\binom{n}{k}, \binom{n}{k-1}\right) = \left((n+1-k)\binom{n}{k}, k\binom{n}{k}\right) = \binom{n}{k}(n+1-k, k)$$

and from the fact that $(n + 1 - k, k) = (n + 1, k)$.

(c) Let

$$A = \left\{x \in \mathbb{Z} \;\middle|\; \frac{x}{n+2-k}\binom{n}{k-1} \text{ is an integer}\right\}.$$

Obviously, $x = n + 2 - k \in A$. Since

$$\frac{k-1}{n+2-k}\binom{n}{k-1} = \frac{(k-1)n!}{(n+2-k)(k-1)!\,(n+1-k)!} = \binom{n}{k-2},$$

then $x = k - 1 \in A$. Hence, any linear combination of $(k - 1)$ and $(n+2-k)$ also belongs to A. In particular, $(n+2-k, k-1)$ belongs also to A and since $(n + 2 - k, k - 1) = (n + 1, k - 1)$, we obtain the result.

REMARK: Parts (a) and (b) of this problem could have been solved as in part (c).

(60) (*Putnam, 1984*). It is clear that

$$f(n + 2) - f(n + 1) = (n + 2)! = (n + 2)(n + 1)! = (n + 2)\big(f(n + 1) - f(n)\big).$$

Therefore, choosing $P(x) = x + 3$ and $Q(x) = -x - 2$, the result follows.

(61) We only need to observe that

$$(2^{3(n+1)+3} - 7(n+1) - 8) - (2^{3n+3} - 7n - 8) = 7(2^{3n+3} - 1)$$

and that $7|2^{3n+3} - 1$, and thereafter use induction.

REMARK: This result follows also from Problem 54. Indeed,

$$7^2|\{(7+1)^{3n} - 7n - (7+1)\} = 2^{3n+3} - 7n - 8.$$

(62) We have $a = 10q + r$, $0 \le r < 10$. Therefore, we must have that $10|r^{10} + 1$, and this is why we must have $r = 3$ or 7.

(63) The answer is YES. Indeed,

$$\begin{aligned} 2^{2n} - 1 &= 4^n - 1 = (3+1)^n - 1 \\ &= 3^n + \binom{n}{1}3^{n-1} + \binom{n}{2}3^{n-2} + \cdots + \binom{n}{n-1}3 + 3^0 - 1 \\ &= 3^n + \binom{n}{1}3^{n-1} + \binom{n}{2}3^{n-2} + \cdots + \binom{n}{n-1}3, \end{aligned}$$

an expression which is divisible by 3.

(64) Let $6k + 5$ be an integer. To show that this integer can be written in the form $3m - 1$, we must find an integer m such that $6k + 5 = 3m - 1$. To do so, it is enough to choose $m = 2k + 2$, thus ending the proof of the first part. For the second part, let $3k - 1$ be an integer. Can one find, for each positive integer k, an integer m such that $3k - 1 = 6m + 5$, that is such that $6m = 3k - 6$? The answer is NO, because if k is odd, it is clear that it is impossible to find such an integer m.

(65) The answer is YES. Indeed, if $n = 8k + 7 = 6\ell + 5$ for certain integers k and ℓ, then $4k = 3\ell - 1$, which happens if and only if $\ell = 3, 7, 11, 15, \ldots$, that is when ℓ is of the form $4m + 3$. Hence, all numbers n of the form

$$n = 6\ell + 5 = 6(4m + 3) + 5 = 24m + 23$$

are automatically of the two required forms, and, of course, there are infinitely many of them.

(66) Let $k \in \mathbb{N}$; then $M_k = 2p_2p_3 \cdots p_k + 1 = 2(2r + 1) + 1 = 4r + 3$. But we know that each perfect square is of the form $4r$ or $4r + 1$, and certainly not of the form $4r + 3$.

(67) Let $N = n^2 = m^3$ for certain positive integers n and m. We easily see that n^2 is of the form $7k$, $7k+1$, $7k+2$ or $7k+4$, while m^3 is of the form $7k$, $7k + 1$ or $7k + 6$. Hence, N must be of the form $7k$ or $7k + 1$.

(68) Since x^2 and y^2 are of the form $4n + 1$, we see that $x^2 + y^2$ is of the form $4m + 2$. But each perfect square is of the form $4k$ or of the form $4k + 1$. Thus the result.

(69) The Binomial Theorem gives

$$(n+1)^n - 1 = \sum_{k=1}^{n} n^k \binom{n}{k} = n^2 + \sum_{k=2}^{n} \binom{n}{k} n^k,$$

and since $\binom{n}{k}$ is an integer, the result is immediate.

(70) By using the Binomial Theorem, we obtain

$$\begin{aligned} n^k - 1 &= [(n-1)+1]^k - 1 \\ &= (n-1)^k + k(n-1)^{k-1} + \cdots + k(n-1), \end{aligned}$$

and we observe that all the terms of this last expression are divisible by $(n-1)^2$ except perhaps the term $k(n-1)$, thus the result. The more general case can be treated in a similar manner, by considering the relation $n^k - a^k = ((n-a)+a)^k - a^k$.

(71) (a) By using the Binomial Theorem, we find

$$a^n = (a-b+b)^n = (a-b)^n + \binom{n}{1}(a-b)^{n-1}b + \cdots + \binom{n}{n-2}(a-b)^2 b^{n-2} + \binom{n}{n-1}(a-b)b^{n-1} + b^n.$$

Hence, there exists an integer K such that

$$\frac{a^n - b^n}{a-b} = (a-b)^{n-1} + \binom{n}{1}(a-b)^{n-2}b + \cdots + \binom{n}{n-2}(a-b)b^{n-2} + nb^{n-1} = K(a-b) + nb^{n-1}.$$

It follows that

$$\left(\frac{a^n - b^n}{a-b}, a-b\right) = \left(nb^{n-1}, a-b\right). \tag{1}$$

Similarly,

$$b^n = (a-(a-b))^n = a^n - \binom{n}{1}a^{n-1}(a-b) + \cdots + (-1)^{n-1}\binom{n}{n-1}a(a-b)^{n-1} + (-1)^n(a-b)^n,$$

and therefore, we find

$$\frac{a^n - b^n}{a-b} = na^{n-1} + L(a-b)$$

for a certain integer L. Hence,

$$\left(\frac{a^n - b^n}{a-b}, a-b\right) = \left(na^{n-1}, a-b\right). \tag{2}$$

Using the equations (1) and (2), we obtain

$$\left(\frac{a^n - b^n}{a-b}, a-b\right) = \left(n(a,b)^{n-1}, a-b\right).$$

Indeed, let $d = (na^{n-1}, a-b) = (nb^{n-1}, a-b)$ and $g = \big(n(a,b)^{n-1}, a-b\big)$. Since $d | na^{n-1}$ and $d | nb^{n-1}$, it follows that

$$d | (na^{n-1}, nb^{n-1}) = n(a^{n-1}, b^{n-1}) = n(a,b)^{n-1}.$$

By using this relation and the fact that $d | (a-b)$, we have $d | g$. Conversely, since $g | n(a,b)^{n-1}$, then $g | na^{n-1}$; and since $g | (a-b)$, it follows that $g | (na^{n-1}, a-b) = d$. Hence, $g = d$, which gives the result.

(b) Setting $b = -B$, we have

$$\frac{a^n + b^n}{a+b} = \frac{a^n - B^n}{a-B}.$$

Part (a) allows us to conclude that

$$\left(\frac{a^n + b^n}{a+b}, a+b\right) = \left(\frac{a^n - B^n}{a-B}, a-B\right)$$
$$= \left(n(a,B)^{n-1}, a-B\right) = \left(n(a,b)^{n-1}, a+b\right),$$

as was required to prove.

(c) Part (b) allows us to conclude that

$$\left(\frac{a^p + b^p}{a+b}, a+b\right) = (p, a+b) = 1 \text{ or } p$$

depending whether p divides $a+b$ or not.

(72) If $2|n$ and $n > 2$, then $(n-1)!$ is even while $n^k - 1$ is odd. If $2\not| n$ and $n > 5$, then $n-1$ is even and

$$n - 1 = 2\frac{n-1}{2} \mid (n-2)!$$

so that $(n-1)^2|(n-1)!$. Since $(n-1)! = n^k - 1$, we must have $(n-1)^2|n^k-1$, and using Problem 70, we must have $(n-1)|k$ and therefore $k \geq n-1$. In this case, for $n > 5$,

$$n^k - 1 \geq n^{n-1} - 1 > (n-1)!\,.$$

Hence, the only possible cases yielding a solution are $n = 2$, $n = 3$ and $n = 5$, the corresponding values of k then being 1, 1 and 2.

(73) (a) Since $b = aq_1 + r_1$ and since $a > r_1$, $q \geq 1$, then $b = aq_1 + r_1 > r_1q_1 + r_1 \geq 2r_1$. Similarly, $a = r_1q_2 + r_2 > r_2q_2 + r_2 \geq 2r_2$. Finally, for each $k \geq 1$,

$$r_k = r_{k+1}q_{k+2} + r_{k+2}, \quad 0 < r_{k+2} < r_{k+1},$$

so that

$$r_k = r_{k+1}q_{k+2} + r_{k+2} > r_{k+2}q_{k+2} + r_{k+2} \geq 2r_{k+2}.$$

(b) Since

$$b > 2r_1 > 2^2r_3 > 2^3r_5 > \ldots > 2^{(j+1)/2}r_j \geq 2^{(j+1)/2} > 2^{j/2},$$

we conclude that $j < 2\log b/\log 2$, and the result follows.

(74) Consider the numbers of the form $n = 3^k$, $k = 0, 1, 2, 3, \ldots$, so that

$$(*)\ 2^n + 1 = 2^{3^k} + 1 = 2^{3^{k-1}3} + 1 = \left(2^{3^{k-1}} + 1\right)\left(\left(2^{3^{k-1}}\right)^2 - 2^{3^{k-1}} + 1\right).$$

We will show that

(i) the first factor on the right-hand side is divisible by 3^{k-1}, while

(ii) the second factor is divisible by 3.

From this, it follows that the left-hand side of $(*)$ is divisible by $3^{k-1}\cdot 3 = 3^k = n$.

To show (i), we will show by induction that $2^{3^m} + 1$ is divisible by 3. It is clear that this result is true for $m = 1$, since $3|9$. Assume now that $2^{3^{m-1}} + 1$ is divisible by 3. Thus, if we set $x = 2^{3^{m-1}}$, we may write

$$2^{3^m}+1=\left(2^{3^{m-1}}\right)^3+1=x^3+1=(x+1)(x^2-x+1)$$
$$=(2^{3^{m-1}}+1)(x^2-x+1),$$

an expression divisible by 3, because of our induction hypothesis.

In order to show (ii), we only need to observe that if a is an odd positive integer (namely, here $a=3^{k-1}$), then $(2^a)^2-2^a+1$ is divisible by 3, which is indeed the case since

$$(2^a)^2-2^a+1=4^a-2^a+1\equiv 1-2+1=0 \pmod 3.$$

(75) Using Euclidean division, we can write $m=nq+r$, $0\le r<n$. Hence,

$$a^m-1=a^{nq+r}-a^{nq}+a^{nq}-1=a^{nq}(a^r-1)+(a^n)^q-1.$$

In light of Problem 8, $a^n-1|a^{nq}-1$, so that we have

$$a^n-1|a^m-1 \iff a^n-1|a^{nq}(a^r-1).$$

But $(a^n-1,a^{nq})=1$, meaning that a^n-1 must divide a^r-1, and this is true provided $r\ne 0$. Finally, the result follows.

(76) We only need to observe that $N_n=10^{n-1}+10^{n-2}+\cdots+10+1=(10^n-1)/9$ and to use the fact that $10^n-1|10^m-1$ if and only if $n|m$ (see the preceding problem).

(77) Any number in the sequence is of the form $100k+11=4(25k+2)+3$, where k is a nonnegative integer (made up entirely of the digit "1"). Then, any integer in the sequence leaves 3 as a remainder when it is divided by 4 and therefore cannot be a perfect square, since it is well known that any perfect square is of the form $4r$ or $4r+1$.

(78) The required number is $2^6\cdot 3^6$. For the general case, the smallest number is $2^{[n,m]}\cdot 3^{[n,m]}$.

(79) Since $371=370+1$, and since 370 is already in the list, it is obvious that 371 is this fourth number.

(80) There are

$$1000-\left[\frac{1000}{2}\right]-\left[\frac{1000}{3}\right]-\left[\frac{1000}{5}\right]+\left[\frac{1000}{6}\right]+\left[\frac{1000}{10}\right]$$
$$+\left[\frac{1000}{15}\right]-\left[\frac{1000}{30}\right]=266$$

such numbers.

(81) We are looking for $a,b\in\mathbb{N}$ such that

$$p=a^2-b^2=(a+b)(a-b).$$

But since p is a prime number, we must have $a-b=1$ and $a+b=p$. Hence,

$$a=\frac{p+1}{2} \quad\text{and}\quad b=\frac{p-1}{2}.$$

We have thus obtained that

$$p=\left(\frac{p+1}{2}\right)^2-\left(\frac{p-1}{2}\right)^2.$$

(82) The result follows immediately from the fact that the system of equations $a-b=1$ and $a+b=p$ has a solution, namely $a=(p+1)/2$, $b=(p-1)/2$.

(83) The answer is YES. However, uniqueness does not hold because for instance $21=11^2-10^2=5^2-2^2$.

(84) This follows from the fact that $7=10-3$ is a divisor of 10^9-3^9.

(85) Let m be this integer. Then,

$$m=2k+1=a^2+b^2, \qquad a \text{ odd}, \ b \text{ even}.$$

We then have that there exist nonnegative integers M and N such that $a=2M+1$ and $b=2N$. Hence, $a^2=4M^2+4M+1=4K+1$ for a certain integer K, and $b^2=4N^2$. We have therefore established that

$$m=a^2+b^2=4K+1+4N^2=4n+1,$$

for a certain integer n, as required.

(86) If a and b are odd, that is $a=2m+1$ and $b=2n+1$, say, then we have $a^2+b^2=4k+2=c^2$, which is impossible since any perfect square is of the form $4k$ or $4k+1$.

