# 7

# Integrals Over Paths and Surfaces

*I hold in fact: (1) That small portions of space are of a nature analogous to little hills on a surface which is on the average flat. (2) That this property of being curved or distorted is continually passed on from one portion of space to another after the manner of a wave. (3) That this variation of curvature of space is really what happens in that phenomenon which we call the motion of matter whether ponderable or ethereal. (4) That in this physical world nothing else takes place but this variation, subject, possibly, to the law of continuity.*

*W. K. Clifford (1870)*

In Chapter 5, we studied integration over regions in $\mathbb{R}^2$ and $\mathbb{R}^3$. In this chapter, we study integration over paths and surfaces. This is basic to an understanding of Chapter 8, in which we discuss the basic relation between vector differential calculus (Chapter 4) and vector integral calculus (this chapter), a relation that generalizes the fundamental theorem of calculus to several variables. This generalization is summarized in the theorems of Green, Gauss, and Stokes.

## 7.1 The Path Integral

This section introduces the concept of a path integral; this is one of the several ways in which integrals of functions of one variable can be generalized to functions of several variables. Besides those in Chapter 5, there are other generalizations, to be discussed in later sections.

Suppose we are given a scalar function $f: \mathbb{R}^3 \to \mathbb{R}$, so that $f$ sends points in $\mathbb{R}^3$ to real numbers. It will be useful to define the integral of such a function $f$ along a path

$\mathbf{c}\colon I = [a, b] \to \mathbb{R}^3$, where $\mathbf{c}(t) = (x(t), y(t), z(t))$. To relate this notion to something tangible, suppose that the image of $\mathbf{c}$ represents a wire. We can let $f(x, y, z)$ denote the mass density at $(x, y, z)$ and the integral of $f$ will be the total mass of the wire. By letting $f(x, y, z)$ indicate temperature, we can also use the integral to determine the average temperature along the wire. We first give the formal definition of the path integral and then, after the following example, further motivate it.

**DEFINITION: Path Integrals** The ***path integral***, or the ***integral of*** $f(x, y, z)$ ***along the path*** $\mathbf{c}$, is defined when $\mathbf{c}\colon I = [a, b] \to \mathbb{R}^3$ is of class $C^1$ and when the composite function $t \mapsto f(x(t), y(t), z(t))$ is continuous on $I$. We define this integral by the equation

$$\int_{\mathbf{c}} f\, ds = \int_a^b f(x(t), y(t), z(t)) \|\mathbf{c}'(t)\|\, dt.$$

Sometimes $\int_{\mathbf{c}} f\, ds$ is denoted

$$\int_{\mathbf{c}} f(x, y, z)\, ds$$

or

$$\int_a^b f(\mathbf{c}(t)) \|\mathbf{c}'(t)\|\, dt.$$

If $\mathbf{c}(t)$ is only piecewise $C^1$ or $f(\mathbf{c}(t))$ is piecewise continuous, we define $\int_{\mathbf{c}} f\, ds$ by breaking $[a, b]$ into pieces over which $f(\mathbf{c}(t))\|\mathbf{c}'(t)\|$ is continuous, and summing the integrals over the pieces.

When $f = 1$, we recover the definition of the arc length of $\mathbf{c}$. Also note that $f$ need only be defined on the image curve $C$ of $\mathbf{c}$ and not necessarily on the whole space in order for the preceding definition to make sense.

**EXAMPLE 1** Let $\mathbf{c}$ be the helix $\mathbf{c}\colon [0, 2\pi] \to \mathbb{R}^3, t \mapsto (\cos t, \sin t, t)$ (see Figure 2.4.9), and let $f(x, y, z) = x^2 + y^2 + z^2$. Evaluate the integral $\int_{\mathbf{c}} f(x, y, z)\, ds$.

SOLUTION First we compute $\|\mathbf{c}'(t)\|$:

$$\|\mathbf{c}'(t)\| = \sqrt{\left[\frac{d(\cos t)}{dt}\right]^2 + \left[\frac{d(\sin t)}{dt}\right]^2 + \left[\frac{dt}{dt}\right]^2} = \sqrt{\sin^2 t + \cos^2 t + 1} = \sqrt{2}.$$

Next, we substitute for $x$, $y$, and $z$ in terms of $t$ to obtain

$$f(x, y, z) = x^2 + y^2 + z^2 = \cos^2 t + \sin^2 t + t^2 = 1 + t^2$$

along $\mathbf{c}$. Inserting this information into the definition of the path integral yields

$$\int_{\mathbf{c}} f(x,y,z)\,ds = \int_0^{2\pi} (1+t^2)\sqrt{2}\,dt = \sqrt{2}\left[t+\frac{t^3}{3}\right]_0^{2\pi} = \frac{2\sqrt{2}\pi}{3}(3+4\pi^2). \quad \blacktriangle$$

To motivate the definition of the path integral, we shall consider "Riemann-like" sums $S_N$ in the same general way we did to define arc length in Section 4.2. For simplicity, let $\mathbf{c}$ be of class $C^1$ on $I$. Subdivide the interval $I=[a,b]$ by means of a partition

$$a = t_0 < t_1 < \cdots < t_N = b.$$

This leads to a decomposition of $\mathbf{c}$ into paths $\mathbf{c}_i$ (Figure 7.1.1) defined on $[t_i, t_{i+1}]$ for $0 \le i \le N-1$. Denote the arc length of $\mathbf{c}_i$ by $\Delta s_i$; thus,

$$\Delta s_i = \int_{t_i}^{t_{i+1}} \|\mathbf{c}'(t)\|\,dt.$$

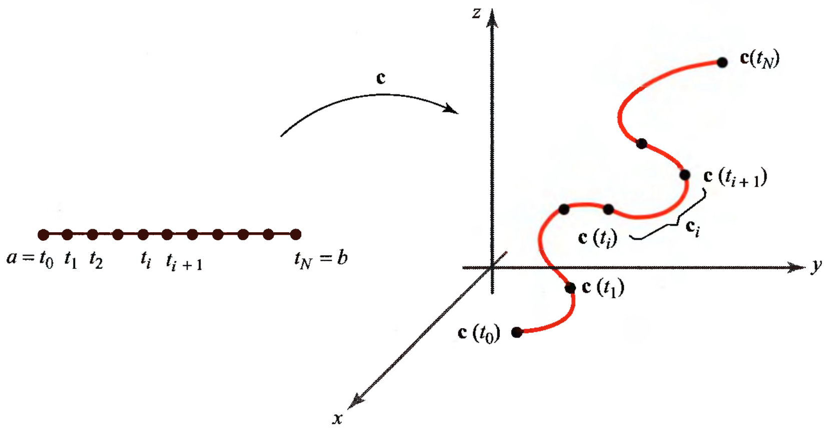



Figure 7.1.1 Breaking $\mathbf{c}$ into smaller $\mathbf{c}_i$.

When $N$ is large, the arc length $\Delta s_i$ is small and $f(x,y,z)$ is approximately constant for points on $\mathbf{c}_i$. We consider the sums

$$S_N = \sum_{i=0}^{N-1} f(x_i, y_i, z_i)\,\Delta s_i,$$

where $(x_i, y_i, z_i) = \mathbf{c}(t)$ for some $t \in [t_i, t_{i+1}]$. By the mean-value theorem we know that $\Delta s_i = \|\mathbf{c}'(t_i^*)\|\Delta t_i$, where $t_i \le t_i^* \le t_{i+1}$ and $\Delta t_i = t_{i+1} - t_i$. From the theory of

Riemann sums, it can be shown that

$$\lim_{N\to\infty} S_N = \lim_{N\to\infty} \sum_{i=0}^{N-1} f(x_i, y_i, z_i)\|\mathbf{c}'(t_i^*)\|\Delta t_i = \int_I f(x(t), y(t), z(t))\|\mathbf{c}'(t)\|\, dt$$
$$= \int_{\mathbf{c}} f(x, y, z)\, ds.$$

## The Path Integral for Planar Curves

An important special case of the path integral occurs when the path $\mathbf{c}$ describes a plane curve. Suppose that all points $\mathbf{c}(t)$ lie in the $xy$ plane and $f$ is a real-valued function of two variables. The path integral of $f$ along $\mathbf{c}$ is

$$\int_{\mathbf{c}} f(x, y)\, ds = \int_a^b f(x(t), y(t))\sqrt{x'(t)^2 + y'(t)^2}\, dt.$$

When $f(x, y) \geq 0$, this integral has a geometric interpretation as the "area of a fence." We can construct a "fence" with base the image of $\mathbf{c}$ and with height $f(x, y)$ at $(x, y)$ (Figure 7.1.2). If $\mathbf{c}$ moves only once along the image of $\mathbf{c}$, the integral $\int_{\mathbf{c}} f(x, y)\, ds$ represents the area of a side of this fence. Readers should try to justify this interpretation for themselves, using an argument like the one used to justify the arc-length formula.

