Calculus of Several Variables

Functions of Several Variables

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Aims and Objectives

- Introduce the notion of a function of two variables
- Graphing functions of two variables: surface
- The notion of a level curve and its applications to economics

Contents

Functions of Two Variables

- If a certain manufacturer determines that
  x units of a particular commodity
  can be sold domestically for 40 € per unit,
  and y units can be sold to foreign markets for 60 € per unit,
- then the total revenue obtained from all the sales is given by

$$R = 40x + 60y$$

Functions of Two Variables. (Continued)

Functions of Two Variables

A *function* f *of two independent variables* x and y
is a rule that assignes to each ordered pair (x, y)
in a given set D (the *domain* of f)
exactly one real number, denoted by $f(x, y)$.

Functions of Two Variables. (Continued)

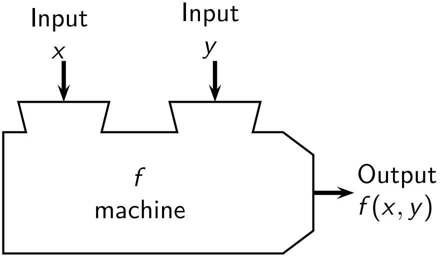

Figure: A function of two variables as a machine

Functions of Two Variables. (Continued)

Example

Suppose

$$f(x, y) = \frac{3x^2 + 5y}{x - y}.$$

1. Find the domain of f.
2. Compute $f(1, -2)$.

Functions of Two Variables. (Continued)

Solution.

1. Since division by any real number except zero is possible, the domain is the set of all ordered pairs (x, y) with $x - y \neq 0$ or $x \neq y$.
   Geometrically, the set of all points in the xy-plane except for those on the line $x = y$.
2. $f(1, -2) = \frac{3 \cdot 1^2 + 5 \cdot (-2)}{1 - (-2)} = -\frac{7}{3}$.

$\square$

Applications

Example

A sports store carries two kinds of tennis rackets,
the Venus Williams and the Martina Hingis autograph brands.
The consumer demand for each depends not only on its own price,
but also on the price of the competing brand.
If the Williams brand sells for x euros per racket
and the Hingis brand for y euros per racket,
the demands for them will be $D_1 = 300 - 20x + 30y$ rackets per year and $D_2 = 200 + 40x - 10y$ rackets per year, respecitvely.
Express the store's total annual revenue from the sale of these as a function of the prices x and y.

Applications. (Continued)

Solution.

$$
\begin{aligned}
R(x, y) &= D_1 x + D_2 y \\
&= (300 - 20x + 30y)x + (200 + 40x - 10y)y \\
&= 300x + 200y + 70xy - 20x^2 - 10y^2.
\end{aligned}
$$

$\square$

Applications. (Continued)

Example

Suppose that at a certain factory, output is given by the *Cobb-Douglas production function* $Q(K, L) = 60K^{1/3}L^{2/3}$ units, where K is the capital investment meassured in units of 1,000 € and L is the size on the labor force meassured in worker-hours.

1. Compute the output if the capital investment is 512,000 € and 1,000 hours of worker-force are used.
2. Show that the output in part 1 will double if both the capital investement and the size of the labor force are doubled.

Applications. (Continued)

Solution.

1. Put $K = 512$ and $L = 1{,}000$:

$$Q(512, 1{,}000) = 60 \cdot 512^{1/3} \cdot 1{,}000^{2/3}$$
$$= 60 \cdot 8 \cdot 100 = 48{,}000$$

units.

2. Evaluate $Q(K, L)$ with $K = 2 \cdot 512$ and $L = 2 \cdot 1{,}000$:

$$Q(2 \cdot 512, 2 \cdot 1{,}000) = 60 \cdot (2 \cdot 512)^{1/3} \cdot (2 \cdot 1{,}000)^{2/3}$$
$$= 60 \cdot 2^{1/3} \cdot 512^{1/3} \cdot 2^{2/3} \cdot 1{,}000^{2/3}$$
$$= 2 \cdot (60 \cdot 512^{1/3} \cdot 1{,}000^{2/3}) = 2Q(512, 1{,}000). \quad \square$$

Graphs of Functions of Two Variables

- The *graph* of a function of two variables $z = f(x, y)$ is the set fo all triples (x, y, z) such that (x, y) is in the domain of the function and $z = f(x, y)$.
- To "picture" such graphs we need a *three-dimensional coordinate system*.

Graphs of Functions of Two Variables. (Continued)

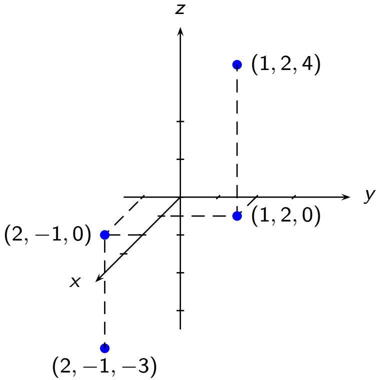

Figure: A three-dimensional coordinate system

Graphs of Functions of Two Variables. (Continued)

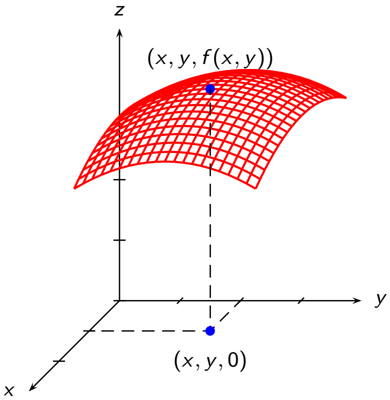

Figure: The graph of $z = f(x, y)$.