(87) (*TYCM, March 85*). For $k=0$, we have $10^k-1=10^0-1=0=0^3$, a cube. If $k<0$, then 10^k-1 is not an integer. Hence, assume that $k\geq 1$ and that $10^k-1=n^3$. Setting $N_k=\frac{1}{9}(10^k-1)$, we then have

$$N_k=\underbrace{11\ldots1}_{k}=10^{k-1}+10^{k-2}+\cdots+10+1.$$

But, for $j\geq 1$, there exists a constant $A\geq 1$ such that $10^j=(3^2+1)^j=3A+1$, which allows us to conclude that there exists a constant $M\geq 1$ such that $N_k=3M+k$. Since $9|(10^k-1)=n^3$, it follows that $27|10^k-1$. Therefore $3|N_k$ so that $k=3r$ for a certain positive integer r. We have thus established that $10^{3r}-1$ and 10^{3r} are two consecutive cubes (with $r\geq 1$), a contradiction. Hence, the only integer k such that 10^k-1 is a cube is $k=0$.

(88) Since $a|42n+37-6(7n+4)=13$, the result follows.

(89) By hypothesis, we have that $(a+b)/ab$ is an integer and therefore that $ab|(a+b)$. Since $a|a+b$ and $a|a$, it follows that $a|b$. Moreover, $b|a+b$ and $b|b$ imply $b|a$. Now clearly, $a|b$ and $b|a$, with a,b positive, implies that $a=b$. It then follows that, for $1/a+1/a=2/a$ to be an integer, we must have $a|2$, which means that $a=1$ or 2.

(90) We write $a=4A$, $b=4B$ where $(A,B)=1$. Then, $(a^2,b^3)=16(A^2,4B^3)$, and since $(A^2,B^3)=1$, we conclude that each common divisor of A^2 and $4B^3$ must be 1 or 4. Hence, the possible values of (a^2,b^3) are 16 and 64.

(91) Set $d_1=(3a+5b,5a+8b)$ and $d=(a,b)$. Since $d_1|(3a+5b)$ and $d_1|(5a+8b)$, then $d_1|\Big(8(3a+5b)-5(5a+8b)\Big)$, that is $d_1|a$. In a similar way, we obtain that $d_1|b$ and consequently $d_1|d$.

Since $d=(a,b)$, it follows that $d|(3a+5b)$ and $d|(5a+8b)$ and therefore $d|d_1$. Since $d|d_1$ and $d_1|d$, we conclude that $d=d_1$.

The general case can he handled in a similar way, and we obtain $(ma+nb,ra+sb)=(a,b)$ when $ms-nr=1$.

(92) Let $r = (d, m)$ and set $s = d/r$. Since $r|m$, it is enough to show that $s|n$. Letting $M = m/r$, then

$$\left(\frac{d}{r}, \frac{m}{r}\right) = (s, M) = 1.$$

Since $d|mn$, there exists an integer t such that $dt = mn$, so that $rst = Mrn$, that is $s|Mn$. Since $(s, M) = 1$, it follows that $s|n$. Let $d' = (r, s)$; then $d'|r$ and $r|m$ imply that $d'|m$ and $d'|n$. We therefore have $d'|(m, n) = 1$, so that $(r, s) = 1$.

(93) We have that $d|(a + b) + (a - b) = 2a$ and therefore that $d|a$ since d is odd. Similarly, we have that $d|(a + b) - (a - b) = 2b$ and therefore that $d|b$ since d is odd. Hence, $d|(a, b)$.

(94) Since $19^2 = 361$, each composite number ≤ 360 is divisible by a prime number ≤ 17. Since there are only seven prime numbers ≤ 17, it follows by the Pigeonhole Principle that at least two of these given eight composite numbers must be divisible by the same prime number.

(95) Since $ab = r^2 = q_1^{2\alpha_1} \cdots q_k^{2\alpha_k}$ for certain prime numbers $q_1, q_2, \ldots, q_k$ and certain positive integers $\alpha_1, \alpha_2, \ldots, \alpha_k$ and since $(a, b) = 1$, it is clear that some of the $q_i^{2\alpha_i}$'s will be factors of a while the others will be factors of b, thus establishing that a and b are perfect squares.

(96) If it were true, it would follow from Problem 95 that n and $n + 1$ are two consecutive perfect squares, which is not possible.

(97) The only possible values are 1, 2, 7 and 14. Indeed, if $d = (n, n + 14)$, then $d|14$.

(98) The first statement is true because it is equivalent to $3|(n - 1)n(n + 1)$. The second statement is false: simply take $n = 4$. The third statement is true because it is easily shown to be equivalent to $8|4n(n+1)$. The fourth statement is true because $2|n(n + 1)$ and $3|n(n + 1)(n + 2)$.

(99) The answer is YES. Indeed, on the one hand, $3|n(n + 1)(n + 2)$, while on the other hand, one of the two numbers n and $n + 2$ is divisible by 4, the other by 2.

(100) Since we have $n^2 + 47 = n^2 + 48 - 1$, it is enough to show that $24|n^2 - 1$. First of all, any positive integer n is of one of the following six forms: $6k$, $6k + 1$, $6k + 2$, $6k + 3$, $6k + 4$, $6k + 5$. Since $(n, 2) = (n, 3) = 1$, it is clear that n can only be of the form $6k + 1$ or $6k + 5$, in which case it is immediate that $n^2 - 1$ is divisible by 24.

(101) Let $a = dr$ and $b = ds$ where $(r, s) = 1$. Dividing each of the integers a, $2a, \ldots, ba$ by b, we obtain the quotients

$$(*) \qquad \frac{r}{s}, 2\frac{r}{s}, \ldots, (b-1)\frac{r}{s}, ds\frac{r}{s}.$$

Since $(r, s) = 1$, the only integers amongst $(*)$ are those whose numerator is a multiple of s. Since $b = ds$, this will happen exactly d times.

(102) Part (a) is immediate. For part (b), it is sufficient to observe that $(x_0 + b)a + (y_0 - a)b = d$.

(103) Let $d = (a, mn)$; then $d|a$ and $d|mn$. Since $(m, n) = 1$, using Problem 92, we have $d = rs$, $(r, s) = 1$, $r|m$ and $s|n$. But $rs|a$ implies $r|a$ and $s|a$. It follows that $r|(a, m)$ and $s|(a, n)$, so that $d|(a, m)(a, n)$. To complete the proof, we must now show that $(a, m)(a, n)|d$. Let $d_2 = (a, m)$ and

$d_1 = (a,n)$; then $d_2|a$ and $d_2|mn$, so that $d_2|(a,mn)$. But $d_1|a$ and $d_1|mn$ imply $d_1|(a,mn)$. Since $(m,n) = 1$, it follows that $(d_1,d_2) = 1$, which allows us to conclude that $d_1 d_2|(a,mn)$.

(104) Let $d = (n^2+3n+2, 6n^3+15n^2+3n-7)$. We then have that $d|6n(n^2+3n+2) - (6n^3+15n^2+3n-7) = 3n^2+9n+7$. Now since $d|n^2+3n+2$, it follows that $d|3n^2+9n+7-3(n^2+3n+2) = 1$ and therefore $d = 1$.

(105) The first three problems can be solved in a similar way. For (a), we proceed as follows. Let $d = (a+b, a-b)$, so that $d|2b$ and $d|2a$. We then have $d|(2b,2a) = 2(a,b)$ so that $d|2$, which proves that $d = 1$ or 2. Finally, for (d), it is sufficient to notice that $a^2 - 3ab + b^2 = (a+b)^2 - 5ab$.

(106) (a) Set $d = (a^3+b^3, a^3-b^3)$. We then have that $d|2a^3$ and $d|2b^3$, and since $(a,b) = 1$, we have $d|2$. Therefore, $d = 1$ or $d = 2$. More precisely, when a and b are of opposite parity, we find the value 1, while if a and b are of the same parity, we obtain the value 2.

(b) We have

$$\begin{aligned}(a^2-b^2, a^3-b^3) &= (a-b)\Big(a+b, a^2-ab+b^2\Big)\\ &= (a-b)\Big(a+b, (a+b)^2-3ab\Big) = (a-b)\Big(a+b, 3ab\Big).\end{aligned}$$

Let $d = (a+b, 3ab)$, so that $d|3b(a+b) - 3ab = 3b^2$ and $d|3a(a+b) - 3ab = 3a^2$, and therefore $d|3$. It follows that $(a^2-b^2, a^3-b^3) = a-b$ or $3(a-b)$. More precisely, the value is $a-b$ if $3\not|(a+b)$ and $3(a-b)$ if $3|(a+b)$.

(107) (a) False. Indeed, $(2,3) = (2,5) = 1$ even though $6 = [2,3] \neq [2,5] = 10$.

(b) True. It is enough to show that: $(a,b) = g \Longrightarrow (a^2,b^2) = g^2$. We know that if $(A,B) = 1$, then $(A^2,B^2) = 1$. But by hypothesis we have $a = Ag$ and $b = Bg$ with $(A,B) = 1$. It follows that $a^2 = A^2g^2$ and $b^2 = B^2g^2$, which means that $(a^2,b^2) = (A^2g^2, B^2g^2) = g^2(A^2,B^2) = g^2$.

(c) True. Indeed, let $g = (a,b)$ and $h = (a,b,c)$. It is clear that $h|g$. Therefore, it follows that $g|h$. But $g = (a,b) = (a,c)$, which implies that $g|a$, $g|b$ and $g|c$. It follows that $g|(a,b,c) = h$, as was to be shown.

(108) (a) The statement is true. Indeed, let $(a,b) = d$, so that $a = dA$ and $b = dB$ with $(A,B) = 1$. Therefore, $(A^n,B^n) = 1$, and since $a^n|b^n$, we obtain $A^nd^n|B^nd^n$, that is $A^n|B^n$. Hence, $A^n|(A^n,B^n) = 1$, which shows that $A = 1$ and therefore that $d = a$. It follows that $b = dB = aB$, which proves the statement.

One can also prove this result by writing $a = \prod p_i^{a_i}$ and $b = \prod p_i^{b_i}$, and then using the fact that $a^n|b^n$ to obtain that $na_i \leq nb_i$; that is $a_i \leq b_i$ for each i, so that $a|b$.

(b) The statement is true because $a^m|b^n$ implies $a^n\, a^{m-n}|b^n$ and therefore $a^n|b^n$. From part (a), we draw the conclusion.

(c) False. Indeed, $(2^3)^2|(2^2)^3$, although $2^3 \not| 2^2$.

(109) We have $(a,b) = 1 \iff$ there exist $x,y \in \mathbb{Z}$ such that $ax+by = 1$. Since $c|a$, there exists $q \in \mathbb{Z}$ such that $a = qc$; therefore, $ax+by = qcx+by = 1$ $\iff (c,b) = 1$.

(110) We have $(a,bc)=1 \iff$ there exist $x,y\in\mathbb{Z}$ such that $ax+bcy=1$. We therefore obtain that $(a,b)=1$ and $(a,c)=1$.

(111) Assuming that $(a,b)=1$, we must show that $1=(a+b,ab)$. Setting $d=(a+b,ab)$, we obtain that $d|a^2$ and $d|b^2$, so that $d|(a^2,b^2)=1$. The more general case $(a,b)=d>1$ can be obtained from the first part, using the fact that $(a/d,b/d)=1$.

Let a and b be the integers such that $a+b=186$ and $[a,b]=1440$. Since $(a,b)=(a+b,[a,b])$ and since $(186,1440)=6$, then $a=6A$ and $b=6B$ where $(A,B)=1$. This leads to $A+B=31$ and $[A,B]=240$, and since $(A,B)=1$, we have $AB=240$. We then have $A(31-A)=240$ and therefore $A=15$ or $A=16$. The other two numbers are therefore $90=2\cdot 3^2\cdot 5$ and $96=2^5\cdot 3$.

(112) (a) Let $d=(a,bc)$ and $g=(a,(a,b)c)$. We have that $g|a$ and $g|(ac,bc)$ and therefore that $g|bc$. Consequently, $g|d$. But $d|a$ and $d|bc$ imply that $d|(ac,bc)=(a,b)c$ and therefore that $d|g$. Hence, $d=g$.

(b) From part (a), we have

$$(a,bc)=(a,c(a,b))=(a,(a,c)(a,b)).$$

(113) Indeed, $(c,ab)=(c,(a,c)b)=(c,b)$.

(114) It is enough to show that the two numbers are both powers of the same prime number. Assume that p^α divides either one of the two expressions. Then, $p^\alpha|mn$, and since $(m,n)=1$, then either $p^\alpha|m$ and $(p,n)=1$ or else $p^\alpha|n$ and $(p,m)=1$. Since both cases are identical, we can assume that $p^\alpha|m$, in which case we must have $(p,n)=1$. Therefore,

$$\begin{aligned} p^\alpha|(ma+nb,mn) &\iff p^\alpha|(ma+nb)\\ &\iff p^\alpha|b \iff p^\alpha|(b,m) \iff p^\alpha|(a,n)(b,m). \end{aligned}$$

Setting $b=a$, we obtain $(a(m+n),mn)=(a,n)(a,m)$. Since $(m,n)=1$ also implies $(m+n,mn)=1$, we conclude that $(a,mn)=(a,n)(a,m)$, thus also obtaining the result of Problem 103.

(115) The answer is NO. This follows from the identity

$$\binom{n}{s}\binom{s}{r}=\binom{n}{r}\binom{n-r}{s-r}$$

and from the fact that both quantities $\binom{s}{r}$ and $\binom{n-r}{n-s}$ are larger than 1.
REMARK: This problem remained unsolved until one thought about using the above identity (see Guy [**16**], B31). P. Erdős and G. Szekeres [**11**] asked if the largest prime factor of the greatest common divisor of $\binom{n}{r}$ and $\binom{n}{s}$ is always larger than r, the only counter-example with $r>3$ being

$$\left(\binom{n}{r},\binom{n}{s}\right)=\left(\binom{28}{5},\binom{28}{14}\right)=2^3\cdot 3^3\cdot 5>5.$$

(116) Let $(a,b)=d$ with $[a,b]-(a,b)=143$. First of all, it is clear that $d|143$. We must therefore examine the possibilities $d=1$, $d=11$, $d=13$ and $d=143$. Set $a=Ad$ and $b=Bd$ with $(A,B)=1$.

If $d=1$, then $[a,b]-(a,b)=143$ becomes $AB-1=143$ and since $(A,B)=1$, we have $A=a=16$ and $B=b=9$, as well as $A=a=1$ and $B=b=144$.

If $d = 11$, then $AB - 1 = 13$ and therefore $A = 2$ and $B = 7$ (which gives $a = 22$ and $b = 77$), as well as $A = 1$ and $B = 14$ (which gives $a = 11$ and $b = 154$).

If $d = 13$, we obtain $a = 39$ and $b = 52$.

If $d = 143$, we have $a = 143$ and $b = 286$.