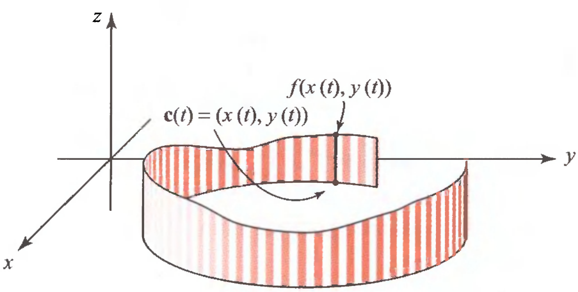



Figure 7.1.2 The path integral as the area of a fence.

**EXAMPLE 2** Tom Sawyer's aunt has asked him to whitewash both sides of the old fence shown in Figure 7.1.3. Tom estimates that for each 25 ft$^2$ of whitewashing he lets someone do for him, the willing victim will pay 5 cents. How much can Tom hope to earn, assuming his aunt will provide whitewash free of charge?

SOLUTION From Figure 7.1.3, the base of the fence in the first quadrant is the path $\mathbf{c}$: $[0, \pi/2] \to \mathbb{R}^2$, $t \mapsto (30\cos^3 t, 30\sin^3 t)$, and the height of the fence at $(x, y)$ is $f(x, y) = 1 + y/3$. The area of one side of the half of the fence is equal to the *integral* $\int_{\mathbf{c}} f(x, y)\, ds = \int_{\mathbf{c}} (1 + y/3)\, ds$. Because $\mathbf{c}'(t) = (-90\cos^2 t \sin t, 90\sin^2 t \cos t)$, we

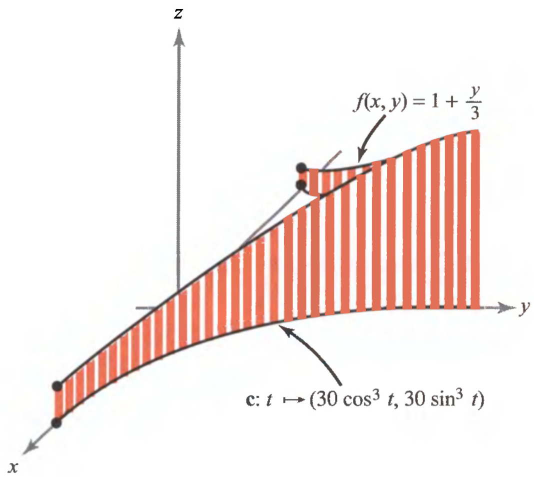



**Figure 7.1.3** Tom Sawyer's fence.

have $\|\mathbf{c}'(t)\| = 90\sin t\cos t$. Thus, the integral is

$$\begin{aligned}\int_{\mathbf{c}}\left(1+\frac{y}{3}\right)ds &= \int_0^{\pi/2}\left(1+\frac{30\sin^3 t}{3}\right)90\sin t\cos t\,dt\\ &= 90\int_0^{\pi/2}(\sin t + 10\sin^4 t)\cos t\,dt\\ &= 90\left[\frac{\sin^2 t}{2} + 2\sin^5 t\right]_0^{\pi/2} = 90\left(\frac{1}{2}+2\right) = 225,\end{aligned}$$

which is the area in the first quadrant. Hence, the area of one side of the fence is 450 ft$^2$. Because both sides are to be whitewashed, we must multiply by 2 to find the total area, which is 900 ft$^2$. Dividing by 25 and then multiplying by 5, we find that Tom could realize as much as \$1.80 for the job. ▲

This concludes our study of integration of *scalar* functions over paths. In the next section we shall turn our attention to the integration of *vector fields* over paths, and we shall see many further applications of the path integral in Chapter 8, when we study vector analysis.

## Supplement to Section 7.1: The Total Curvature of a Curve

Exercises 12 to 17 of Section 4.2 described the notions of curvature $\kappa$ and torsion $\tau$ of a smooth curve $C$ in space. If $\mathbf{c}\colon [a, b] \to C \subset \mathbb{R}^3$ is a unit-speed parametrization of $C$, so that $\|\mathbf{c}'(t)\| = 1$, then the ***curvature*** $\kappa(p)$ at $p \in C$ is defined by $\kappa(p) = \|\mathbf{c}''(t)\|$, where $p = \mathbf{c}(t)$. A result of differential geometry is that two unit-speed curves with the same curvature and torsion can be obtained from one another by a rigid rotation, translation, or reflection.

The curvature $\kappa: C \to \mathbb{R}$ is a real-valued function on the set $C$, so we define the ***total curvature*** as its path integral over $C$: $\int_C \kappa\, ds$. There are some surprising facts that mathematicians have been able to prove about the total curvature. For one thing, if $C$ is a closed [that is, $\mathbf{c}(a) = \mathbf{c}(b)$] planar curve, then

$$\int_C \kappa\, ds \geq 2\pi,$$

and equals $2\pi$ only when $C$ is a circle. If $C$ is a closed space curve with

$$\int_C \kappa\, ds \leq 4\pi,$$

then $C$ is "unknotted"; that is, $C$ can be continuously deformed (without ever intersecting itself) into a planar circle. Therefore, for knotted curves,

$$\int_C \kappa\, ds > 4\pi.$$

See Figure 7.1.4.

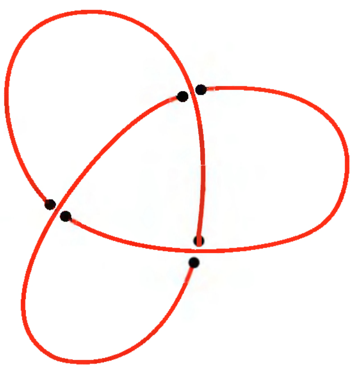


**Figure 7.1.4** A knotted curve in $\mathbb{R}^3$.

The formal statement of this fact is known as the *Fary–Milnor* theorem. Legend has it that John Milnor, a contemporary of John Nash's[1] at Princeton University, was asleep in a math class as the professor wrote three *unsolved* knot theory problems on the blackboard. At the end of the class, Milnor (still an undergraduate) woke up and, thinking the blackboard problems were assigned as homework, quickly wrote

[1]John Nash is the subject of Sylvia Nasar's best-selling biography, *A Beautiful Mind*, a fictionalized version of which was made into a movie in 2001.

them down. The following week he turned in the solution to all three problems—one of which was a proof of the Fary–Milnor theorem! Some years later, he was appointed a professor at Princeton, and in 1962 he was awarded (albeit for other work) a Fields medal, mathematics' highest honor, generally regarded as the mathematical Nobel Prize.

## EXERCISES

**1.** Let $f(x, y, z) = y$ and $\mathbf{c}(t) = (0, 0, t)$, $0 \leq t \leq 1$. Prove that $\int_{\mathbf{c}} f\, ds = 0$.

**2.** Evaluate the following path integrals $\int_{\mathbf{c}} f(x, y, z)\, ds$, where

(a) $f(x, y, z) = x + y + z$ and $\mathbf{c}$: $t \mapsto (\sin t, \cos t, t)$, $t \in [0, 2\pi]$
(b) $f(x, y, z) = \cos z$, $\mathbf{c}$ as in part (a)

**3.** Evaluate the following path integrals $\int_{\mathbf{c}} f(x, y, z)\, ds$, where

(a) $f(x, y, z) = \exp \sqrt{z}$, and $\mathbf{c}$: $t \mapsto (1, 2, t^2)$, $t \in [0, 1]$
(b) $f(x, y, z) = yz$, and $\mathbf{c}$: $t \mapsto (t, 3t, 2t)$, $t \in [1, 3]$

**4.** Evaluate the integral of $f(x, y, z)$ along the path $\mathbf{c}$, where

(a) $f(x, y, z) = x \cos z$, $\mathbf{c}$: $t \mapsto t\mathbf{i} + t^2\mathbf{j}$, $t \in [0, 1]$
(b) $f(x, y, z) = (x + y)/(y + z)$, and $\mathbf{c}$: $t \mapsto \left(t, \frac{2}{3}t^{3/2}, t\right)$, $t \in [1, 2]$

**5.** Let $f$: $\mathbb{R}^3 \backslash \{xz \text{ plane}\} \to \mathbb{R}$ be defined by $f(x, y, z) = 1/y^3$. Evaluate $\int_{\mathbf{c}} f(x, y, z)\, ds$, where $\mathbf{c}$: $[1, e] \to \mathbb{R}^3$ is given by $\mathbf{c}(t) = (\log t)\mathbf{i} + t\mathbf{j} + 2\mathbf{k}$.