Several Surfaces in Three-Dimensional Space

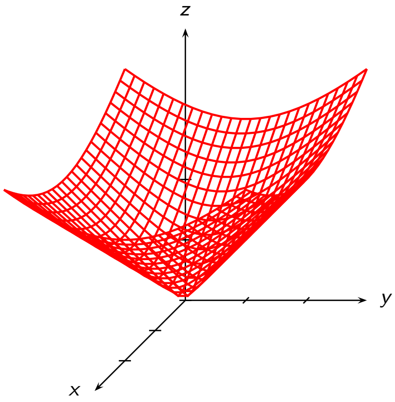

Figure: A cone $z = \sqrt{x^2 + y^2}$.

Several Surfaces in Three-Dimensional Space. (Continued)

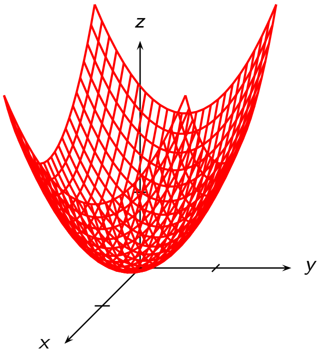

Figure: A paraboloid $z = x^2 + y^2$.

Several Surfaces in Three-Dimensional Space. (Continued)

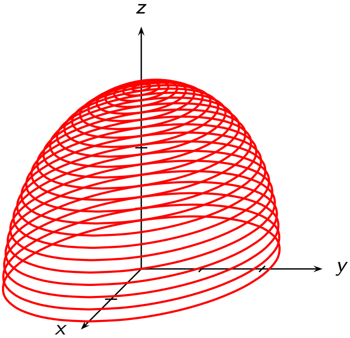

Figure: An ellipsoid $z = \sqrt{9 - 3x^2 - 2y^2}$.

Functions of Two Variables

- When the plane $z = C$ intercepts the surface $z = f(x, y)$, the result is a curve in space.
- The corresponding set of points (x, y) in the xy plane that satisfy $f(x, y) = C$ is called the *level curve* of f at C.
- By sketching members of the family of level curves for different values of C, we obtain a useful representation of the surface $z = f(x, y)$.

The Curve of Constant Product (Isoquant)

- In economics, if the output $Q(x, y)$ of a production process is determined by two inputs x and y, then the level curve $Q(x, y) = C$ is called the *curve of constant product* C (*isoquant*).
- A consumer who is considering a purchase of a number of units of each of the two commodities is associated with a *utility function* $U(x, y)$, which measures the total satisfaction (utility) the consumer derives from having x units of the first and y units of the second commodity.
- A level curve $U(x, y) = C$ of the utility function is called an *indifference curve,* and gives all of the combinations of x and y which lead to the same level of consumer satisfacton.

Indifference Curves

Example

Suppose the utility derived by a consumer from x units of one commodity and y units of a second commodity is given by the utility function $U(x, y) = x^{3/2}y$.
If the consumer currently owns $x = 16$ units of the first and $y = 20$ units of the second commodity, find the consumer's current level of utility and sketch the corresponding indfference curve.

Indifference Curves. (Continued)

Solution.

The current level of utility is

$$U(16,20) = 16^{3/2} \cdot 20 = 1{,}280$$

and the correspondent indifference curve is

$$x^{3/2} y = 1{,}280.$$

The curve $x^{3/2} y = 1{,}280$ and several other curves of the family of indifference curves are shown in the following figure. □

Indifference Curves. (Continued)

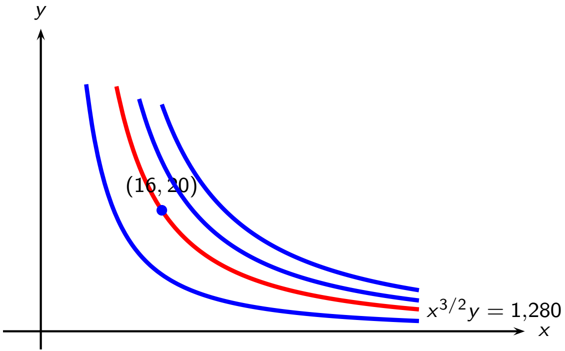

Figure: Indifference curves for the utility function $U(x, y) = x^{3/2}y$.

For Further Reading

- http://fberisha.netfirms.com
- Homework: Exercises from teaching materials
- L. D. Hofmann, G. L. Bradley, *Calculus – for business, economics and life sciences,* pp. 502–519.

Summary

- Functions of two varables: $z = f(x, y)$
- The graph: surface
- Level curve: $f(x, y) = C$
- Applications to economics:
  - Constant-production curve (isoquant)
  - Indifference curve