The only six possible (ordered pairs) solutions are therefore

$$\{a, b\} = \{1, 144\},\ \{9, 16\},\ \{11, 154\},\ \{22, 77\},\ \{39, 52\},\ \{143, 286\}.$$

(117) For the first part we proceed as follows. Let $d = (a, b, c)$ and $d_1 = ((a, b), c)$. Since $d|a$ and $d|b$, we have $d|(a, b)$. Similarly, $d|c$; hence, $d|d_1$. On the other hand, $d_1|(a, b)$; it follows that $d_1|a$ and $d_1|b$, and since $d|c$, this shows that $d_1|d$. Since $d|d_1$ and $d_1|d$, we have $d = d_1$.

For the second part, we proceed in the following manner. Let $M = [a, b, c]$, $m = [a, b]$ and $m_1 = [m, c]$. From the definition of m_1, it follows that $m|m_1$ and $c|m_1$. Consequently, $a|m_1$, $b|m_1$ and $c|m_1$; that is $[a, b, c]|m_1$. Conversely, $M = [a, b, c]$ implies $a|M$, $b|M$ and $c|M$, and therefore $[a, b]|M$ and $c|M$. This allows us to conclude that $m_1 = [[a, b], c]|M$ and the result follows.

More generally, we have

$$(a_1, a_2, \ldots, a_n) = (a_1, (a_2, \ldots, a_n)) \text{ and } [a_1, a_2, \ldots, a_n] = [a_1, [a_2, \ldots, a_n]].$$

By using Euclid's algorithm, we obtain

$$\begin{aligned}
132 &= 102 \cdot 1 + 30, \\
102 &= 30 \cdot 3 + 12, \\
30 &= 12 \cdot 2 + 6, \\
12 &= 6 \cdot 2.
\end{aligned}$$

It follows from this that $(132, 102) = 6$ and therefore that

$$(132, 102, 36) = ((132, 102), 36) = (6, 36) = 6.$$

Using the above system of equations starting at the second one from the bottom and moving up, we obtain successively

$$\begin{aligned}
6 &= 30 - 12 \cdot 2 = 30 - (102 - 3 \cdot 30) \cdot 2 = 7 \cdot 30 - 2 \cdot 102 \\
&= 7 \cdot (132 - 102) - 102 \cdot 2 = 7 \cdot 132 - 9 \cdot 102.
\end{aligned}$$

On the other hand, since $7 \cdot 6 + (-1) \cdot 36 = 6$, we obtain that

$$7 \cdot (7 \cdot 132 - 9 \cdot 102) - 36 = 6 \Rightarrow 49 \cdot 132 + (-63) \cdot 102 + (-1) \cdot 36 = 6.$$

We may thus choose $x = 49$, $y = -63$ and $z = -1$.

(118) It is easy to see that $(n, n+1, n+2) = ((n, n+1), n+2) = (1, n+2) = 1$.

Since $(n, n+1) = 1$, it follows that $[n, n+1, n+2] = [[n, n+1], n+2] = [n(n+1), n+2]$. Since $(n(n+1), n+2) = (n, n+2) = 1$ or 2, then

$$[n, n+1, n+2] = \begin{cases} n(n+1)(n+2) & \text{if } n \text{ is odd}, \\ n(n+1)(n+2)/2 & \text{if } n \text{ is even}. \end{cases}$$

(119) We know that $(*)$ $(ab, c)[ab, c] = abc$. Since $(a, b) = 1$, it follows that $[a, b] = ab$, and since $(a, c) = (b, c) = 1$, we have $(ab, c) = 1$. Therefore, $(*)$ becomes $[ab, c] = abc$, so that $[[a, b], c] = abc$. By using Problem 117, we reach the conclusion.

(120) The answer is YES. If $(a,b)=1$, we have $(a^2,b^2)=1$; hence, using Problem 117, we obtain

$$(a^2,ab,b^2)=((a^2,b^2),ab)=(1,ab)=1.$$

(121) The answer is YES. If $(a,b)=1$, then $(a^2,b^2)=1$, $[a^2,b^2]=a^2b^2$ and Problem 120 allows us to obtain that $(a^2,ab,b^2)=1$. Consequently, from Problem 117, we have

$$[a^2,ab,b^2]=[\,[a^2,b^2],ab]=[a^2b^2,ab]=a^2b^2=[a^2,b^2].$$

For the general case $(a,b)=d$, it is enough to redo the last part with $(a/d,b/d)=1$.

(122) The answer is YES. We set $h=(a,b,c)$ and $g=((a,b),(a,c))$, and we easily show that $g|h$ and $h|g$.

(123) The answer is NO. It is enough to consider the counter-example provided by choosing $a=6$, $b=3$ and $c=15$.

(124) *This problem was stated by the mathematician Jean-Henri Lambert (1728–1777).* Letting $(m,n)=e$ and using the fact that $[m,n](m,n)=mn$, we have

$$\begin{aligned} d^{[m,n]}-1 &= (d^m-1+1)^{n/e}-1=(d^m-1)^{n/e} \\ &\quad +\binom{n/e}{1}(d^m-1)^{n/e-1}+\cdots+\binom{n/e}{n/e-1}(d^m-1) \end{aligned}$$

and we conclude that $a|d^{[m,n]}-1$. Similarly,

$$\begin{aligned} d^{[m,n]}-1 &= (d^n-1+1)^{m/e}-1=(d^n-1)^{m/e} \\ &\quad +\binom{m/e}{1}(d^n-1)^{m/e-1}+\cdots+\binom{m/e}{m/e-1}(d^n-1) \end{aligned}$$

and we obtain that $b|d^{[m,n]}-1$. Since $(a,b)=1$, the result follows.

(125) Assume that $(m,n)=1$, $m>n$. We will show that

$$(1)\qquad (a^m-1,a^n-1)=a-1.$$

Since $(a,b+ma)=(a,b)$, we have

$$(a^m-1,a^n-1)=(a^m-1-(a^n-1),a^n-1)=(a^m-a^n,a^n-1).$$

Since $(a^n,a^n-1)=1$, this shows that

$$(a^m-1,a^n-1)=(a^n(a^{m-n}-1),a^n-1)=(a^{m-n}-1,a^n-1).$$

Without any loss in generality, we may assume that $m>n$, in which case we can write $m=nq+r$, $0\le r<n$, so that

$$(a^m-1,a^n-1)=(a^r-1,a^n-1).$$

Then, writing $n=rs+t$, $0\le t<r$, we obtain

$$(a^r-1,a^n-1)=(a^r-1,a^t-1),$$

and so on until we arrive at $(a-1,a-1)=a-1$, which proves (1). Assume now that $d=(m,n)>1$. Since $(m/d,n/d)=1$, we are brought back to the first case, and we thus have

$$\left((a^d)^{m/d}-1,(a^d)^{n/d}-1\right)=a^d-1,$$

which takes care of the first part of the problem. For the other cases mentioned in the second part (which by the way cover also the first part), we proceed in the following manner. First letting $(m,n)=1$ and $u=\pm 1$, $v=\pm 1$, we have

$$\begin{aligned}(a^m+u,a^n+v) &= (a^m+u-uv(a^n+v),a^n+v)\\ &= (a^m-uva^n,a^n+v)=(a^{m-n}-uv,a^n+v),\end{aligned}$$

since $(a^n,a^n\pm 1)=1$. Continuing this process, we obtain

$$(a^m+1,a^n+1)=(a^m+1,a^n-1)=(a+1,a+1) \text{ or } (a+1,a-1)$$

according to the parities of m and n. More precisely, we have the following: Since $(a+1)|(a^k+1)$ for k odd and since $(a+1)|(a^k-1)$ for k even, it follows that taking into account the fact that $a+1$ cannot divide a^k+1 if k is even unless $a=1$, we obtain that

$$(a^m+1,a^n+1)=\begin{cases} a+1 & \text{if } mn \text{ is odd,}\\ 1 & \text{if } mn \text{ is even and } a \text{ is even,}\\ 2 & \text{if } mn \text{ is even and } a \text{ is odd}\end{cases}$$

and that

$$(a^m+1,a^n-1)=\begin{cases} 1 & \text{if } n \text{ is odd and } a \text{ is even,}\\ 2 & \text{if } n \text{ is odd and } a \text{ is odd,}\\ a+1 & \text{if } n \text{ is even.}\end{cases}$$

When $d=(m,n)>1$, we can proceed essentially as we did for the first case. To find the value of (a^m-b^m,a^n-b^n), we may assume that $(a,b)=1$ and $a>b$. In this case, set $d=(m,n)$, $u=a^d-b^d$ and $v=(a^m-b^m,a^n-b^n)$. Since $d|m$, it follows that $u|(a^m-b^m)$, and since $d|n$, we also have $u|(a^n-b^n)$ and we obtain that $u|v$. Then, we only need to show that $v|u$. Choose integers $x>0$ and $y>0$ such that $mx-ny=d$. It is clear that

$$a^{mx}=a^{ny+d}=a^{ny}(b^d+u),$$

and therefore

$$a^{mx}-b^{mx}=a^{ny}(b^d+u)-b^{ny+d}=b^d(a^{ny}-b^{ny})+ua^{ny}.$$

Since $v|(a^m-b^m)$, we have $v|(a^{mx}-b^{mx})$ and similarly $v|(a^{ny}-b^{ny})$, and the last equation allows one to obtain that $v|ua^{ny}$. Since $(a,b)=1$, we have $(v,a)=1$. Indeed, every common divisor of a and v divides a^m and a^m-b^m, and therefore divides b^m, and since $(a,b)=1$, we have $(a,v)=1$. Finally, $v|ua^{ny}$ implies $v|u$ and the result follows.

(126) Let a and b be two arbitrary integers, and set $x=5a$ and $y=5b$. In order to have $x+y=5a+5b=40$, we must have $a+b=8$. Moreover, to have $(x,y)=5$, we must have $(a,b)=1$. Therefore, it remains to show that it is possible to find infinitely many relatively prime pairs of integers a and b such that $a+b=8$. To do so, it is enough to choose, for example, $a=3+2t$ and $b=5-2t$, where $t\in\mathbb{Z}$.

(127) There are four possible pairs: $a=15$, $b=90$; $a=90$, $b=15$; $a=30$, $b=45$; $a=45$, $b=30$. For the general case, we proceed in the following way. Since $(a,b)=d$, there exist integers A and B such that $a=dA$, $b=dB$, where $(A,B)=1$. But $[a,b]=m$ implies that $[dA,dB]=d\,[A,B]=dAB=m$. Hence, the system of equations $(a,b)=d$, $[a,b]=m$

has solutions if and only if $d|m$. These solutions will be the same as that of $AB = m/d$ where $(A, B) = 1$. For each prime number p dividing m/d, we cannot have both $p|A$ and $p|B$. Therefore, either A contains the largest power of p which divides m/d, or else A does not have p as a divisor. Hence, for each prime factor p of m/d, we have two choices for the pair $\{A, B\}$, and therefore in total as many pairs as m/d has distinct prime factors, that is as many as $2^{\omega(m/d)}$.

(128) This follows from the fact that $3|m$ and $3|n$ while $3 \not| 101$.

(129) (a) We observe that

$$a^{2^{m-r}} \cdot a^{2^{m-r}} = a^{2^{m-r+1}}.$$

It follows that

$$\begin{aligned} (*) \qquad a^{2^m} - 1 &= (a^{2^{m-1}} + 1)(a^{2^{m-1}} - 1) \\ &= (a^{2^{m-1}} + 1)(a^{2^{m-2}} + 1) \cdots (a^2 + 1)(a + 1)(a - 1). \end{aligned}$$

Hence, if $m > n$, $a^{2^n} + 1$ is a divisor of $a^{2^m} - 1$, as required.

(b) Note that $a^{2^m} - 1 = (a^{2^m} + 1) - 2$ and that this integer is divisible by each of the factors on the right-hand side of $(*)$. Let $d = (a^{2^m} + 1, a^{2^n} + 1)$. We may assume that $n < m$, and therefore $a^{2^n} + 1 | (a^{2^m} + 1) - 2$, which implies $d | (a^{2^m} + 1) - 2$. Therefore, $d|2$ so that $d = 1$ or $d = 2$.

(130) (*AMM, Vol. 75, 1971, p. 201*). Let d be the greatest common divisor of the given numbers. In particular, d divides the sum of these numbers and since (see Problem 17 (b))

$$\binom{2n}{1} + \binom{2n}{3} + \binom{2n}{5} + \cdots + \binom{2n}{2n-1} = 2^{2n-1},$$

it follows that d must be of the form 2^a. If $n = 2^k r$, where r is an odd integer and k a nonnegative integer, then since $\binom{2n}{1} = 2^{k+1} r$, it follows that any common divisor of the given numbers cannot be larger than 2^{k+1}. To show that 2^{k+1} divides all these numbers, we first write, for $m = 1, 3, \ldots, 2n - 1$,

$$\binom{2n}{m} = \binom{2^{k+1} r}{m} = \frac{2^{k+1} r}{m} \binom{2^{k+1} r - 1}{m - 1}.$$

Since the binomial coefficients are integers and since m is an odd number, we have

$$\binom{2n}{m} = \binom{2^{k+1} r}{m} = 2^{k+1} M,$$

where M is an integer and $m = 1, 3, \ldots, 2n - 1$. This proves that 2^{k+1} is the greatest common divisor of the given numbers.

(131) Each a_i can be written in the form $a_i = 2^{\alpha_i} b_i$, where $\alpha_i \geq 0$ and b_i is odd. Let $B = \{b_1, b_2, \ldots, b_{n+1}\}$. We have $b_i \leq 2n$, $i = 1, 2, \ldots, n+1$. But there exist only n odd numbers $\leq 2n$; hence, there exist j, k such that $b_j = b_k$. Then, consider the two integers

$$a_j = 2^{\alpha_j} b_j \quad \text{and} \quad a_k = 2^{\alpha_k} b_k.$$

It is clear that $\alpha_j \neq \alpha_k$ (since $b_j = b_k$). If $\alpha_j < \alpha_k$, then $a_j | a_k$. If $\alpha_k < \alpha_j$, then $a_k | a_j$. In each case, the result follows.

(132) (*Contribution of Imre Kátai, Budapest*). Let

$$S := \sum_{j=1}^{n}(a_j^{(1)} + a_j^{(2)})(a_j^{(1)} + a_j^{(3)}).$$

Then,

$$S = \sum_{j=1}^{n}(a_j^{(1)})^2 + \sum_{j=1}^{n} a_j^{(1)}a_j^{(3)} + \sum_{j=1}^{n} a_j^{(2)}a_j^{(1)} + \sum_{j=1}^{n} a_j^{(2)}a_j^{(3)} = n.$$

But since it is clear that the expression $(a_j^{(1)}+a_j^{(2)})(a_j^{(1)}+a_j^{(3)})$ is a multiple of 4, the result follows.