**6.** (a) Show that the path integral of $f(x, y)$ along a path given in polar coordinates by $r = r(\theta)$, $\theta_1 \leq \theta \leq \theta_2$, is

$$\int_{\theta_1}^{\theta_2} f(r\cos\theta, r\sin\theta)\sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta.$$

(b) Compute the arc length of the path $r = 1 + \cos\theta$, $0 \leq \theta \leq 2\pi$.

**7.** Let $f(x, y) = 2x - y$, and consider the path $x = t^4$, $y = t^4$, $-1 \leq t \leq 1$.

(a) Compute the integral of $f$ along this path and interpret the answer geometrically.
(b) Evaluate the arc-length function $s(t)$ and redo part (a) in terms of $s$ (you may wish to consult Exercise 2, Section 4.2).

*Exercises 8 to 11 are concerned with the application of the path integral to the problem of defining the average value of a scalar function along a path. Define the number*

$$\frac{\int_{\mathbf{c}} f(x, y, z)\, ds}{l(\mathbf{c})}$$

*to be the **average value** of $f$ along $\mathbf{c}$. Here $l(\mathbf{c})$ is the length of the path:*

$$l(\mathbf{c}) = \int_{\mathbf{c}} \|\mathbf{c}'(t)\|\, dt.$$

*(This is analogous to the average of a function over a region defined in Section 6.3.)*

**8.** (a) Justify the formula $[\int_{\mathbf{c}} f(x, y, z)\, ds]/l(\mathbf{c})$ for the average value of $f$ along $\mathbf{c}$ using Riemann sums.

(b) Show that the average value of $f$ along $\mathbf{c}$ in Example 1 is $(1 + \frac{4}{3}\pi^2)$.

(c) In Exercise 2(a) and (b) above, find the average value of $f$ over the given curves.

**9.** Find the average $y$ coordinate of the points on the semicircle parametrized by $\mathbf{c}\colon [0, \pi] \to \mathbb{R}^3, \theta \mapsto (0, a\sin\theta, a\cos\theta); a > 0$.

**10.** Suppose the semicircle in Exercise 9 is made of a wire with a uniform density of 2 grams per unit length.

(a) What is the total mass of the wire?

(b) Where is the center of mass of this configuration of wire? (Consult Section 6.3.)

**11.** Let $\mathbf{c}$ be the path given by $\mathbf{c}(t) = (t^2, t, 3)$ for $t \in [0, 1]$.

(a) Find $l(\mathbf{c})$, the length of the path.

(b) Find the average $y$ coordinate along the path $\mathbf{c}$.

**12.** If $f\colon [a, b] \to \mathbb{R}$ is piecewise continuously differentiable, let the *length of the graph* of $f$ on $[a, b]$ be defined as the length of the path $t \mapsto (t, f(t))$ for $t \in [a, b]$.

(a) Show that the length of the graph of $f$ on $[a, b]$ is

$$\int_a^b \sqrt{1 + [f'(x)]^2}\, dx.$$

(b) Find the length of the graph of $y = \log x$ from $x = 1$ to $x = 2$.

**13.** Find the mass of a wire formed by the intersection of the sphere $x^2 + y^2 + z^2 = 1$ and the plane $x + y + z = 0$ if the density at $(x, y, z)$ is given by $\rho(x, y, z) = x^2$ grams per unit length of wire.

**14.** Evaluate $\int_{\mathbf{c}} f\, ds$ where $f(x, y, z) = z$ and $\mathbf{c}(t) = (t\cos t, t\sin t, t)$ for $0 \le t \le t_0$.

**15.** Write the following limit as a path integral of $f(x, y, z) = xy$ over some path $\mathbf{c}$ on $[0, 1]$ and evaluate:

$$\operatorname*{limit}_{N\to\infty} \sum_{i=1}^{N-1} t_i^2\left(t_{i+1}^2 - t_i^2\right),$$

where $t_1, \ldots, t_N$ is a partition of $[0, 1]$.

**16.** Consider paths that connect the points $A = (0, 1)$ and $B = (1, 0)$ in the $xy$ plane, as in Figure 7.1.5.[2]

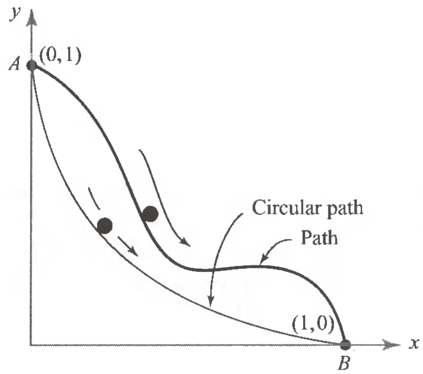



**Figure 7.1.5** A curve joining the points $A$ and $B$.

Galileo contemplated the following question: does a bead falling under the influence of gravity from a point $A$ to a point $B$ along a curve do so in *the least possible time* if that curve is a circular arc? For any given path, the time of transit $T$ is a path integral

$$T = \int \frac{dt}{v},$$

where the bead's velocity is $v = \sqrt{2gy}$, where $g$ is the gravitational constant. In 1697, Johann Bernoulli challenged the mathematical world to find the path in which the bead would roll from $A$ to $B$ in the least time. This solution would determine whether Galileo's considerations had been correct.

(a) Calculate $T$ for the straight-line path $y = 1 - x$.
(b) Write a formula for $T$ for Galileo's circular path, given by $(x - 1)^2 + (y - 1)^2 = 1$.

Incidentally, Newton was the first to send his solution [which turned out to be a cycloid—the same curve (inverted) that we studied in Example 2.4.4], but he did so anonymously. Bernoulli was not fooled, however. When he received the solution, he immediately knew its author, exclaiming, "I know the Lion from his paw." While the solution of this problem is a cycloid, it is known in the literature as the *brachistrochrone*. This was the beginning of the important field called the *calculus of variations*.

# 7.2 Line Integrals

We now consider the problem of integrating a *vector field* along a path. We will begin by considering the notion of *work* to motivate the general definition.

## Work Done by Force Fields

If **F** is a force field in space, then a test particle (for example, a small unit charge in an electric force field or a unit mass in a gravitational field) will experience the force **F**.

[2] We thank Tanya Leise for suggesting this exercise.

Suppose the particle moves along the image of a path **c** while being acted upon by **F**. A fundamental concept is the *work done* by **F** on the particle as it traces out the path **c**. If **c** is a straight-line displacement given by the vector **d** and if **F** is a constant force, then the work done by **F** in moving the particle along the path is the dot product $\mathbf{F} \cdot \mathbf{d}$:

$$\mathbf{F} \cdot \mathbf{d} = (\text{magnitude of force}) \times (\text{displacement in direction of force}).$$

If the path is curved, we can imagine that it is made up of a succession of infinitesimal straight-line displacements or that it is *approximated* by a finite number of straight-line displacements. Then (as in our derivation of the formulas for the path integral in the preceding section) we are led to the following formula for the work done by the force field **F** on a particle moving along a path $\mathbf{c}\colon [a, b] \to \mathbb{R}^3$:

$$\text{work done by } \mathbf{F} = \int_a^b \mathbf{F}(\mathbf{c}(t)) \cdot \mathbf{c}'(t)\, dt.$$

We can further justify this derivation as follows. As $t$ ranges over a small interval $t$ to $t + \Delta t$, the particle moves from $\mathbf{c}(t)$ to $\mathbf{c}(t + \Delta t)$, a vector displacement of $\Delta\mathbf{s} = \mathbf{c}(t + \Delta t) - \mathbf{c}(t)$ (see Figure 7.2.1).