(133) In order to show that a series made up of nonnegative real numbers converges, we only need to bound it by a series which converges. So let $\ell(n)$ be the number of digits of the positive integer n, in its decimal representation. We first observe that, for each positive integer r, we have

$$\sum_{\substack{n\in A\\ \ell(n)=r}} 1 = 8\cdot 9^{r-1}.$$

Hence, it follows from this that

$$\sum_{n\in A}\frac{1}{n} = \sum_{r=1}^{\infty}\sum_{\substack{n\in A\\ \ell(n)=r}}\frac{1}{n} < \sum_{r=1}^{\infty}\frac{1}{10^{r-1}}\sum_{\substack{n\in A\\ \ell(n)=r}} 1$$
$$= \sum_{r=1}^{\infty}\frac{8\cdot 9^{r-1}}{10^{r-1}} = 8\sum_{r=1}^{\infty}\left(\frac{9}{10}\right)^{r-1} = 80,$$

from which the result follows.

(134) If $a > b$, it is clear that $a - b \geq (a,b)$, and we know that $(a,b)[a,b] = ab$. Hence, since

$$(u_{n+1} - u_n)[u_{n+1}, u_n] \geq (u_{n+1}, u_n)[u_{n+1}, u_n] = u_{n+1}\cdot u_n,$$

we obtain

$$\frac{1}{[u_{n+1}, u_n]} \leq \frac{u_{n+1} - u_n}{u_{n+1}\cdot u_n} = \frac{1}{u_n} - \frac{1}{u_{n+1}}.$$

Therefore the series is bounded above by a convergent series and this is why it converges.

(135) (a) With MAPLE, we have

```
> for i from 3 by 2 to 525 do
> if isprime(2^i-1)
> then print(2^i-1, ' is a prime number ')
> else fi; od;
```

(b) With MAPLE, we may use

```
> nextprime(10∧(100)+1);
```

We thus obtain the integer $10^{100} + 267$.

(136) With MAPLE, the program below enumerates the first N (here $N = 120$) prime numbers.

```
> for i to 120 do
> p.i :=ithprime(i) od;
```

For example, $p.(1..120)$ gives the first 120 prime numbers.

(137) In order to find four consecutive integers with the same number of prime factors, we must use the function Ω. First we type in

```
> readlib(ifactors):  with(numtheory):
```

and thereafter, we type in the following instructions:

```
> Omega:=n->sum(ifactors(n)[2][i][2],
> i=1..nops(factorset(n))):
> for n to 1000 do if Omega(n)=Omega(n+1) and
> Omega(n+1)=Omega(n+2) and Omega(n+2)=Omega(n+3)
> then print(n) else fi; od;
```

To find four consecutive integers having the same number of divisors, it is enough to type in the instructions

```
> with(numtheory):
> for n to 1000 do
> if tau(n)=tau(n+1) and tau(n+1)=tau(n+2)
> and tau(n+2)=tau(n+3)
> then print(n) else fi; od;
```

(138) (a) With the procedure "return", the search is easily done:

```
> return:=proc(n::integer)
> local m,s;
> m:=n; s:=0;
> while m<>0 do
> s:=10*s+irem(m,10);
> m:=iquo(m,10) od; s end:
```

And for our problem, we have the following procedure:

```
> invp:=proc(N::integer) local n;
> for n from 1 to N do if isprime(n)
> and isprime(return(n)) then
> print (n) fi; od; end:
```

Without the procedure "return", we may proceed as follows:

```
> invp:=proc(N)
> for j from 169 to N do
> L:=convert(ithprime(j),base,10);# N <= 1229
> if type(1000*L[1]+100*L[2]+10*L[3]+L[4],prime)=true
> then print(ithprime(j)) else fi; od; end:
```

(b)

```
> invp:=proc(N::integer)
> local n;
> for n from 1 to N do
> if isprime(n)=false then elif
> type(sqrt(return(n)),integer)=true
> then print(n) else fi; od; end:
```

If we do not use the procedure "return", we may proceed as follows:

```
> invp:=proc(N) local j, L;
> for j from 26 to N
> do L:=convert(ithprime(j),base,10);# N <= 168
> if type(sqrt(100*L[1]+10*L[2]+L[3]),integer)=true then
> print(ithprime(j)) else fi; od; end:
```

or the following procedure:

```
> invp:=proc(N) local j, L;
> for j from 169 to N
> do L:=convert(ithprime(j),base,10);# N <= 1229
> if type(sqrt(1000*L[1]+100*L[2]+10*L[3]+L[4]),
> integer)=true
> then print(ithprime(j)) else fi; od; end:
```

(139) With MAPLE:

```
> for n from 3 by 2 to 10000 do
> if isprime(n) and isprime(n+2) and isprime(n+6)
> then print(n) else fi; od;
```

(140) We prove this result using induction on k. The result is immediate for $k = 2$. Assume that the result is true for a certain integer $k > 2$, that is for which we have $p_k < 2^k$. It is enough to show that $p_{k+1} < 2^{k+1}$. From Bertrand's Postulate, there exists a prime number between p_k and $2p_k$, in which case $p_{k+1} < 2p_k$, and the result is proved.

(141) It is enough to show that $d \not\equiv 2, 4 \pmod 6$. First of all, assume that $d = 2$: if $p_k \equiv 1 \pmod 3$, then $p_{k+1} = p_k + 2 \equiv 0 \pmod 3$, contradicting the fact that p_{k+1} is prime; similarly if $p_k \equiv 2 \pmod 3$, then $p_{k-1} = p_k - 2 \equiv 0 \pmod 3$, contradicting the fact that p_{k-1} is prime. The same type of contradiction emerges when we assume that $d = 4$. If $d = 6k+2$ or $6k+4$ with $k \geq 1$, the same argument works. For $d = 6$, it is $p_{16} = 53$; for $d = 12$, it is $p_{47} = 211$; for $d = 18$, it is $p_{2285} = 20\,201$.

REMARK: It is interesting to observe that the gap $d = 24$ is reached earlier than might be expected in the sequence of prime numbers, namely with $p_{1939} = 16\,787$.

(142) This statement follows from the fact that each of the listed numbers is a perfect square, since

$$12321 = 111^2, \quad 1234321 = 1111^2, \ldots, \quad 12345678987654321 = 111111111^2.$$

(143) Let $k \geq 2$. Since each number $\leq n_k$ is either 1, a prime number or else a composite number, it is clear that

$$n_k = 1 + \pi(n_k) + k. \tag{1}$$

By using MATHEMATICA and the program

```
n=1;Do[n=n+1;While[PrimePi[n]!=n-10^a-1,n++]; Print[10^a,
" ",n],{a,1,3}]
```

we obtain the table

10	18
100	133
1000	1197

This reveals that $n_{10} = 18$, $n_{100} = 133$ and $n_{1000} = 1197$. For values of k larger than 1000, and to accelerate the computations, one can use the approximation (guaranteed by the Prime Number Theorem) $\pi(x) \approx \frac{x}{\log x} + \frac{x}{\log^2 x}$, so that (1) gives

$$n_k \approx \frac{n_k}{\log n_k} + \frac{n_k}{\log^2 n_k} + k$$

and therefore that

$$n_k\left(1-\frac{1}{\log n_k}-\frac{1}{\log^2 n_k}\right)\approx k, \tag{2}$$

which in particular means that

$$\log n_k \approx \log k. \tag{3}$$

Combining (2) and (3), we obtain the approximation

$$n_k \approx k\cdot\left(1-\frac{1}{\log k}-\frac{1}{\log^2 k}\right)^{-1}. \tag{4}$$

Setting $s_k(n) := 1+\pi(n)+k-n$, it follows that if a number n satisfies $s_k(n)=0$, then $n=n_k$.

First consider the case $k=10^4$. From (4), we have as a first approximation $n_{10^4}\approx 11\,369$. By using MATHEMATICA and the program

```
n=11369;While[(a=s[n])!=0,n=n+a];Print[n]
```

where $s(n)=s_{1000}(n)$, we obtain that $n_{10000}=11\,374$. Similarly, with the approximation $n_{10^5}\approx 110\,425$, we obtain that $n_{10^5}=110\,487$. The following is the table giving the values of n_{10^α} for $1\le\alpha\le 10$.

α	n_{10^α}
1	18
2	133
3	1 197
4	11 374
5	110 487

α	n_{10^α}
6	1 084 605
7	10 708 555
8	106 091 745
9	1 053 422 339
10	10 475 688 327

(144) Let $k\ge 2$. Setting $r=[\log n_k/\log 2]$, it is clear that the number n_k satisfies

$$\sum_{i=1}^{r}\pi(n_k^{1/i})=k.$$

From this relation and the approximation $\pi(x)\approx\dfrac{x}{\log x}$ (guaranteed by the Prime Number Theorem), it follows that

$$\frac{n_k}{\log n_k}\approx k,$$

so that $\log n_k\approx\log k+\log\log n_k\approx\log k$ and therefore that

$$n_k\approx k\log n_k\approx k\log k,$$

which gives a starting point for the computation of the exact value of n_k.

Using MATHEMATICA and the program

```
Do[k = 10^j; n = Floor[N[k*Log[k]]];
While[r = Floor[N[Log[n]/Log[2]]];
s=Sum[PrimePi[n^(1/i)],{i,1,r}];(a=k-s)!=0,n=n+a];
Print[j,"->", n,"=",FactorInteger[n]],{j,3,10}]
```

we finally obtain the following table:

α	n_{10^α}
1	16
2	419
3	7 517
4	103 511
5	1 295 953

α	n_{10^α}
6	15 474 787
7	179 390 821
8	2 037 968 761
9	22 801 415 981
10	252 096 677 813

(145) If n is even, then 2^n+n^2 is also even and therefore not a prime. It follows that $n \equiv 1, 3$ or 5 modulo 6. If $n = 6k+1$ for a certain nonnegative integer k, then $2^n = 2^{6k+1} \equiv 2 \pmod 3$ and $n^2 \equiv 1 \pmod 3$; in this case, we have that $2^n + n^2 \equiv 2+1 \equiv 0 \pmod 3$. Similarly, if $n = 6k+5$ for a certain nonnegative integer k, we easily show that $3|2^n+n^2$. Therefore, the only way that 2^n+n^2 can be a prime number is that $n \equiv 3 \pmod 6$.

Thus, by considering all the positive integers $n < 100$ of the form $n = 6k+3$ and using a computer, we easily find that the only prime numbers of the form 2^n+n^2, with $n < 100$, are those corresponding to $n = 1, 9, 15, 21, 33$.

(146) We will show that if n is of the form $n = 3k+1$ or $n = 3k+2$ with $k \geq 1$, then $7|a_n$. Moreover, we will show that if n is of the form $n = 3(3k+1)$ or $n = 3(3k+2)$ with $k \geq 1$, then $73|a_n$. Finally, since $3|a_n$ if n is even, it will follow that, for a_n to be prime, n must be an odd multiple of 9.

So let $n = 3k+a$, with $a = 1$ or 2. Since $8^k \equiv 1 \pmod 7$ for each integer $k \geq 1$, we have

$$\begin{aligned} a_n = 2^n(2^n+1)+1 &= 2^{3k+a}(2^{3k+a}+1)+1 \\ &= 8^k 2^a(8^k 2^a+1)+1 \equiv 2^a(2^a+1)+1 \pmod 7. \end{aligned}$$

But

$$2^a(2^a+1)+1 = \begin{cases} 7 \equiv 0 \pmod 7 & \text{if } a=1, \\ 21 \equiv 0 \pmod 7 & \text{if } a=2, \end{cases}$$

which establishes our first statement.

Let us now assume that $n = 3(3k+a)$ with $a = 1$ or 2. Since $2^9 \equiv 1 \pmod{73}$, we have

$$\begin{aligned} a_n = 2^n(2^n+1)+1 &= 2^{9k+3a}(2^{9k+3a}+1)+1 \\ &= (2^9)^k 2^{3a}((2^9)^k 2^{3a}+1)+1 \equiv 2^{3a}(2^{3a}+1)+1 \pmod{73}. \end{aligned}$$

But

$$2^{3a}(2^{3a}+1)+1 = \begin{cases} 2^3(2^3+1)+1 = 73 \equiv 0 \pmod{73} & \text{if } a=1, \\ 2^6(2^{12}+1)+1 = 4161 \equiv 0 \pmod{73} & \text{if } a=2, \end{cases}$$

which establishes our second statement.

Having observed that a_n is prime for $n = 1, 3$ and 9, and then considering all the numbers of the form $n = 9(2k+1)$, we obtain using a computer that a_n is composite for each integer n, $10 \leq n < 1000$.

(147) That number is $n = 9$; we then have $a_9 = 326981 = 79 \cdot 4139$.

(148) First observe that it follows from Wilson's Theorem that

$$\left[\frac{(j-1)!+1}{j} - \left[\frac{(j-1)!}{j}\right]\right] = \begin{cases} 1 & \text{if } j \text{ is prime}, \\ 0 & \text{if } j \text{ is composite}. \end{cases}$$

Hence, to obtain the formula of Mináс and Willans, we only need to prove that

$$p_n = 2 + \sum_{m=2}^{2^n} \left[\left[\frac{n}{1+\pi(m)}\right]^{1/n}\right].$$

But we easily prove that

$$\left[\left[\frac{n}{1+\pi(m)}\right]^{1/n}\right] = \begin{cases} 1 & \text{if } \pi(m) \le n-1, \\ 0 & \text{otherwise.} \end{cases}$$

Now, as m varies from 2 to 2^n, we have that $\pi(m) \le n-1$ for $m = 2, 3, \ldots, p_n - 1$, that is a total of $p_n - 2$ numbers. Therefore,

$$2 + \sum_{m=2}^{2^n} \left[\left[\frac{n}{1+\pi(m)}\right]^{1/n}\right] = 2 + p_n - 2 = p_n,$$

as was to be shown.