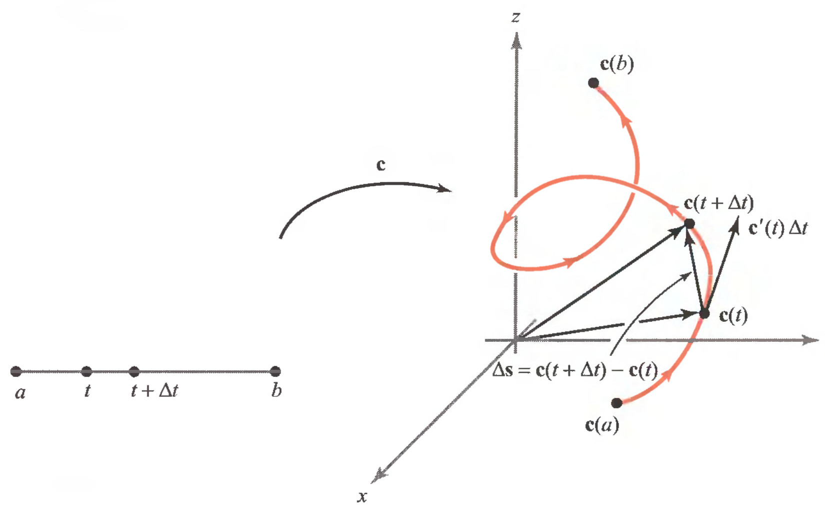



Figure 7.2.1 For small $\Delta t$, $\Delta\mathbf{s} = \mathbf{c}(t + \Delta t) - \mathbf{c}(t) \approx \mathbf{c}'(t)\Delta t$.

From the definition of the derivative, we get the approximation $\Delta\mathbf{s} \approx \mathbf{c}'(t)\Delta t$. The work done in going from $\mathbf{c}(t)$ to $\mathbf{c}(t + \Delta t)$ is therefore approximately

$$\mathbf{F}(\mathbf{c}(t)) \cdot \Delta\mathbf{s} \approx \mathbf{F}(\mathbf{c}(t)) \cdot \mathbf{c}'(t)\Delta t.$$

If we subdivide the interval $[a, b]$ into $n$ equal parts $a = t_0 < t_1 < \cdots < t_n = b$, with $\Delta t = t_{i+1} - t_i$, then the work done by $\mathbf{F}$ is approximately

$$\sum_{i=0}^{n-1} \mathbf{F}(\mathbf{c}(t_i)) \cdot \Delta \mathbf{s} \approx \sum_{i=0}^{n-1} \mathbf{F}(\mathbf{c}(t_i)) \cdot \mathbf{c}'(t_i)\Delta t.$$

As $n \to \infty$, this approximation becomes better and better, and so it is reasonable to take as our definition of work to be the limit of the sum just given as $n \to \infty$. This limit is given by the integral

$$\int_a^b \mathbf{F}(\mathbf{c}(t)) \cdot \mathbf{c}'(t)\, dt.$$

## Definition of the Line Integral

The previous discussion of work motivates the following definition.

**DEFINITION: Line Integrals** Let $\mathbf{F}$ be a vector field on $\mathbb{R}^3$ that is continuous on the $C^1$ path $\mathbf{c}\colon [a, b] \to \mathbb{R}^3$. We define $\int_{\mathbf{c}} \mathbf{F} \cdot d\mathbf{s}$, the ***line integral*** of $\mathbf{F}$ along $\mathbf{c}$, by the formula

$$\int_{\mathbf{c}} \mathbf{F} \cdot d\mathbf{s} = \int_a^b \mathbf{F}(\mathbf{c}(t)) \cdot \mathbf{c}'(t)\, dt;$$

that is, we integrate the dot product of $\mathbf{F}$ with $\mathbf{c}'$ over the interval $[a, b]$.

As is the case with scalar functions, we can also define $\int_{\mathbf{c}} \mathbf{F} \cdot d\mathbf{s}$ if $\mathbf{F}(\mathbf{c}(t)) \cdot \mathbf{c}'(t)$ is only piecewise continuous.

For paths $\mathbf{c}$ that satisfy $\mathbf{c}'(t) \neq \mathbf{0}$, there is another useful formula for the line integral: Namely, if $\mathbf{T}(t) = \mathbf{c}'(t)/\|\mathbf{c}'(t)\|$ denotes the unit tangent vector, we have

$$\begin{aligned}
\int \mathbf{F} \cdot d\mathbf{s} &= \int_a^b \mathbf{F}(\mathbf{c}(t)) \cdot \mathbf{c}'(t)\, dt && \text{(by definition)} \\
&= \int_a^b \left[\mathbf{F}(\mathbf{c}(t)) \cdot \frac{\mathbf{c}'(t)}{\|\mathbf{c}'(t)\|}\right] \|\mathbf{c}'(t)\|\, dt && \text{(canceling } \|\mathbf{c}'(t)\|) \qquad (1) \\
&= \int_a^b [\mathbf{F}(\mathbf{c}(t)) \cdot \mathbf{T}(t)] \|\mathbf{c}'(t)\|\, dt.
\end{aligned}$$

This formula says that $\int_{\mathbf{c}} \mathbf{F} \cdot d\mathbf{s}$ is equal to something that looks like the path integral of the tangential component $\mathbf{F}(\mathbf{c}(t)) \cdot \mathbf{T}(t)$ of $\mathbf{F}$ along $\mathbf{c}$. In fact, the last part of formula (1) is analogous to the path integral of a scalar function $f$ along $\mathbf{c}$.[3]

To compute a line integral in any particular case, one can either use the original definition or integrate the tangential component of $\mathbf{F}$ along $\mathbf{c}$, as prescribed by formula (1), whichever is easier or more appropriate.

**EXAMPLE 1** Let $\mathbf{c}(t) = (\sin t, \cos t, t)$ with $0 \leq t \leq 2\pi$. Let the vector field $\mathbf{F}$ be defined by $\mathbf{F}(x, y, z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$. Compute $\int_{\mathbf{c}} \mathbf{F} \cdot d\mathbf{s}$.

SOLUTION Here, $\mathbf{F}(\mathbf{c}(t)) = \mathbf{F}(\sin t, \cos t, t) = (\sin t)\mathbf{i} + (\cos t)\mathbf{j} + t\mathbf{k}$, and $\mathbf{c}'(t) = (\cos t)\mathbf{i} - (\sin t)\mathbf{j} + \mathbf{k}$. Therefore,

$$\mathbf{F}(\mathbf{c}(t)) \cdot \mathbf{c}'(t) = \sin t \cos t - \cos t \sin t + t = t,$$

and so

$$\int_{\mathbf{c}} \mathbf{F} \cdot d\mathbf{s} = \int_0^{2\pi} t\, dt = 2\pi^2. \quad \blacktriangle$$

Another common way of writing line integrals is

$$\int_{\mathbf{c}} \mathbf{F} \cdot d\mathbf{s} = \int_{\mathbf{c}} F_1\, dx + F_2\, dy + F_3\, dz,$$

where $F_1$, $F_2$, and $F_3$ are the components of the vector field $\mathbf{F}$. We call the expression $F_1\, dx + F_2\, dy + F_3\, dz$ a ***differential form***.[4] By *definition*, the integral of a differential form along a path $\mathbf{c}$, where $\mathbf{c}(t) = (x(t), y(t), z(t))$, is

$$\int_{\mathbf{c}} F_1\, dx + F_2\, dy + F_3\, dz = \int_a^b \left( F_1 \frac{dx}{dt} + F_2 \frac{dy}{dt} + F_3 \frac{dz}{dt} \right) dt = \int_{\mathbf{c}} \mathbf{F} \cdot d\mathbf{s}.$$

Note that we may think of $d\mathbf{s}$ as the differential form $d\mathbf{s} = dx\mathbf{i} + dy\mathbf{j} + dz\mathbf{k}$. Thus, the differential form $F_1\, dx + F_2\, dy + F_3\, dz$ may be written as the dot product $\mathbf{F} \cdot d\mathbf{s}$.

**EXAMPLE 2** Evaluate the line integral

$$\int_{\mathbf{c}} x^2\, dx + xy\, dy + dz,$$

where $\mathbf{c}\colon [0, 1] \to \mathbb{R}^3$ is given by $\mathbf{c}(t) = (t, t^2, 1) = (x(t), y(t), z(t))$.

---

[3] If $\mathbf{c}$ does not intersect itself (that is, if $\mathbf{c}(t_1) = \mathbf{c}(t_2)$ implies $t_1 = t_2$), then each point P on $C$ (the image curve of $\mathbf{c}$) can be written uniquely as $\mathbf{c}(t)$ for some $t$. If we define $f(\mathrm{P}) = f(\mathbf{c}(t)) = \mathbf{F}(\mathbf{c}) \cdot \mathbf{T}(t)$, $f$ is a function on $C$; by definition, its path integral along $\mathbf{c}$ is given by formula (1) and there is no difficulty in literally interpreting $\int_{\mathbf{c}} \mathbf{F} \cdot d\mathbf{s}$ as a path integral. If $\mathbf{c}$ intersects itself, we cannot define $f$ as a function on $C$ as before (why?); however, in this case it is still useful to think of the right side of formula (1) as a path integral.

[4] See Section 8.6 for a brief discussion of the general theory of differential forms.