(149) We use an induction argument. The result is true for $n = 1$ and for $n = 2$. So let $n \ge 3$. Assume that the result is true for all natural numbers $\le n-1$ and let us show that it implies that it must be true for n. Let $P_n = \prod_{p \le n} p$. First of all, if n is even, then $P_n = P_{n-1}$, so that the result is true for n. Let us examine the case where n is odd, that is $n = 2k+1$ for a certain positive integer k. It follows that each prime number p such that $k+2 \le p \le 2k+1$ is a divisor of the number

$$(*) \qquad \binom{2k+1}{k} = \frac{(2k+1)(2k)(2k-1)(2k-2)\cdots(k+2)}{1 \cdot 2 \cdot 3 \cdots k}.$$

Since

$$2^{2k+1} = (1+1)^{2k+1} > \binom{2k+1}{k} + \binom{2k+1}{k+1} = 2\binom{2k+1}{k},$$

we obtain

$$\binom{2k+1}{k} < 4^k.$$

It follows that the product of all the prime numbers p such that $k+2 \le p \le 2k+1$ is a divisor of $\binom{2k+1}{k}$ and therefore smaller than 4^k. On the other hand, using the induction hypothesis, we have that $P_{k+1} \le 4^{k+1}$. This is why

$$P_n = P_{2k+1} = \prod_{p \le k+1} p \cdot \prod_{k+2 \le p \le 2k+1} p < 4^{k+1} \cdot 4^k = 4^{2k+1} = 4^n,$$

as was to be shown.

(150) Let $m = (a, c)$. Then, there exist two integers u and v such that $(u, v) = 1$ and such that $a = mu$ and $c = mv$. Hence, since $ab = cd$, we have $mub = mvd$ and therefore $ub = vd$. Since $(u, v) = 1$, we have $u|d$ and this is why there exists an integer n such that $d = nu$. Since $ub = vnu$, we therefore have $b = nv$. It follows from these relations that

$$\begin{aligned} a^2 + b^2 + c^2 + d^2 = m^2u^2 + n^2v^2 + m^2v^2 + n^2u^2 &= m^2(u^2+v^2) \\ +n^2(u^2+v^2) &= (u^2+v^2)(m^2+n^2), \end{aligned}$$

a product of two integers larger than 1.

(151) Since

$$4n^3 + 6n^2 + 4n + 1 = n^4 + 4n^3 + 6n^2 + 4n + 1 - n^4 = (n+1)^4 - n^4$$
$$= ((n+1)^2 - n^2)((n+1)^2 + n^2) = (2n+1)(2n^2 + 2n + 1),$$

the product of two integers larger than 1, the result follows.

(152) First of all, since $p + q$ is even, we can write

$$(*) \qquad p + q = 2 \cdot \frac{p+q}{2}.$$

Since $\frac{p+q}{2}$ is an integer located between the two consecutive prime numbers p and q, it must be composite, that is the product of at least two prime numbers, and this is why the right-hand side of $(*)$ has at least three prime factors.

(153) The answer is YES. We look for positive integers n, a, b and c such that

$$\frac{n}{2} = a^2, \quad \frac{n}{3} = b^3, \quad \frac{n}{5} = c^5.$$

It is sufficient to find integers a, b and c such that

$$2a^2 = 3b^3 = 5c^5.$$

The task is therefore to find integers α_i, β_i and γ_i $(i = 1, 2, 3)$ such that

$$2\left(2^{\alpha_1}3^{\beta_1}5^{\gamma_1}\right)^2 = 3\left(2^{\alpha_2}3^{\beta_2}5^{\gamma_2}\right)^3 = 5\left(2^{\alpha_3}3^{\beta_3}5^{\gamma_3}\right)^5.$$

To do so, we must find integers α_i, β_i and γ_i $(i = 1, 2, 3)$ such that

$$2\alpha_1 + 1 = 3\alpha_2 = 5\alpha_3, \quad 2\beta_1 = 3\beta_2 + 1 = 5\beta_3, \quad 2\gamma_1 = 3\gamma_2 = 5\gamma_3 + 1.$$

We easily find

$$\alpha_1 = 7, \alpha_2 = 5, \alpha_3 = 3, \quad \beta_1 = 5, \beta_2 = 3, \beta_3 = 2, \quad \gamma_1 = 3, \gamma_2 = 2, \gamma_3 = 1.$$

We then obtain that $n = 2(2^7 \cdot 3^5 \cdot 5^3)^2 = 30\,233\,088\,000\,000$ serves our purpose.

(154) This follows from the identity

$$n^{42} - 27 = (n^{14})^3 - 3^3 = (n^{14} - 3)(n^{28} + 3n^{14} + 3^2).$$

(155) We proceed by contradiction by assuming that there does not exist any prime number in the interval $]x, 2x]$, in which case we have $\theta(2x) = \theta(x)$. By using the inequalities $0.73x < \theta(x) < 1.12x$, we would then have

$$1.46x = 2(0.73)x < \theta(2x) = \theta(x) < 1.12x,$$

a contradiction.

(156) We proceed by induction. First of all, for $n = 4$, the result is true, since $121 = 11^2 = p_5^2 < p_1p_2p_3p_4 = 210$. Assume that the inequality $p_k^2 < p_1p_2\cdots p_{k-1}$ is true for a certain integer $k \geq 5$. By using Bertrand's Postulate in the form $p_{k+1} < 2p_k$, we then have

$$p_{k+1}^2 < 4p_k^2 < 4p_1p_2\cdots p_{k-1} < p_1p_2\cdots p_k,$$

and the result then follows by induction.

(157) If there exist $q, r, a \in \mathbb{N}$ such that $q^r = (q^{r/2})^2 = a^2$, where r is even and $q^r = p + m^2$ with p prime and $m \in \mathbb{N}$, then $a^2 - m^2 = p$, so that $(a-m)(a+m) = p$. Since p is prime, we must have $a-m = 1$ and $a+m = p$, and therefore $m = a - 1$ and $p = 2a - 1$. Hence, if $2a - 1 = 2q^{r/2} - 1$ is composite, q^r cannot be written as $p + m^2$, as was to be shown.

(158) For $p = 3$, the result is immediate. Assume that $p \geq 5$. If $p = 3k + 1$ for a certain positive integer k, then $8k + 1 = 24k + 9$, a multiple of 3. Otherwise, that is if $p = 3k - 1$ for a certain positive integer k, then $8p - 1 = 24k - 9$, a multiple of 3, which contradicts the fact that $8p - 1$ is prime. In both cases, the result is proved.

(159) If a positive integer of the form $3k + 2$ has no prime factor of the form $3k+2$, then all its prime factors are of the form $3k+1$. Since the product of two integers of the form $3k + 1$ is of the form $3k + 1$, the result follows.

Since each product of prime numbers of the form $4k+1$ is of the same form and since each product of prime numbers of the form $6k+1$ is of the same form, the result follows.

(160) (a) We have $23 = 3 \cdot 3! + 2 \cdot 2! + 1 \cdot 1!$ and $57 = 2 \cdot 4! + 1 \cdot 3! + 1 \cdot 2! + 1 \cdot 1!$.

(b) To find the Cantor expansion of a positive integer n, we proceed as follows. Let m be the largest positive integer such that $m! \leq n$ and let a_m be the largest positive integer such that $a_m \cdot m! \leq n$. It is clear that $0 < a_m \leq m$; otherwise, this would contradict the maximal choice of m. If $a_m \cdot m! = n$, then the Cantor expansion is given by $n = a_m \cdot m!$. Otherwise, that is if $a_m \cdot m! < n$, let $d_1 = n - a_m \cdot m! > 0$, let m_1 be the largest positive integer such that $m_1! \leq d_1$ and let a_{m_1} be the largest positive integer such that $a_{m_1} \cdot m_1! \leq d_i$. As above, we have $a < a_{m_1} \leq m_1$. If $a_{m_1} \cdot m_1! = d_1$; then the Cantor expansion is given by $n = a_m \cdot m! + a_{m_1} \cdot m_1!$, where $0 < a_{m_1} \leq m_1 < m$. If $a_{m_1} \cdot m_1! < d_1$, then we set $d_2 = d_1 - a_{m_1} \cdot m_1!$ and we let m_2 be the largest positive integer such that $m_2! \leq d_2$. And so on. We thus build a sequence of positive integers $m > m_1 > m_2 > \ldots$ with the corresponding integers $0 < a_{m_i} \leq m_i$. Since the sequence of m_i's is decreasing, it must have an end. Let us show the uniqueness of this representation. Assume that for $0 \leq a_j, b_j \leq j$, we have

$$n = a_m m! + \cdots + a_1 1! = b_m m! + \cdots + b_1 1!,$$

that is $(a_m - b_m)m! + \cdots + (a_1 - b_1)1! = 0$. If both expansions are different, then there exists a smaller integer j such that $1 \leq j < m$ and $a_j \neq b_j$. Hence,

$$j!\left((a_m - b_m)\frac{m!}{j!} + \cdots + (a_{j+1} - b_{j+1})(j+1) + (a_j - b_j)\right) = 0$$

and therefore

$$\begin{aligned} b_j - a_j &= (a_m - b_m)\frac{m!}{j!} + \cdots + (a_{j+1} - b_{j+1})(j+1) \\ &= (j+1)\left((a_m - b_m)\frac{m!}{(j+1)!} + \cdots + (a_{j+1} - b_{j+1})\right), \end{aligned}$$

which implies that $(j+1) | (b_j - a_j)$. Since $0 \leq a_j, b_j \leq j$, it follows that $a_j = b_j$, a contradiction.

(161) (*TYCM, Vol. 19, 1988, p. 191*). The expression in the statement can be written as

$$\frac{(p-1)!+1}{p}+\frac{(-1)^d d!(p-1)!+1}{p+d}.$$

Since $(p+d-1)! = (p+d-1)(p+d-2)\cdots(p+d-d)(p-1)!$, we have $(p+d-1)! \equiv (-1)^d d!(p-1)! \pmod{p+d}$, and it follows that the expression in the statement is an integer if and only if

$$(1) \qquad \frac{(p-1)!+1}{p}+\frac{(p+d-1)!+1}{p+d}$$

is an integer. From Wilson's Theorem, if p and $p+d$ are two prime numbers, then each of the terms of (1) is an integer, which proves the necessary condition.

Conversely, assume that expression (1) is an integer. If p or $p+d$ is not a prime, then by Wilson's Theorem, at least one of the terms of (1) is not an integer. This implies that none of the terms of (1) is an integer or equivalently neither of p and $p+d$ is prime. It follows that both fractions of (1) are in reduced form.

It is easy to see that if a/b and a'/b' are reduced fractions such that $a/b+a'/b' = (ab'+a'b)/(bb')$ is an integer, then $b|b'$ and $b'|b$.

Applying this result to (1), we obtain that $(p+d)|p$, which is impossible. We may therefore conclude that if (1) is an integer, then both p and $p+d$ must be primes.

(162) If $p=3$, then $p+2=5$ is prime and $p^2+2p-8=7$ is prime. It is the only number with this property. Indeed, $p=2$ does not have this property, while if $p>3$, then

$$p^2+2p-8 \equiv 1+2p-2 \equiv 2(p+1) \equiv 0 \pmod 3 \iff 3|(p+1).$$

But for $p>3$, $p=3k\pm 1$, and in each of the cases it is easily seen that at least one of the two numbers $p+2$ and p^2+2p-8 is not a prime.

(163) The answer is YES. If $p=3$, then $p^2+8=17$ is prime and $p^3+4=31$ is prime. It is the only prime number with this property. Indeed, $p=2$ does not have this property, while if $p>3$, then $p\equiv \pm 1 \pmod 3$, in which case $p^2\equiv 1 \pmod 3$, that is $p^2+8\equiv 9\equiv 0 \pmod 3$, so that p^2+8 is not a prime. Thus the result.

(164) (Ribenboim [**28**], p. 145). First assume that the congruence is satisfied. Then $n\neq 2,4$ and $(n-1)!+1\equiv 0 \pmod n$. Thus, using Wilson's Theorem, n is prime. Moreover, $4(n-1)!+2\equiv 0 \pmod{n+2}$; thus, multiplying by $n(n+1)$ we obtain

$$4[(n+1)!+1]+2n^2+2n-4\equiv 0 \pmod{n+2}$$

and therefore

$$4[(n+1)!+1]+(n+2)(2n-2)\equiv 0 \pmod{n+2};$$

hence, $4[(n+1)!+1]\equiv 0 \pmod{n+2}$. This is why, using Wilson's Theorem, $n+2$ is also prime.

Conversely, if n and $n+2$ are prime, then $n\neq 2$ and

$$\begin{aligned}(n-1)!+1 &\equiv 0 \pmod n,\\ (n+1)!+1 &\equiv 0 \pmod{n+2}.\end{aligned}$$

But $n(n+1) = (n+2)(n-1)+2$, and this is why $2(n-1)!+1 = k(n+2)$, where k is an integer. From the relation $(n-1)! \equiv -1 \pmod n$, we obtain $2k+1 \equiv 0 \pmod n$. Now, $2(n-1)!+1 = k(n+2)$ is equivalent to $4(n-1)!+2 \equiv 0 \equiv -(n+2) \pmod{n+2}$. Moreover, $4(n-1)!+2 \equiv 4k \equiv -2 \equiv -(n+2) \pmod n$. Hence, $4(n-1)!+2 \equiv -(n+2) \pmod{(n(n+2))}$; that is $4\Big((n-1)!+1\Big)+n \equiv 0 \pmod{n(n+2)}$.

(165) The prime number $p = 3$ is the only one with this property, because if $p > 3$, then $p = 2k+1$ for a certain integer $k \geq 2$, in which case

$$2^p = 2^{2k+1} = 4^k \cdot 2 \equiv 2 \pmod 3$$

while

$$p^2 \equiv 1 \pmod 3,$$

so that

$$2^p + p^2 \equiv 0 \pmod 3.$$

(166) The answer is $p = 19$. Indeed, $17p+1 = a^2 \Rightarrow 17p = (a-1)(a+1)$. We then have $17 = a-1$ and $p = a+1$ or $17 = a+1$ and $p = a-1$. The first case yields $a = 18$ and $p = 19$, while the second case yields $a = 16$ and $p = 15$, which is to be rejected. Thus the result.

(167) (a) The possible values of (a^2, b) are p and p^2.
(b) The only possible value of (a^2, b^2) is p^2.
(c) The possible values of (a^3, b) are p, p^2 and p^3.
(d) The possible values of (a^3, b^2) are p^2 and p^3.

(168) (a) The only possible value is p^3.
(b) The only possible value is p.
(c) The only possible value is p.
(d) The possible values are p^2, p^3, p^4 and p^5.

(169) We have $(a^2b^2, p^4) = p^4$ and $(a^2+b^2, p^4) = p^2$.

(170) (a) True. (b) True. (c) True.
(d) False. Indeed, we have $13 | 2^2+3^2$ and $13 | 3^2+2^2$, while $13 \nmid 2^2+2^2 = 8$.

(171) It is an immediate consequence of Theorem 12.

(172) Let

$$\begin{cases} a = p_1^{\alpha_1} \cdots p_r^{\alpha_r}, \\ b = p_1^{\beta_1} \cdots p_r^{\beta_r}, \\ c = p_1^{\gamma_1} \cdots p_r^{\gamma_r}. \end{cases}$$

From Theorem 12,

$$(a,b,c) = p_1^{\min(\alpha_1,\beta_1,\gamma_1)} \cdots p_r^{\min(\alpha_r,\beta_r,\gamma_r)}$$

and

$$[a,b,c] = p_1^{\max(\alpha_1,\beta_1,\gamma_1)} \cdots p_r^{\max(\alpha_r,\beta_r,\gamma_r)}.$$

To prove the result, we proceed by contradiction. Assume for example that $(a,b) > 1$. Using the fact that $(a,b,c)[a,b,c] = abc$, it follows, using the above notation, that

$$\min(\alpha_i,\beta_i,\gamma_i) + \max(\alpha_i,\beta_i,\gamma_i) = \alpha_i + \beta_i + \gamma_i \qquad (i = 1,2,\ldots,r).$$

But it is easy to prove that for the sum of three nonnegative integers to be equal to the sum of the smallest and of the largest of these same three numbers, at least two of these numbers must be 0. But this contradicts the fact that $(a,b)>1$, an inequality which means that there exists an i_0 $(1\le i_0\le r)$ for which $\min(\alpha_{i_0},\beta_{i_0})\ge 1$. Hence, the result.

(173) We use Theorem 12 and the fact that

$$\min\{\alpha_i,\beta_i,\gamma_i\}=\min\{\alpha_i,\beta_i\}+\min\{\beta_i,\gamma_i\}+\min\{\alpha,\gamma_i\}\\ -\min\{\alpha_i+\beta_i,\beta_i+\gamma_i,\alpha_i+\gamma_i\}.$$

The second part follows from the first part and Problem 171.

(174) Let

$$\begin{cases} a=p_1^{\alpha_1}\cdots p_r^{\alpha_r},\\ b=p_1^{\beta_1}\cdots p_r^{\beta_r},\\ c=p_1^{\gamma_1}\cdots p_r^{\gamma_r}.\end{cases}$$

Since $[a,b]=\prod_{i=1}^r p_i^{\max\{\alpha_i,\beta_i\}}$ and $(a,b)=\prod_{i=1}^r p_i^{\min\{\alpha_i,\beta_i\}}$, it is enough to show that, for each i,

$$2\max\{\alpha_i,\beta_i,\gamma_i\}-\max\{\alpha_i,\beta_i\}-\max\{\beta_i,\gamma_i\}-\max\{\gamma_i,\alpha_i\}\\ =2\min\{\alpha_i,\beta_i,\gamma_i\}-\min\{\alpha_i,\beta_i\}-\min\{\beta_i,\gamma_i\}-\min\{\gamma_i,\alpha_i\}.$$

Without loosing in generality, we may assume that, for a given i, $\alpha_i\ge\beta_i\ge\gamma_i$, from which the result easily follows.

(175) Let $a=\prod_{i=1}^r p_i^{a_i}$, $b=\prod_{i=1}^r p_i^{b_i}$, $c=\prod_{i=1}^r p_i^{c_i}$. Without loosing in generality, we may assume that $a_i\le b_i\le c_i$. The equation of the statement allows one to conclude that $c_i+a_i=\frac{1}{2}(a_i+b_i+c_i)$ and therefore that $a_i+c_i=b_i$, which implies, since $c_i\ge b_i$, that $b_i=c_i$ and $a_i=0$. This means that in order for the relation to be true, for the same prime number, two of the exponents must be equal while the third one should be 0. Hence, we can choose $a=2^1\cdot 3^1\cdot 5^0=6$, $b=2^1\cdot 3^0\cdot 5^1=10$ and $c=2^0\cdot 3^1\cdot 5^1=15$. Note that the numbers 42, 70, 15 will also do.

(176) The left inequality is obvious. To prove the right equality, first observe that

$$\#n=[1,2,\ldots,n]=\prod_{p\le n}p^{\delta_p},$$

where p^{δ_p} is the largest power of p not exceeding n. In other words, δ_p is defined implicitly by the inequalities $p^{\delta_p}\le n<p^{\delta_p+1}$. It follows successively that

$$\delta_p\log p\le\log n<(\delta_p+1)\log p,$$

$$\delta_p\le\frac{\log n}{\log p}<\delta_p+1,$$

$$\delta_p=\left[\frac{\log n}{\log p}\right].$$

We have thus established that

$$\#n=\prod_{p\le n}p^{[\log n/\log p]},$$

which was to be shown.

(177) It is easy to see that

$$(p, p+r) = \begin{cases} p & \text{if } p|r, \\ 1 & \text{if } p \nmid r \end{cases}$$

and that

$$[p, p+r] = \begin{cases} p+r & \text{if } p|r, \\ p(p+r) & \text{if } p \nmid r. \end{cases}$$

(178) We have that $p|(8ad - bd) - (8bc - bd) = 8(ad - bc)$, and therefore that $p|(ad - bc)$, since p is odd.

(179) Since p is odd, it is clear that $p + p + 2 = 2(p + 1)$ is a multiple of 4. On the other hand, since $p + 2$ is prime, the prime number p must be of the form $3k + 2$. It follows that $p + p + 2 = 2p + 2 = 2(3k + 2) + 2 = 6k + 6$, a multiple of 3. Hence, the result.

(180) Set $n = pr$, where p is a prime number. If $p \neq r$, then p and r show up as factors in the product $(n - 1)!$ and therefore n divides $(n - 1)!$. If $r = p$, then $n = p^2$ and

$$(n - 1)! = (p^2 - 1)(p^2 - 2) \cdots p \cdots 1.$$

Hence, in order for $(n-1)!$ to be divisible by p^2, we must have that $(n-1)!$ contains the factors p and $2p$; that is we must have $p^2 - \alpha = 2p$ for a certain integer $\alpha \geq 2$. But this is possible only if we choose $\alpha = p(p-2)$ (provided that $p > 2$). If $p = 2$, that is $n = 4$, it is clear that the result does not hold.

(181) If it is the case, we will have

$$\left.\frac{n(n+1)}{2} \,\right|\, n!, \quad \text{that is} \quad \left.\frac{n+1}{2} \,\right|\, (n-1)!.$$

This means that we are looking for the positive integers n for which there exists a positive integer $M = M(n)$ such that

$$(n - 1)! = M\frac{(n+1)}{2}, \quad \text{that is} \quad 2\frac{(n-1)!}{(n+1)} = M.$$

If $n + 1 = p$, with p prime, then $n = p - 1$, in which case M is not an integer. Therefore, $n + 1$ must be composite; that is $n + 1 = pr$, where p is prime. If p and r can be chosen in such a way that $p \neq r$, then p and r will show up as factors in the product $(n - 1)!$, implying that M is an integer. If the only possible choice for p and r is $p = r$, then we have $n + 1 = r^2$ and

$$2(n - 1)! = 2(p^2 - 2)! = 2(p^2 - 2)(p^2 - 3) \cdots p \cdots 1.$$

Hence, in order for $2(n - 1)!$ to be divisible by p^2, we must have that $2(n-1)!$ contains the factors p and $2p$; that is we must have $(p^2 - \alpha) = 2p$ for a certain integer $\alpha \geq 2$. But this is possible only if $\alpha = p(p - 2)$ (provided $p > 2$). It follows that the result is true for all integers n such that $n \neq p - 1$, p being an odd prime number.

(182) (*AMM, Vol. 81, 1974, p. 778*). From the solution of Problem 181, we have that if $n > 5$ is a composite integer, $(n-2)!/n$ is an even integer and therefore that

$$\sin \frac{\pi}{2} n \left(\frac{(n-2)!}{n} \right) = 0.$$

On the other hand, if $n = p$ is prime, then by Wilson's Theorem, $(p-2)! \equiv -(p-1)! \equiv 1 \pmod{p}$, in which case there exists an integer k such that $(p-2)! = kp + 1$ and therefore

$$\left(\frac{(p-2)!}{p}\right) = k = \frac{(p-2)! - 1}{p}.$$

Hence, if $p > 5$, then $4|(p-2)!$ and therefore

$$\sin \frac{\pi}{2} p \left(\frac{(p-2)!}{p}\right) = \sin \frac{\pi}{2}((p-2)! - 1) = \sin\left(\frac{\pi}{2}(p-2)! - \frac{\pi}{2}\right) = -1.$$

These two cases allow us to conclude that for $n > 5$, the term indexed by n in the sum is 0 if n or $n+2$ is composite and is $(-1)(-1) = 1$ if n and $n+2$ are prime. Note that the term "2" is necessary in order to count the pairs of twin primes $(3,5)$ and $(5,7)$.

(183) Assume that there does not exist any prime p between n and $n!$. Then, consider the number $N = n! - 1$. If N is prime, we have found a prime number between n and $n!$, a contradiction. If N is composite, then there exists a prime number p such that $p \le n$ and $p|(n! - 1)$; but since $p|n!$, we must have $p|1$, again a contradiction. Thus, the result.

(184) The answer is NO. Since $1 + 2 + 2^2 + \cdots + 2^n = 2^{n+1} - 1$, it is easy to check that $1 + 2 + \cdots + 2^8$ can be written as

$$2^9 - 1 = (2^3)^3 - 1 = 8^3 - 1 = (8-1)(8^2 + 8 + 1),$$

a composite number, while the preceding number, that is 255, is also composite.

REMARK: The prime numbers of the form $2^k - 1$ are called Mersenne primes, and it is not known if there exist infinitely many of them. See the next problem.

(185) It is easy to see that

$$a^n - 1 = (a-1)(a^{n-1} + a^{n-2} + \cdots + a + 1)$$

where the second factor is larger than 1. This implies that $a - 1 = 1$; that is $a = 2$. Moreover, if n is composite, then there exist integers r and s such that $n = rs$, $r > 1$, $s > 1$, and therefore

$$a^n - 1 = 2^{rs} - 1 = (2^r - 1)(2^{r(s-1)} + \cdots + 2^r + 1),$$

where each factor is larger than 1, which contradicts the fact that $a^n - 1$ is prime.

(186) If a is odd, then $a^n + 1 \ge 4$ and is an even integer, hence not prime. On the other hand, if n has an odd factor $r > 1$, then there exists a positive integer m such that $n = mr$, in which case

$$a^n + 1 = (a^m + 1)(a^{m(r-1)} - a^{m(r-2)} + \cdots - a^m + 1).$$

Since $r \ge 3$, both factors are larger than 1, and this contradicts the fact that $a^n + 1$ is prime. Hence, n has no odd factor larger than 1 and n must be of the form 2^r.

(187) (*TYCM, Vol. 13, 1982, p. 208*). We reduce these expressions modulo 3. Since $2^{2^z} + 1$ with $z > 0$ is of the form $2^{2t} + 1$ with $t > 0$, it follows that

$$2^{2^z} + 1 = 2^{2t} + 1 = \left(2^2\right)^t + 1 \equiv 2 \pmod{3}.$$

But $2^x - 1 \equiv 0$ or $1 \pmod 3$ depending whether x is even or odd. Hence, $(2^x - 1)(2^y - 1) \equiv 0$ or $1 \pmod 3$, and since $2^{2^z} + 1 \equiv 2 \pmod 3$, the result follows.

(188) The result is true for $n = 1$, since in this case it is easy to check that $F_0 = 2^{2^0} + 1 = 3$ and $F_1 - 2 = 2^{2^1} + 1 - 2 = 5 - 2 = 3$. Assume that the result is true for $n = k$ and let us show that it implies that the result is then true for $n = k + 1$. Indeed, by the induction hypothesis, we have

$$\begin{aligned} F_0F_1F_2\cdots F_{k-1}F_k &= (F_k - 2)F_k = \left(2^{2^k} - 1\right)\left(2^{2^k} + 1\right) \\ &= 2^{2^{k+1}} - 1 = \left(2^{2^{k+1}} + 1\right) - 2 = F_{k+1} - 2, \end{aligned}$$

as required.

(189) Assume the contrary, that is that there exist two integers $m > n \geq 0$ such that $(F_m, F_n) = d > 1$. Then, using Problem 188, we have

$$(*) \qquad F_0F_1\cdots F_{m-1} = F_m - 2.$$

Since F_n is one of the factors on the left of $(*)$, it follows that $d|2$. But since each Fermat number is odd, it is impossible to have $d = 2$. Hence, $d = 1$, and the result follows.

(190) With the help of a computer, we find that this number is 29 341.

(191) From Problem 189, all Fermat numbers are pairwise relatively prime. Each Fermat number therefore introduces in its factorization at least one new prime number. As a consequence, the Fermat numbers generate infinitely many prime numbers.

(192) To prove part (a), we proceed by induction. First of all, it is clear that $3^2|2^{3^1} + 1$. Assuming that $3^k|f_{k-1}$ for some $k \geq 2$, we will show that this implies that $3^{k+1}|f_k$. Using the fact that $a^3 + 1 = (a+1)(a^2 - a + 1)$, we have

$$(*) \qquad \begin{aligned} f_k &= 2^{3^k} + 1 = \left(2^{3^{k-1}}\right)^3 + 1 \\ &= \left(2^{3^{k-1}} + 1\right)\left(\left(2^{3^{k-1}}\right)^2 - \left(2^{3^{k-1}}\right) + 1\right) \\ &= A \cdot B, \end{aligned}$$

say. The expression A is divisible by 3^k because of the induction hypothesis. It therefore remains to show that $3|B$. But $B = a^2 - a + 1$, where $a = 2^{3^{k-1}} \equiv 2 \pmod 3$. It follows that

$$B = a^2 - a + 1 \equiv 2^2 - 2 + 1 \equiv 0 \pmod 3,$$

as required.

To prove part (b), we only need to observe that $f_{n-1}|f_n$, as is implied by the second line of $(*)$.

(193) (This problem can be found on page 64 of the book of D.J. Newman [**24**]). Consider the arithmetic progression $15k+7$, $k = 1, 2, \ldots$, which by Dirichlet's Theorem contains infinitely many prime numbers. If $p = 15k + 7$, it is clear that $p - 2 = 15k + 5$ is a multiple of 5 and that $p + 2 = 15k + 9$ is a multiple of 3, which proves the result.

(194) Since

$$2^{16} = 2^{2^4} = 65536 \equiv 154 \pmod{641},$$

we have

$$2^{32} = (2^{16})^2 \equiv (154)^2 = 23716 \equiv 640 \equiv -1 \pmod{641},$$

and the result follows.