SOLUTION We compute $dx/dt = 1$, $dy/dt = 2t$, $dz/dt = 0$; therefore,

$$\int_{\mathbf{c}} x^2\,dx + xy\,dy + dz = \int_0^1 \left([x(t)]^2 \frac{dx}{dt} + [x(t)y(t)]\frac{dy}{dt}\right) dt$$

$$= \int_0^1 (t^2 + 2t^4)\,dt = \left[\frac{1}{3}t^3 + \frac{2}{5}t^5\right]_0^1 = \frac{11}{15}. \quad \blacktriangle$$

**EXAMPLE 3** Evaluate the line integral

$$\int_{\mathbf{c}} \cos z\,dx + e^x\,dy + e^y\,dz,$$

where the path $\mathbf{c}$ is defined by $\mathbf{c}(t) = (1, t, e^t)$ and $0 \le t \le 2$.

SOLUTION We compute $dx/dt = 0$, $dy/dt = 1$, $dz/dt = e^t$, and so

$$\int_{\mathbf{c}} \cos z\,dx + e^x\,dy + e^y\,dz = \int_0^2 (0 + e + e^{2t})\,dt$$

$$= \left[et + \frac{1}{2}e^{2t}\right]_0^2 = 2e + \frac{1}{2}e^4 - \frac{1}{2}. \quad \blacktriangle$$

**EXAMPLE 4** Let $\mathbf{c}$ be the path

$$x = \cos^3\theta, \qquad y = \sin^3\theta, \qquad z = \theta, \qquad 0 \le \theta \le \frac{7\pi}{2}$$

(see Figure 7.2.2). Evaluate the integral $\int_{\mathbf{c}}(\sin z\,dx + \cos z\,dy - (xy)^{1/3}\,dz)$.

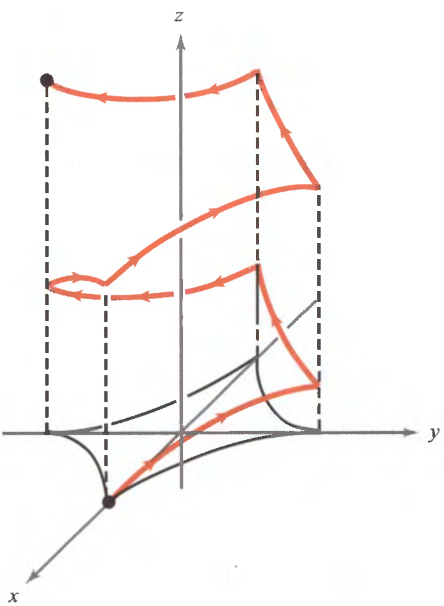



Figure 7.2.2 The image of the path $x = \cos^3\theta, y = \sin^3\theta, z = \theta; 0 \le \theta \le 7\pi/2$.

SOLUTION In this case, we have

$$\frac{dx}{d\theta} = -3\cos^2\theta\sin\theta, \qquad \frac{dy}{d\theta} = 3\sin^2\theta\cos\theta, \qquad \frac{dz}{d\theta} = 1,$$

so the integral is

$$\int_{\mathbf{c}} \sin z\,dx + \cos z\,dy - (xy)^{1/3}\,dz$$

$$= \int_0^{7\pi/2} (-3\cos^2\theta\sin^2\theta + 3\sin^2\theta\cos^2\theta - \cos\theta\sin\theta)\,d\theta.$$

The first two terms cancel, and so we get

$$-\int_0^{7\pi/2} \cos\theta\sin\theta\,d\theta = -\left[\frac{1}{2}\sin^2\theta\right]_0^{7\pi/2} = -\frac{1}{2}. \quad ▲$$

**EXAMPLE 5** Suppose $\mathbf{F}$ is the force vector field $\mathbf{F}(x, y, z) = x^3\mathbf{i} + y\mathbf{j} + z\mathbf{k}$. Parametrize the circle of radius $a$ in the $yz$ plane by letting $\mathbf{c}(\theta)$ have components

$$x = 0, \qquad y = a\cos\theta, \qquad z = a\sin\theta, \qquad 0 \le \theta \le 2\pi.$$

Because $\mathbf{F}(\mathbf{c}(\theta)) \cdot \mathbf{c}'(\theta) = 0$, the force field $\mathbf{F}$ is normal to the circle at every point on the circle, so $\mathbf{F}$ will not do any work on a particle moving along the circle (Figure 7.2.3).

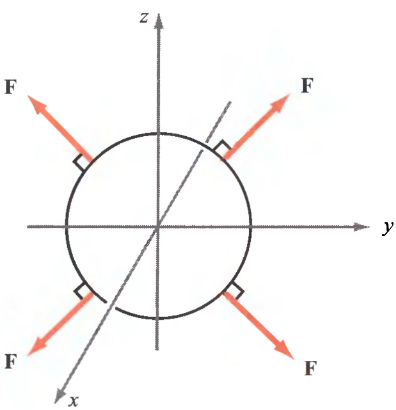



Figure 7.2.3 A vector field $\mathbf{F}$ normal to a circle in the $yz$ plane.

We can verify by direct computation that the work done by $\mathbf{F}$ is zero:

$$W = \int_{\mathbf{c}} \mathbf{F} \cdot d\mathbf{s} = \int_{\mathbf{c}} x^3\,dx + y\,dy + z\,dz$$

$$= \int_0^{2\pi} (0 - a^2\cos\theta\sin\theta + a^2\cos\theta\sin\theta)\,d\theta = 0. \quad ▲$$

**EXAMPLE 6** If we consider the field and curve of Example 4, we see that the work done by the field is $-\frac{1}{2}$, a negative quantity. This means that the field impedes movement along the path. ▲

## Reparametrizations

The line integral $\int_{\mathbf{c}} \mathbf{F} \cdot d\mathbf{s}$ depends not only on the field $\mathbf{F}$ but also on the path $\mathbf{c}\colon [a, b] \to \mathbb{R}^3$. In general, if $\mathbf{c}_1$ and $\mathbf{c}_2$ are two different paths in $\mathbb{R}^3$, $\int_{\mathbf{c}_1} \mathbf{F} \cdot d\mathbf{s} \neq \int_{\mathbf{c}_2} \mathbf{F} \cdot d\mathbf{s}$. On the other hand, we shall see that it is true that $\int_{\mathbf{c}_1} \mathbf{F} \cdot d\mathbf{s} = \pm \int_{\mathbf{c}_2} \mathbf{F} \cdot d\mathbf{s}$ for every vector field $\mathbf{F}$ if $\mathbf{c}_1$ is what we call a ***reparametrization*** of $\mathbf{c}_2$; roughly speaking, this means that $\mathbf{c}_1$ and $\mathbf{c}_2$ are different descriptions of the same geometric curve.

**DEFINITION** Let $h\colon I \to I_1$ be a $C^1$ real-valued function that is a one-to-one map of an interval $I = [a, b]$ onto another interval $I_1 = [a_1, b_1]$. Let $\mathbf{c}\colon I_1 \to \mathbb{R}^3$ be a piecewise $C^1$ path. Then we call the composition

$$\mathbf{p} = \mathbf{c} \circ h\colon I \to \mathbb{R}^3$$

a ***reparametrization*** of $\mathbf{c}$.

This means that $\mathbf{p}(t) = \mathbf{c}(h(t))$, and so $h$ changes the variable; alternatively, one can think of $h$ as changing the speed at which a point moves along the path. Indeed, observe that $\mathbf{p}'(t) = \mathbf{c}'(h(t))h'(t)$, so that the velocity vector for $\mathbf{p}$ equals that for $\mathbf{c}$ but is multiplied by the scalar factor $h'(t)$.

It is implicit in the definition that $h$ must carry endpoints to endpoints; that is, either $h(a) = a_1$ and $h(b) = b_1$, or $h(a) = b_1$ and $h(b) = a_1$. We thus distinguish two types of reparametrizations. If $\mathbf{c} \circ h$ is a reparametrization of $\mathbf{c}$, then either

$$(\mathbf{c} \circ h)(a) = \mathbf{c}(a_1) \qquad \text{and} \qquad (\mathbf{c} \circ h)(b) = \mathbf{c}(b_1)$$

or

$$(\mathbf{c} \circ h)(a) = \mathbf{c}(b_1) \qquad \text{and} \qquad (\mathbf{c} \circ h)(b) = \mathbf{c}(a_1).$$

In the first case, the reparametrization is said to be ***orientation-preserving***, and a particle tracing the path $\mathbf{c} \circ h$ moves in the *same direction* as a particle tracing $\mathbf{c}$. In the second case, the reparametrization is described as ***orientation-reversing***, and a particle tracing the path $\mathbf{c} \circ h$ moves in the *opposite direction* to that of a particle tracing $\mathbf{c}$ (Figure 7.2.4).

For example, if $C$ is the image of a path $\mathbf{c}$, as shown in Figure 7.2.5, that is, $C = \mathbf{c}([a_1, b_1])$, and if $h$ is orientation-preserving, then $\mathbf{c} \circ h(t)$ will go from $\mathbf{c}(a_1)$ to $\mathbf{c}(b_1)$ as $t$ goes from $a$ to $b$; and if $h$ is orientation-reversing, $\mathbf{c} \circ h(t)$ will go from $\mathbf{c}(b_1)$ to $\mathbf{c}(a_1)$ as $t$ goes from $a$ to $b$.

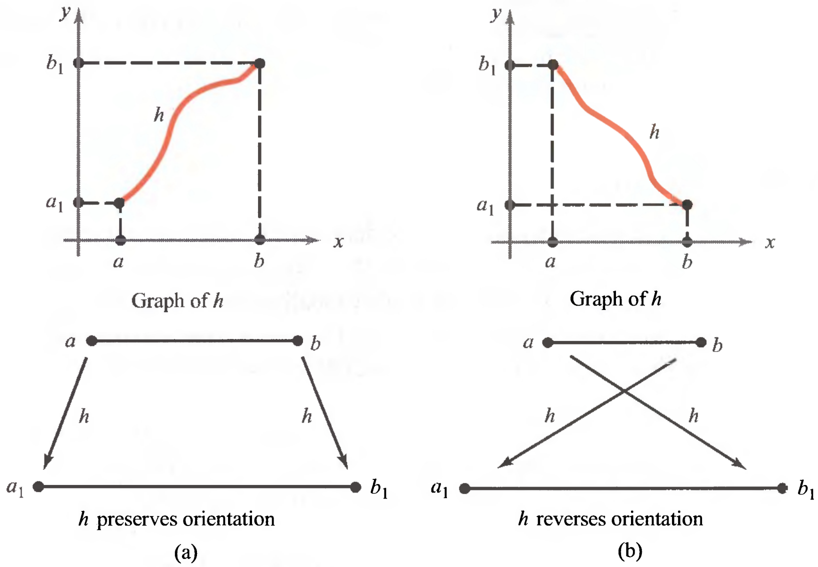



**Figure 7.2.4** Illustrating (a) an orientation-preserving reparametrization, and (b) an orientation-reversing reparametrization.

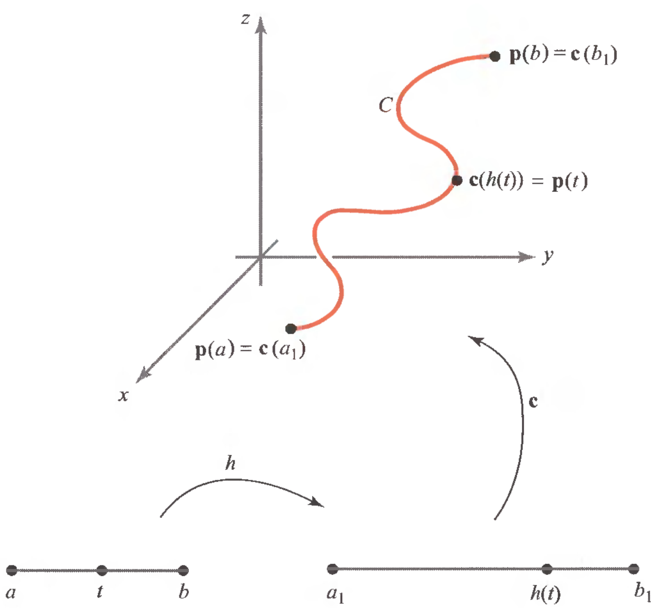



**Figure 7.2.5** The path $\mathbf{p} = \mathbf{c} \circ h$ is a reparametrization of $\mathbf{c}$.

**EXAMPLE 7** Let $\mathbf{c}: [a, b] \to \mathbb{R}^3$ be a piecewise $C^1$ path. Then:

(a) The path $\mathbf{c}_{\mathrm{op}}: [a, b] \to \mathbb{R}^3, t \mapsto \mathbf{c}(a + b - t)$, is reparametrization of $\mathbf{c}$ corresponding to the map $h: [a, b] \to [a, b], t \mapsto a + b - t$; we call $\mathbf{c}_{\mathrm{op}}$ the ***opposite path*** to $\mathbf{c}$. This reparametrization is orientation-reversing.

(b) The path $\mathbf{p}: [0, 1] \to \mathbb{R}^3, t \mapsto \mathbf{c}(a + (b - a)t)$, is an orientation-preserving reparametrization of $\mathbf{c}$ corresponding to a change of coordinates $h: [0, 1] \to [a, b], t \mapsto a + (b - a)t$. ▲

**THEOREM 1: Change of Parametrization for Line Integrals** Let $\mathbf{F}$ be a vector field continuous on the $C^1$ path $\mathbf{c}: [a_1, b_1] \to \mathbb{R}^3$, and let $\mathbf{p}: [a, b] \to \mathbb{R}^3$ be a reparametrization of $\mathbf{c}$. If $\mathbf{p}$ is orientation-preserving, then

$$\int_{\mathbf{p}} \mathbf{F} \cdot d\mathbf{s} = \int_{\mathbf{c}} \mathbf{F} \cdot d\mathbf{s},$$

and if $\mathbf{p}$ is orientation-reversing, then

$$\int_{\mathbf{p}} \mathbf{F} \cdot d\mathbf{s} = -\int_{\mathbf{c}} \mathbf{F} \cdot d\mathbf{s}.$$

*PROOF* By hypothesis, we have a map $h$ such that $\mathbf{p} = \mathbf{c} \circ h$. By the chain rule,

$$\mathbf{p}'(t) = \mathbf{c}'(h(t))h'(t),$$

and so

$$\int_{\mathbf{p}} \mathbf{F} \cdot d\mathbf{s} = \int_a^b [\mathbf{F}(\mathbf{c}(h(t))) \cdot \mathbf{c}'(h(t))]h'(t)\, dt.$$

Changing variables with $s = h(t)$, this becomes

$$\int_{h(a)}^{h(b)} \mathbf{F}(\mathbf{c}(s)) \cdot \mathbf{c}'(s)\, ds$$

$$= \begin{cases} \displaystyle\int_{a_1}^{b_1} \mathbf{F}(\mathbf{c}(s)) \cdot \mathbf{c}'(s)\, ds = \int_{\mathbf{c}} \mathbf{F} \cdot d\mathbf{s} & \text{if } \mathbf{p} \text{ is orientation-preserving} \\ \displaystyle\int_{b_1}^{a_1} \mathbf{F}(\mathbf{c}(s)) \cdot \mathbf{c}'(s)\, ds = -\int_{\mathbf{c}} \mathbf{F} \cdot d\mathbf{s} & \text{if } \mathbf{p} \text{ is orientation-reversing.} \end{cases}$$ ■

Theorem 1 also holds for piecewise $C^1$ paths, as may be seen by breaking up the intervals into segments on which the paths are of class $C^1$ and summing the integrals over the separate intervals.

Thus, if it is convenient to reparametrize a path when evaluating an integral, Theorem 1 assures us that the value of the integral will not be affected, except possibly for the sign, depending on the orientation.

**EXAMPLE 8** Let $\mathbf{F}(x, y, z) = yz\mathbf{i} + xz\mathbf{j} + xy\mathbf{k}$ and $\mathbf{c}\colon [-5, 10] \to \mathbb{R}^3$ be defined by $t \mapsto (t, t^2, t^3)$. Evaluate $\int_{\mathbf{c}} \mathbf{F} \cdot d\mathbf{s}$ and $\int_{\mathbf{c}_{\text{op}}} \mathbf{F} \cdot d\mathbf{s}$.