(195) First of all, it is clear that $F_2 = 2^{2^2} + 1 = 17 \equiv 7 \pmod{10}$. Therefore, it is enough to show that if $F_k \equiv 7 \pmod{10}$ for a certain $k \geq 2$, then $F_{k+1} \equiv 7 \pmod{10}$. Indeed, we have by the induction hypothesis that

$$\begin{aligned} F_{k+1} = 2^{2^{k+1}} + 1 &= \left(2^{2^k}\right)^2 + 1 \\ &= \left((2^{2^k} + 1) - 1\right)^2 + 1 \equiv (7-1)^2 + 1 = 37 \equiv 7 \pmod{10}, \end{aligned}$$

as required.

(196) We proceed by contradiction in assuming that 2^k divides an integer $m \in E \setminus \{2^k\}$, in which case $m = 2^k r$ for a certain integer $r > 1$, implying that 2^{k+1} is in the set E, which contradicts the minimal choice of 2^k.

For the second part of the problem, assume that the given sum is an integer M. The smallest common multiple of the elements of E must be of the form $2^k m$, where m is an odd number. Multiplying the sum by $m2^{k-1}$, we obtain

$$m2^{k-1} \sum_{j=1}^{n} \frac{1}{j} = m2^{k-1} M.$$

But when the left-hand side is expanded, one of the n terms is equal to $m/2$ while all the others are integers, which yields a contradiction since m is odd.

(197) We have

$$2^{5n} - 1 = (2^5)^n - 1 = (2^5 - 1)(2^{5(n-1)} + 2^{5(n-2)} + \cdots + 2^5 + 1),$$

which implies that the number $2^{5n} - 1$ is divisible by 31 for any positive integer n. Hence, $p = 31$ will serve our purpose.

(198) We have $M_1 = 3$, $M_2 = 7$, $M_3 = 31$, $M_4 = 211$, $M_5 = 2\,311$, which are all prime numbers, while $M_6 = 59 \cdot 509$ and $M_7 = 19 \cdot 97 \cdot 277$ are composite. REMARK: Using the MAPLE program

```
> for k from 8 to 10 do print(
> M(k) = ifactor(product(ithprime(i), i=1..k)+1)) od;
```

we obtain $M_8 = 347 \cdot 27953$, $M_9 = 317 \cdot 703763$ and $M_{10} = 331 \cdot 571 \cdot 34231$.

As of 2006, we still don't know if the sequence $\{M_k\}$ contains infinitely many prime numbers; with the help of a computer, we can nevertheless easily establish that the only values of $k < 1000$ for which M_k is a prime number are: 1, 2, 3, 4, 5, 11, 75, 171, 172, 384, 457, 616 and 643.

(199) Any prime number dividing $p_1 p_2 \cdots p_r + 1$ is distinct from any of the primes $p_1, p_2, \ldots, p_r$; hence, it follows that

$$p_{r+1} \leq p_1 p_2 \cdots p_r + 1,$$

and using an induction argument, we obtain that

$$p_{r+1} \leq 2^{2^0} \cdot 2^{2^1} \cdot 2^{2^2} \cdots 2^{2^{r-1}} + 1 < 2^{2^r},$$

which proves the result.

(200) Let $x \geq 3$. Choose $r \in \mathbb{N}$ such that

$$e^{e^{r-1}} < x \leq e^{e^r}. \tag{1}$$

We easily observe that such a choice of r is unique. The left inequality of (1), the fact that $\pi(x)$ is a nondecreasing function and the relation $p_r \leq 2^{2^{r-1}}$ allow us to write

$$\pi(x) \geq \pi(e^{e^{r-1}}) \geq \pi(2^{2^{r-1}}) \geq \pi(p_r) = r. \tag{2}$$

The right inequality of (1) guarantees that

$$r \geq \log\log x. \tag{3}$$

Combining (2) and (3), we obtain the required inequality.

(201) Assume that there is only a finite number of prime numbers of the form $4n+3$. Denote them by

$$q_1 < q_2 < \ldots < q_k$$

and consider the number

$$N = 4q_1q_2\cdots q_k - 1 = 4(q_1q_2\cdots q_k - 1) + 3, \tag{$*$}$$

which is clearly of the form $4n+3$. If N is prime, then we have found a prime number of the form $4n+3$ larger then q_k, thereby yielding a contradiction. If N is composite, then N cannot be the product of only prime numbers of the form $4n+1$ (since N would then also be of the form $4n+1$). Therefore there exists a prime number q of the form $4n+3$ which divides N. If q is equal to one of the q_i's, that would mean, in light of relation $(*)$, that $q|1$, again a contradiction. Hence, $q > q_k$ and the result is proved.

(202) Assume that there is only a finite number of prime numbers of the form $6n+5$. Denote them by

$$q_1 < q_2 < \ldots < q_k$$

and consider the number

$$N = 6q_1q_2\cdots q_k - 1 = 6(q_1q_2\cdots q_k - 1) + 5, \tag{$*$}$$

which is surely of the form $6n+5$. If N is prime, then we have found a prime number of the form $6n+5$ larger than q_k, thereby yielding a contradiction. If N is composite, then N cannot be the product of only prime numbers of the form $6n+1$ (since N would also be of the form $6n+1$). Therefore there exists a prime number q of the form $6n+5$ which divides N. If q is equal to one of the q_i's, that would mean, in light of $(*)$, that $q|1$, again a contradiction. Hence, $q > q_k$ and the result is proved.

(203) It is enough to consider the polynomial

$$f(x) = (x - p_1)(x - p_2)\cdots(x - p_r) + x,$$

where p_k stands for the k-th prime number.

(204) The answer is NO. Consider such a number N with $2k+1$ digits, $k \geq 2$. We first notice that, for each integer $k > 1$,

$$(1 + 10^2 + \cdots + 10^{2k})(10^2 - 1) = (10^{2k+2} - 1) = (10^{k+1} - 1)(10^{k+1} + 1).$$

Hence,

$$1 + 10^2 + \cdots + 10^{2k} = \frac{(10^{k+1} - 1)(10^{k+1} + 1)}{9 \cdot 11}.$$

Since $k > 1$, both factors on the right-hand side, after dividing by 99, have two factors larger than 1, so that the number N is composite. On the other hand, in the particular case $k = 1$, we find the prime number 101.

(205) Let $G_n = 2^{2^n} + 5$. First of all, $G_0 = 2^1 + 5 = 7$, which is prime. We will show that all the other G_n's, that is those with $n \geq 1$, are divisible by 3. To do so, it is enough to prove that $2^{2^n} \equiv 1 \pmod 3$. But this is true since

$$2^{2^n} = \left(2^2\right)^{2^{n-1}} = 4^{2^{n-1}} \equiv 1^{2^{n-1}} = 1 \pmod 3.$$

Clearly, we could have obtained the same result if instead of 5 we would have used a number of the form $3k+2$, except that in this case, one should first check whether $2 + 3k + 2 = 3k + 4$ is prime or not.

(206) The answer is NO. Indeed, the next gap in the list is 14; it occurs when $p_{r+1} - p_r = 127 - 113 = 14$, while the first gaps of 10 and 12 occur respectively when $p_{r+1} - p_r = 149 - 139 = 10$ and $p_{r+1} - p_r = 211 - 199 = 12$.

(207) Let S be this series; then

$$\begin{aligned} S &= \sum_p \left(\frac{1}{p^2} + \frac{1}{p^3} + \frac{1}{p^4} + \cdots\right) = \sum_p \frac{1}{p^2}\left(1 + \frac{1}{p} + \frac{1}{p^2} + \frac{1}{p^3} + \cdots\right) \\ &= \sum_p \frac{1}{p^2}\frac{1}{1 - \frac{1}{p}} = \sum_p \frac{1}{p(p-1)} < \sum_{n=2}^{\infty} \frac{1}{n(n-1)} = 1. \end{aligned}$$

In fact, the exact value of S is $0.773156669\ldots$.

(208) We have successively

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{\mu(n)}{n} f(x^{1/n}) &= \sum_{n=1}^{\infty} \left(\frac{\mu(n)}{n} \sum_{m=1}^{\infty} \frac{1}{m}\pi(x^{1/mn})\right) \\ &= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{\mu(\frac{mn}{m})}{mn}\pi(x^{1/mn}) \\ &= \sum_{r=1}^{\infty} \left(\frac{\pi(x^{1/r})}{r} \sum_{d|r} \mu(d)\right) = \pi(x), \end{aligned}$$

where we used the fact that $\sum_{d|r} \mu(d) = 0$ if $r > 1$ and 1 if $r = 1$.

(209) The result is immediate for $2 \leq n \leq 6$. On the other hand, since

$$\sum_{n \leq m^2 \leq 2n} 1 = \left[\sqrt{2n}\right] - \left[\sqrt{n}\right] + 1 \geq \sqrt{2n} - 1 - \sqrt{n} + 1 = (\sqrt{2} - 1)\sqrt{n} > 1,$$

for $n > 6$, the result follows for each integer $n \geq 2$.

(210) Assume the contrary, that is that $n^3 = p + m^3$. We then have $n^3 - m^3 = p$, which implies that $(n-m)(n^2+mn+m^2) = p$ and therefore that $n - m = 1$ and $n^2 + mn + m^2 = p$. This shows that $n^2 + n(n-1) + (n-1)^2 = p$,

that is $3n^2 - 3n + 1 = p$, which contradicts the fact that $3n^2 - 3n + 1$ is composite.

(211) The prime numbers $p < 10\,000$ of the indicated form are 2, 5, 17, 37, 101, 197, 257, 401, 577, 677, 1297, 1601, 2917, 3137, 4357, 5477, 7057, 8101 and 8837. Given $n \geq 1$, we have $n \equiv 0, 1, 2, 3, 4, 5, 6, 7, 8, 9 \pmod{10}$, in which case $n^2 \equiv 0, 1, 4, 9, 6, 5, 6, 9, 4, 1 \pmod{10}$ and therefore $n^2 + 1 \equiv 1, 2, 5, 0, 7, 6, 7, 0, 5, 2 \pmod{10}$. Since the numbers $n^2 + 1$, congruent to 0, 2, 6 or 5 modulo 10, are composite, we are left with the numbers n for which $n^2 + 1 \equiv 1, 7 \pmod{10}$. Finally, since the numbers $n \equiv 4, 6 \pmod{10}$ are such that $n^2 + 1 \equiv 7 \pmod{10}$ while only the numbers $n \equiv 0 \pmod{10}$ are such that $n^2 + 1 \equiv 1 \pmod{10}$, this explains why the digit 7 seems to appear twice as often.

(212) This follows from the fact that, from Theorem 27, we have

$$n! = \prod_{p \leq n} p^{\sum_{j=1}^{\infty} \left[\frac{n}{p^j}\right]} \leq \prod_{p \leq n} p^{n \sum_{j=1}^{\infty} \frac{1}{p^j}} = \prod_{p \leq n} p^{\frac{n}{p-1}},$$

where we used the fact that

$$\frac{1}{p} + \frac{1}{p^2} + \frac{1}{p^3} + \cdots = \frac{1}{p}\left(1 + \frac{1}{p} + \frac{1}{p^2} + \cdots\right) = \frac{1}{p-1}.$$

(213) Every positive integer $n \geq 6$ can be written as $n = 6k$, $n = 6k + 1$, $n = 6k + 2$, $n = 6k + 3$, $n = 6k + 4$ or $n = 6k + 5$, in which case the corresponding values of $n^2 + 2$ are respectively multiples of 2, multiples of 3, multiples of 2, of the form $6K + 5$, multiples of 2 and multiples of 3. It follows that only those $n = 6k + 3$ (with $n^2 + 2 = 6K + 5$) are possible candidates for ensuring that $n^2 + 2$ is prime, thus the result.

(214) Let $N + i$ be one of these numbers. If i is prime, then $i | N + i$ and $N + i$ is composite. While if i is not prime, then i is divisible by a prime number $p_0 < i < p$, in which case $p_0 | N + i$ and $N + i$ is composite.

(215) We write n as

$$n = a_k 10^k + a_{k-1} 10^{k-1} + \cdots + a_2 10^2 + a_1 10 + a_0,$$

where $k \geq 2$ and where the a_i's are integers satisfying $0 \leq a_i \leq 9$, $a_k \neq 0$. Part (a) is trivial. To prove (b), it is enough to observe that, since $4 | 10^j = 5^j \cdot 2^j$ for each integer $j \geq 2$, it follows that $4 | a_1 10 + a_0$ if and only if $4 | n$. To prove (c), it is enough to observe that, since $8 | 10^j = 5^j \cdot 2^j$ for each integer $j \geq 3$, it follows that $8 | a_2 100 + a_1 10 + a_0 \Leftrightarrow 8 | n$. Therefore it becomes clear that one can generalize this result as follows: *If $n > 1$ is an integer having at least k digits, then $2^k | n$ if and only if the number made up of the last k digits of n is divisible by 2^k.*

(216) (Hlawka [**19**]). Let $n > e^e$ and set $f(n) = \displaystyle\sum_{p | n, p > \log n} 1$. It follows that

$$n \geq \prod_{\substack{p | n \\ p > \log n}} p \geq (\log n)^{f(n)}$$

and therefore that $\log n \geq f(n) \log\log n$; that is $f(n) \leq (\log n)/\log\log n$. On the other hand, for n sufficiently large, we have

$$\log\left(1 - \frac{1}{\log n}\right) \geq -\frac{2}{\log n}.$$

It follows that

$$\begin{aligned} 0 \geq \log P(n) &= \sum_{p|n, p>\log n} \log\left(1-\frac{1}{p}\right) \geq f(n)\log\left(1-\frac{1}{\log n}\right) \\ &\geq -2\frac{f(n)}{\log n} \geq -\frac{2}{\log\log n}. \end{aligned}$$

Hence, $\lim_{n\to\infty} \log P(n) = 0$, and the result is proved.

(217) (*MMAG, April 1992, p. 130*). Assume the contrary, that is that each interval $[n^2, (n+1)^2]$ contains less than 1000 prime numbers. We know that the sum of the reciprocals of the prime numbers diverges. Hence, according to our hypothesis, we have

$$\begin{aligned} +\infty = \sum_p \frac{1}{p} &= \sum_{n=1}^{\infty} \sum_{n^2<p\leq(n+1)^2} \frac{1}{p} < \sum_{n=1}^{\infty} \frac{1}{n^2} \sum_{n^2<p\leq(n+1)^2} 1 \\ &= \sum_{n=1}^{\infty} \frac{1}{n^2}\Big(\pi((n+1)^2) - \pi(n^2)\Big) < 1000 \sum_{n=1}^{\infty}\frac{1}{n^2} < +\infty, \end{aligned}$$

a contradiction.