SOLUTION For the path $\mathbf{c}$, we have $dx/dt = 1$, $dy/dt = 2t$, $dz/dt = 3t^2$, and $\mathbf{F}(\mathbf{c}(t)) = t^5\mathbf{i} + t^4\mathbf{j} + t^3\mathbf{k}$. Therefore,

$$\int_{\mathbf{c}} \mathbf{F} \cdot d\mathbf{s} = \int_{-5}^{10} \left( F_1 \frac{dx}{dt} + F_2 \frac{dy}{dt} + F_3 \frac{dz}{dt} \right) dt = \int_{-5}^{10} (t^5 + 2t^5 + 3t^5)\, dt = [t^6]_{-5}^{10} = 984,375.$$

On the other hand, for

$$\mathbf{c}_{\text{op}}\colon [-5, 10] \to \mathbb{R}^3, t \mapsto \mathbf{c}(5 - t) = (5 - t, (5 - t)^2, (5 - t)^3),$$

we have $dx/dt = -1$, $dy/dt = -10 + 2t = -2(5 - t)$, $dz/dt = -75 + 30t - 3t^2 = -3(5 - t)^2$, and $\mathbf{F}(\mathbf{c}_{\text{op}}(t)) = (5 - t)^5\mathbf{i} + (5 - t)^4\mathbf{j} + (5 - t)^3\mathbf{k}$. Therefore,

$$\int_{\mathbf{c}_{\text{op}}} \mathbf{F} \cdot d\mathbf{s} = \int_{-5}^{10} [-(5 - t)^5 - 2(5 - t)^5 - 3(5 - t)^5]\, dt = [(5 - t)^6]_{-5}^{10} = -984,375. \quad \blacktriangle$$

We are interested in reparametrizations, because if the image of a particular $\mathbf{c}$ can be represented in many ways, we want to be sure that path and line integrals depend only on the image curve and not on the particular parametrization. For example, for some problems the unit circle may be conveniently represented by the map $\mathbf{p}$ given by

$$x(t) = \cos 2t, \qquad y(t) = \sin 2t, \qquad 0 \leq t \leq \pi.$$

Theorem 1 guarantees that any integral computed for this representation will be the same as when we represent the circle by the map $\mathbf{c}$ given by

$$x(t) = \cos t, \qquad y(t) = \sin t, \qquad 0 \leq t \leq 2\pi,$$

because $\mathbf{p} = \mathbf{c} \circ h$, where $h(t) = 2t$, and thus $\mathbf{p}$ is an orientation-preserving reparametrization of $\mathbf{c}$. However, notice that the map $\gamma$ given by

$$\gamma(t) = (\cos t, \sin t), \qquad 0 \leq t \leq 4\pi$$

is *not* a reparametrization of $\mathbf{c}$. Although it traces out the same image (the circle), it does so twice. (Why does this imply that $\gamma$ is not a reparametrization of $\mathbf{c}$?)

The line integral is an *oriented integral*, in that a change of sign occurs (as we have seen in Theorem 1) if the orientation of the curve is reversed. The *path integral* does not have this property. This follows from the fact that changing $t$ to $-t$ (reversing orientation) just changes the sign of $\mathbf{c}'(t)$, not its length. This is one of the differences between line and path integrals. The following theorem, which is proved by the same method as Theorem 1, shows that path integrals are unchanged under reparametrizations—even orientation-reversing ones.

**THEOREM 2: Change of Parametrization for Path Integrals** Let $\mathbf{c}$ be piecewise $C^1$, let $f$ be a continuous (real-valued) function on the image of $\mathbf{c}$, and let $\mathbf{p}$ be any reparametrization of $\mathbf{c}$. Then

$$\int_{\mathbf{c}} f(x, y, z)\, ds = \int_{\mathbf{p}} f(x, y, z)\, ds. \tag{2}$$

## Line Integrals of Gradient Fields

We next consider a useful technique for evaluating certain types of line integrals. Recall that a vector field $\mathbf{F}$ is a *gradient vector field* if $\mathbf{F} = \nabla f$ for some real-valued function $f$. Thus,

$$\mathbf{F} = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}.$$

Suppose $g$ and $G$ are real-valued continuous functions defined on a closed interval $[a, b]$, that $G$ is differentiable on $(a, b)$, and that $G' = g$. Then by the fundamental theorem of calculus

$$\int_a^b g(x)\, dx = G(b) - G(a).$$

Thus, the value of the integral of $g$ depends only on the value of $G$ at the endpoints of the interval $[a, b]$. Because $\nabla f$ represents the derivative of $f$, one can ask whether $\int_{\mathbf{c}} \nabla f \cdot d\mathbf{s}$ is completely determined by the value of $f$ at the endpoints $\mathbf{c}(a)$ and $\mathbf{c}(b)$. The answer is contained in the following *generalization of the fundamental theorem of calculus*.

**THEOREM 3: Line Integrals of Gradient Vector Fields** Suppose that $f\colon \mathbb{R}^3 \to \mathbb{R}$ is of class $C^1$ and that $\mathbf{c}\colon [a, b] \to \mathbb{R}^3$ is a piecewise $C^1$ path. Then

$$\int_{\mathbf{c}} \nabla f \cdot d\mathbf{s} = f(\mathbf{c}(b)) - f(\mathbf{c}(a)).$$

*PROOF* Apply the chain rule to the composite function

$$F\colon t \mapsto f(\mathbf{c}(t))$$

to obtain

$$F'(t) = (f \circ \mathbf{c})'(t) = \nabla f(\mathbf{c}(t)) \cdot \mathbf{c}'(t).$$

The function $F$ is a real-valued function of the variable $t$, and so, by the fundamental theorem of single-variable calculus,

$$\int_a^b F'(t)\, dt = F(b) - F(a) = f(\mathbf{c}(b)) - f(\mathbf{c}(a)).$$

Therefore,

$$\begin{aligned}\int_{\mathbf{c}} \nabla f \cdot d\mathbf{s} = \int_a^b \nabla f(\mathbf{c}(t)) \cdot \mathbf{c}'(t)\, dt &= \int_a^b F'(t)\, dt = F(b) - F(a)\\ &= f(\mathbf{c}(b)) - f(\mathbf{c}(a)). \quad \blacksquare\end{aligned}$$

**EXAMPLE 9** Let $\mathbf{c}$ be the path $\mathbf{c}(t) = (t^4/4, \sin^3(t\pi/2), 0)$, $t \in [0, 1]$. Evaluate

$$\int_{\mathbf{c}} y\, dx + x\, dy$$

(which means $\int_{\mathbf{c}} y\, dx + x\, dy + 0\, dz$).

SOLUTION We recognize $y\, dx + x\, dy$, or equivalently, the vector field $y\mathbf{i} + x\mathbf{j} + 0\mathbf{k}$, as the gradient of the function $f(x, y, z) = xy$. Thus,

$$\int_{\mathbf{c}} y\, dx + x\, dy = f(\mathbf{c}(1)) - f(\mathbf{c}(0)) = \frac{1}{4} \cdot 1 - 0 = \frac{1}{4}. \quad \blacktriangle$$

Obviously, if one can recognize the integrand as a gradient, then evaluation of the integral becomes much easier. For example, the reader should try to work out the integral in Example 9 directly. In one-variable calculus, every integral is, in principle, obtainable by finding an antiderivative. For vector fields, however, this is not always true, because a given vector field need not be a gradient. This point will be examined in detail in Section 8.3, where we derive a test to determine when a vector field $\mathbf{F}$ is a gradient; that is, when $\mathbf{F} = \nabla f$ for some $f$.

## Line Integrals Over Geometric Curves

We have seen how to define path integrals (integrals of scalar functions) and line integrals (integrals of vector functions) over parametrized curves. We have also seen that our work is simplified if we make a judicious choice of parametrization. Because these integrals are independent of the parametrization (except possibly for the sign), it seems natural to express the theory in a way that is independent of the parametrization, and that is thereby more "geometric." We do this briefly and somewhat informally in the following discussion.

DEFINITION We define a ***simple curve*** $C$ to be the image of a piecewise $C^1$ map $\mathbf{c}\colon I \to \mathbb{R}^3$ that is one-to-one on an interval $I$; $\mathbf{c}$ is called a ***parametrization*** of $C$. Thus, a simple curve is one that does not intersect itself (Figure 7.2.6). If $I = [a, b]$, we call $\mathbf{c}(a)$ and $\mathbf{c}(b)$ ***endpoints*** of the curve.

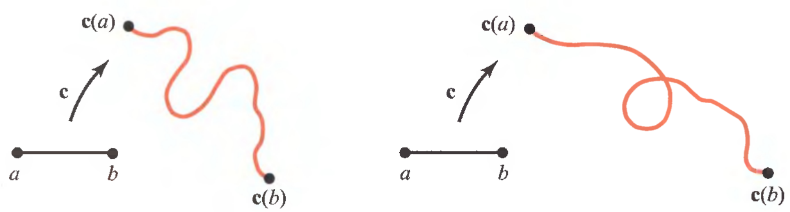



Figure 7.2.6 A simple curve that has no self-intersections is shown on the left. On the right is a curve with a self-intersection, so it is not simple.

Each simple curve $C$ has two orientations or directions associated with it. If P and Q are the endpoints of the curve, then we can consider $C$ as directed either from P to Q or from Q to P. The simple curve $C$ together with a sense of direction is called an ***oriented simple curve*** or ***directed simple curve*** (Figure 7.2.7).