(218) We only need to show that if $x < y$ are any two positive real numbers, then there exist two prime numbers p and q such that $x < p/q < y$. It is obvious that

$$\pi(qy) - \pi(qx) = \pi(qx)\left(\frac{\pi(qy)}{\pi(qx)} - 1\right). \tag{1}$$

On the other hand, using the Prime Number Theorem, we have

$$\lim_{q\to\infty} \frac{\pi(qy)}{\pi(qx)} = \frac{y}{x} > 1. \tag{2}$$

It follows from (1) and (2) that $\lim_{q\to\infty}(\pi(qy) - \pi(qx)) = +\infty$. This means that for q sufficiently large, say $q = q_0$, there exists at least one prime number p_0 such that $q_0 x < p_0 < q_0 y$, in which case we have

$$x < \frac{p_0}{q_0} < y,$$

as required.

(219) Assume that $q_1 < q_2 < \ldots < q_m$ are prime numbers such that

$$\frac{1}{q_1} + \cdots + \frac{1}{q_m} = n$$

for a certain integer $n \geq 1$. Then,

$$\frac{1}{q_1} = n - \frac{1}{q_2} - \cdots - \frac{1}{q_m} = \frac{r}{q_2\cdots q_m},$$

where r is an integer. In this case, the product $q_2 \cdots q_m$ is divisible by q_1, which is impossible.

(220) We have successively

$$\begin{aligned}
\sum_{\substack{n\le x\\ P(n)>\sqrt{x}}} 1 &= \sum_{\sqrt{x}<p\le x}\ \sum_{\substack{n\le x\\ P(n)=p}} 1 = \sum_{\sqrt{x}<p\le x}\ \sum_{\substack{mp\le x\\ P(m)\le p}} 1 = \sum_{\sqrt{x}<p\le x}\ \sum_{\substack{m\le x/p\\ P(m)\le p}} 1\\
&= \sum_{\sqrt{x}<p\le x}\left[\frac{x}{p}\right] = \sum_{\sqrt{x}<p\le x}\frac{x}{p} + \sum_{\sqrt{x}<p\le x}\left(\left[\frac{x}{p}\right]-\frac{x}{p}\right)\\
&= x\log 2 + xR(x) - xR(\sqrt{x}) + S(x),
\end{aligned}$$

say, where $|S(x)| \le \pi(x)$. Relation (3) then follows from (1) and (2). To show the last part, it is sufficient to observe that, since $\log 2 = 0.69\ldots$, then if x is sufficiently large,

$$\frac{1}{x}\sum_{\substack{n\le x\\ P(n)>\sqrt{n}}} 1 > \frac{1}{x}\sum_{\substack{n\le x\\ P(n)>\sqrt{x}}} 1 = \log 2 + T(x) > \frac{2}{3}.$$

(221) Let $p_1, p_2, \ldots, p_r$ be the prime numbers $\le \sqrt{x}$. Then, all odd integers $\le x$ which are not divisible by $p_1, p_2, \ldots, p_r$ are prime numbers. Consequently, $\pi(x) - \pi(\sqrt{x})$ counts the number of prime numbers $> \sqrt{x}$. But the number of positive integers $\le x$ which are divisible by none of the primes $p_1, p_2, \ldots, p_r$ is equal to

$$(*)\qquad [x] - \sum_{1\le i\le r}\left[\frac{x}{p_i}\right] + \sum_{1\le i<j\le r}\left[\frac{x}{p_ip_j}\right] - \sum_{1\le i<j<k\le r}\left[\frac{x}{p_ip_jp_k}\right] + \cdots + (-1)^r\left[\frac{x}{p_1\cdots p_r}\right].$$

Indeed, let n be an integer $\le x$ which is divisible only by the prime numbers $p_1, p_2, \ldots, p_r$; in this case, the sum is equal to

$$1 - \binom{r}{1} + \binom{r}{2} - \cdots + (-1)^r\binom{r}{r} = (1-1)^r = 0,$$

while if n is not divisible by any of the primes $p_1, \ldots, p_r$, then its contribution to the right-hand side is obviously 1.

Remark: Observe that expression $(*)$ can also be written as

$$\sum_{n|p_1p_2\cdots p_r} \mu(n)\left[\frac{x}{n}\right].$$

(222) Let $n > 5$. From Conjecture A, if n is even, there exist two prime numbers p and q such that $n-2 = p+q$, that is $n = p+q+2$; while if n is odd, there exist two prime numbers p and q such that $n-3 = p+q$, so that $n = p+q+3$. In both cases, Conjecture B follows.

Let n be an even integer ≥ 4. From Conjecture B, there exist three prime numbers p, q, r such that $n+2 = p+q+r$. Since $n+2$ is even, it is clear that one of the three prime numbers p, q, r must be even, that is equal to 2. Assume that $r = 2$. It follows that $n = p+q$, which establishes Conjecture A.

(223) (This result is attributed to Mináč; see P. Ribenboim [**29**]). First of all, we observe that if $n \ne 4$ is not a prime number, then $n|(n-1)!$. Indeed, either $n = ab$, with $2 \le a < b \le n-1$, in which case $n|(n-1)!$, or

$n = p^2 \neq 4$, in which case $2 < p \leq n-1 = p^2-1$ and $2p \leq p^2-1$, which implies that $n|2p^2 = p \cdot 2p$, an expression which divides $(n-1)!$.

To prove the stated relation, we analyze separately the cases "j prime" and "j composite".

If j is prime, then by Wilson's Theorem, there exists $k \in \mathbb{N}$ such that $(j-1)!+1 = kj$ so that

$$\left[\frac{(j-1)!+1}{j} - \left[\frac{(j-1)!}{j}\right]\right] = \left[k - \left[k - \frac{1}{j}\right]\right] = [k-(k-1)] = 1.$$

If j is composite, $j \geq 6$, then $j|(j-1)!$ in light of the above observation. Therefore, there exists an integer k such that $(j-1)! = kj$. It follows that

$$\left[\frac{(j-1)!+1}{j} - \left[\frac{(j-1)!}{j}\right]\right] = \left[k + \frac{1}{j} - k\right] = 0.$$

Finally, if $j=4$, we have $\left[\frac{3!+1}{4} - \left[\frac{3!}{4}\right]\right] = 0$, which completes the proof of Mináč's formula.

(224) (a) We have

$$A(n) = \sum_{\substack{m \leq n \\ a|m}} 1 = \sum_{ar \leq n} 1 = \sum_{r \leq \frac{n}{a}} 1 = \left[\frac{n}{a}\right].$$

It is therefore easy to see that the quotient $A(n)/n$ tends to $1/a$ as $n \to \infty$.

(b) We have

$$A(n) = \sum_{\substack{m \leq n \\ a|m}} 1 - \sum_{\substack{m \leq n \\ a|m,\ a_0|m}} 1 = \sum_{ar \leq n} 1 - \sum_{\substack{m \leq n \\ [a,a_0]|m}} 1 = \sum_{r \leq \frac{n}{a}} 1 - \sum_{r \leq \frac{n}{[a,a_0]}} 1 = \left[\frac{n}{a}\right] - \left[\frac{n}{[a,a_0]}\right].$$

It is therefore easy to see that the quotient $A(n)/n$ tends to $\frac{1}{a} - \frac{1}{[a,a_0]}$ as $n \to \infty$.

(c) Using the inclusion-exclusion principle, we have

$$\begin{aligned} n - A(n) &= \sum_{1 \leq i \leq r} \sum_{\substack{m \leq n \\ q_i|m}} 1 - \sum_{1 \leq i < j \leq r} \sum_{\substack{m \leq n \\ q_iq_j|m}} 1 \\ &\quad + \sum_{1 \leq i < j < k \leq r} \sum_{\substack{m \leq n \\ q_iq_jq_k|m}} 1 - \cdots + (-1)^{r+1} \sum_{\substack{m \leq n \\ q_1q_2\cdots q_r|m}} 1 \\ &= \sum_{1 \leq i \leq r} \left[\frac{n}{q_i}\right] - \sum_{1 \leq i < j \leq r} \left[\frac{n}{q_iq_j}\right] + \sum_{1 \leq i < j < k \leq r} \left[\frac{n}{q_iq_jq_r}\right] \\ &\quad - \cdots + (-1)^{r+1} \left[\frac{n}{q_1q_2\cdots q_r}\right]. \end{aligned}$$

It follows that

$$1-\frac{A(n)}{n}=\sum_{1\le i\le r}\frac{1}{n}\left[\frac{n}{q_i}\right]-\sum_{1\le i<j\le r}\frac{1}{n}\left[\frac{n}{q_iq_j}\right]$$
$$+\sum_{1\le i<j<k\le r}\frac{1}{n}\left[\frac{n}{q_iq_jq_r}\right]-\cdots+(-1)^{r+1}\frac{1}{n}\left[\frac{n}{q_1q_2\cdots q_r}\right].$$

But, as $n\to\infty$, this last expression tends to

$$\sum_{1\le i\le r}\frac{1}{q_i}-\sum_{1\le i<j\le r}\frac{1}{q_iq_j}+\sum_{1\le i<j<k\le r}\frac{1}{q_iq_jq_r}-\cdots$$
$$+(-1)^{r+1}\frac{1}{q_1q_2\cdots q_r}=-\prod_{i=1}^{r}\left(1-\frac{1}{q_i}\right)+1,$$

as required.

(225) We will show that

$$\underline{\mathbf{d}}\,\mathcal{A}=\frac{1}{3}<\frac{2}{3}=\overline{\mathbf{d}}\,\mathcal{A}.$$

To do so, we prove that

$$\lim_{k\to\infty}\frac{A(2^{2k+1})}{2^{2k+1}}=\frac{2}{3},\quad\text{while}\quad\lim_{k\to\infty}\frac{A(2^{2k})}{2^{2k}}=\frac{1}{3}.$$

Indeed,

$$\begin{aligned}A(2^{2k+1}) &= \sum_{1\le n<2}1+\sum_{2^2\le n<2^3}1+\sum_{2^4\le n<2^5}1+\cdots+\sum_{2^{2k}\le n<2^{2k+1}}1\\ &= 1+(2^3-2^2)+(2^5-2^4)+\cdots+(2^{2k+1}-2^{2k})\\ &= 1+2^2+2^4+\cdots+2^{2k}\\ &= 1+4+4^2+\cdots+4^k=\frac{4^{k+1}-1}{4-1}=\frac{1}{3}\left(4^{k+1}-1\right).\end{aligned}$$

It follows from this that

$$\lim_{k\to\infty}\frac{A(2^{2k+1})}{2^{2k+1}}=\lim_{k\to\infty}\frac{1}{3}\frac{4^{k+1}-1}{2\cdot4^k}=\frac{2}{3}.$$

On the other hand,

$$A(2^{2k})=\sum_{\substack{n\in\mathcal{A}\\ n\le2^{2k}}}1=\sum_{\substack{n\in\mathcal{A}\\ n<2^{2k-1}}}1=\sum_{\substack{n\in\mathcal{A}\\ n\le2^{2(k-1)+1}}}1=\frac{1}{3}(4^k-1),$$

which implies that

$$\lim_{k\to\infty}\frac{A(2^{2k})}{2^{2k}}=\lim_{k\to\infty}\frac{1}{3}\frac{4^k-1}{4^k}=\frac{1}{3}.$$

(226) To each element $a_i\in\mathcal{A}$, we associate its largest odd divisor d_i. It is clear that all the d_i's are distinct; indeed, if $d_i=d_j$ for two positive integers $i\ne j$, then $a_i|a_j$ or $a_j|a_i$, which is possible only if $i=j$. It follows that $A(2n)\le n$, since there are no more than n odd numbers $\le 2n$. Hence, the result.

(227) To each element $a_i \in \mathcal{A}$, we associate its largest prime factor q_i. Let $B_i = \{n \in \mathbb{N} : p(n) > q_i\}$, where $p(n)$ stands for the smallest prime factor of n. Let also $C_i = a_i B_i = \{a_i \cdot b : b \in B_i\}$. The sets C_i are disjoint; indeed, if $a_i r = a_j s$ (with $q_i \le q_j$), where $r \in B_i$ and $s \in B_j$, then $a_i | a_j$, which is possible only if $i = j$. It follows that, for each positive integer k, we have

$$\sum_{i=1}^{k} \mathbf{d}\, C_i \le 1.$$

From (a) and (c) of Problem 224, we have

$$\mathbf{d}\, C_i = \frac{1}{a_i} \mathbf{d}\, B_i = \frac{1}{a_i} \prod_{p \le q_i} \left(1 - \frac{1}{p}\right),$$

so that

$$\sum_{i=1}^{\infty} \frac{1}{a_i} \prod_{p \le q_i} \left(1 - \frac{1}{p}\right) \le 1. \tag{1}$$

But from Mertens' Theorem, we have that

$$\prod_{p \le q_i} \left(1 - \frac{1}{p}\right) \gg \frac{1}{\log q_i} \ge \frac{1}{\log a_i}. \tag{2}$$

The result then follows by combining (1) and (2).

(228) Part (a) is obvious. To prove (b), first observe that the norm of every element of E is always ≥ 5. Assume that $3 = (a+b\sqrt{-5})(c+d\sqrt{-5})$; taking the norm, we have $9 = (a^2 + 5b^2)(c^2 + 5d^2)$. This is however impossible since both factors on the right-hand side are larger than 5. Hence, 3 is a prime belonging to E. We easily obtain that $29 = (3+2\sqrt{-5})(3-2\sqrt{-5})$ and is therefore a composite number in E. Part (c) follows from the fact that

$$9 = (3 + 0\sqrt{-5}) \cdot (3 + 0\sqrt{-5}) = (2 + \sqrt{-5})(2 - \sqrt{-5}).$$

(229) Since

$$A(n) - A(n-1) = \begin{cases} 1 & \text{if } n \in A, \\ 0 & \text{otherwise,} \end{cases}$$

we have that

$$\begin{aligned} \sum_{\substack{n \le x \\ n \in A}} \frac{1}{n} &= \sum_{2 \le n \le x} \frac{A(n) - A(n-1)}{n} \\ &= \sum_{2 \le n \le x} A(n) \left(\frac{1}{n} - \frac{1}{n+1}\right) + \frac{A(x)}{[x]+1} \\ &= \sum_{n \le x} \frac{A(n)}{n(n+1)} + \frac{A(x)}{[x]+1}. \end{aligned}$$