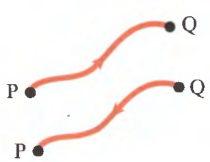



Figure 7.2.7 There are two possible senses of direction on a curve joining P and Q.

**DEFINITION: Simple Closed Curves** By a ***simple closed curve*** we mean the image of a piecewise $C^1$ map $\mathbf{c}: [a, b] \to \mathbb{R}^3$ that is one-to-one on $[a, b)$ and satisfies $\mathbf{c}(a) = \mathbf{c}(b)$ (Figure 7.2.8). If $\mathbf{c}$ satisfies the condition $\mathbf{c}(a) = \mathbf{c}(b)$, but is not necessarily one-to-one on $[a, b)$, we call its image a ***closed curve***. Simple closed curves have two orientations, corresponding to the two possible directions of motion along the curve (Figure 7.2.9).

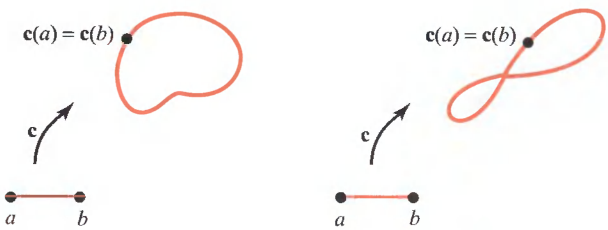



Figure 7.2.8 A simple closed curve (left) and a closed curve that is not simple (right).

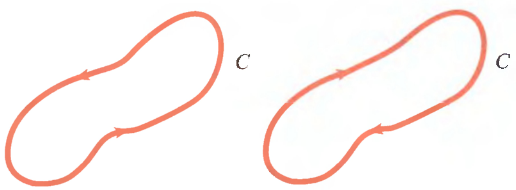



Figure 7.2.9 Two possible orientations for a simple closed curve $C$.

If $C$ is an oriented simple curve or an oriented simple closed curve, we may unambiguously define line integrals along them.

**Line Integrals and Path Integrals Over Oriented Simple Curves and Simple Closed Curves** $C$:

$$\int_C \mathbf{F} \cdot d\mathbf{s} = \int_{\mathbf{c}} \mathbf{F} \cdot d\mathbf{s} \qquad \text{and} \qquad \int_C f\, ds = \int_{\mathbf{c}} f\, ds, \tag{3}$$

where $\mathbf{c}$ is any *orientation-preserving* parametrization of $C$.

These integrals do not depend on the choice of $\mathbf{c}$ as long as $\mathbf{c}$ is one-to-one (except possibly at the endpoints) by virtue of Theorems 1 and 2.[5] The point we want to make here is that *although a curve must be parametrized to make integration along it tractable, it is not necessary to include the parametrization in our notation for the integral.*

[5] We have not proved that any two one-to-one paths $\mathbf{c}$ and $\mathbf{p}$ with the same image must be reparametrizations of each other, but this technical point will be omitted.

**EXAMPLE 10** If $I = [a, b]$ is a closed interval on the $x$ axis, then $I$, as a curve, has two orientations: one corresponding to motion from $a$ to $b$ (left to right) and the other corresponding to motion from $b$ to $a$ (right to left). If $f$ is a real-valued function continuous on $I$, then denoting $I$ with the first orientation by $I^+$ and $I$ with the second orientation by $I^-$, we have

$$\int_{I^+} f(x)\,dx = \int_a^b f(x)\,dx = -\int_b^a f(x)\,dx = -\int_{I^-} f(x)\,dx. \quad \blacktriangle$$

A given simple closed curve can be parametrized in many different ways. Figure 7.2.10 shows $C$ represented as the image of a map $\mathbf{p}$, with $\mathbf{p}(t)$ progressing in a prescribed direction around an oriented curve $C$ as $t$ ranges from $a$ to $b$. Note that $\mathbf{p}'(t)$ points in this direction also. The speed with which we traverse $C$ may vary from parametrization to parametrization, but as long as the orientation is preserved, the integral will not, according to Theorems 1 and 2.

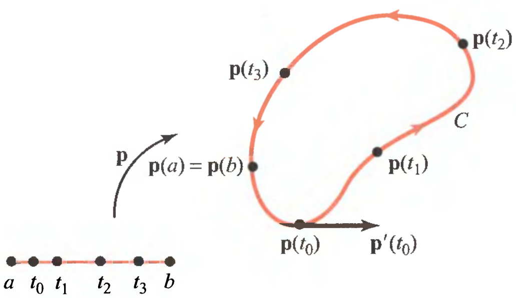



**Figure 7.2.10** As $t$ goes from $a$ to $b$, $\mathbf{p}(t)$ moves around the curve $C$ in some fixed direction.

The following precaution should be noted in regard to these remarks. It is possible to have two mappings $\mathbf{c}$ and $\mathbf{p}$ with the same image, and inducing the same orientation on the image, such that

$$\int_{\mathbf{c}} \mathbf{F} \cdot d\mathbf{s} \neq \int_{\mathbf{p}} \mathbf{F} \cdot d\mathbf{s}.$$

For an example, let $\mathbf{c}(t) = (\cos t, \sin t, 0)$ and $\mathbf{p}(t) = (\cos 2t, \sin 2t, 0)$, $0 \le t \le 2\pi$, with $\mathbf{F}(x, y, z) = (y, 0, 0)$. Then $F_1(x, y, z) = y$, $F_2(x, y, z) = 0$, and $F_3(x, y, z) = 0$, so

$$\int_{\mathbf{c}} \mathbf{F} \cdot d\mathbf{s} = \int_0^{2\pi} F_1(\mathbf{c}(t))\,\frac{dx}{dt}\,dt = -\int_0^{2\pi} \sin^2 t\,dt = -\pi.$$

But $\int_{\mathbf{p}} \mathbf{F} \cdot d\mathbf{s} = -2\int_0^{2\pi} \sin^2 2t\,dt = -2\pi$. Clearly, $\mathbf{c}$ and $\mathbf{p}$ have the same image, namely, the unit circle in the $xy$ plane. Moreover, they traverse the unit circle in the same direction; yet $\int_{\mathbf{c}} \mathbf{F} \cdot d\mathbf{s} \neq \int_{\mathbf{p}} \mathbf{F} \cdot d\mathbf{s}$. The reason for this is that $\mathbf{c}$ is one-to-one, but $\mathbf{p}$ is not ($\mathbf{p}$ traverses the unit circle *twice* in a counterclockwise direction); therefore, $\mathbf{p}$ is not a parametrization of the unit circle as a simple closed curve.

As a consequence of Theorem 1 and generalizing the notation in Example 10, we introduce the following convention:

**Line Integrals Over Curves with Opposite Orientations** Let $C^-$ be the same curve as $C$, but with the opposite orientation. Then

$$\int_C \mathbf{F} \cdot d\mathbf{s} = -\int_{C^-} \mathbf{F} \cdot d\mathbf{s}.$$

We also have:

**Line Integrals Over Curves Consisting of Several Components** Let $C$ be an oriented curve that is made up of several oriented component curves $C_i$, $i = 1, \dots, k$, as in Figure 7.2.11. Then we shall write $C = C_1 + C_2 + \cdots + C_k$. Because we can parametrize $C$ by parametrizing the pieces $C_1, \dots, C_k$ separately, one can prove that

$$\int_C \mathbf{F} \cdot d\mathbf{s} = \int_{C_1} \mathbf{F} \cdot d\mathbf{s} + \int_{C_2} \mathbf{F} \cdot d\mathbf{s} + \cdots + \int_{C_k} \mathbf{F} \cdot d\mathbf{s}. \tag{4}$$

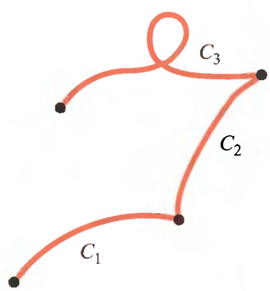



**Figure 7.2.11** A curve can be made up of several components.

One reason for writing a curve as a sum of components is that it may be easier to parametrize the components $C_i$ individually than it is to parametrize $C$ as a whole. If that is the case, formula (4) provides a convenient way of evaluating $\int_C \mathbf{F} \cdot d\mathbf{s}$.

## The $d\mathbf{r}$ Notation for Line Integrals

Sometimes one writes, as we occasionally do later, the line integral using the notation

$$\int_C \mathbf{F} \cdot d\mathbf{r}.$$

The reason is that we think of describing a $C^1$ path $\mathbf{c}$ in terms of a moving *position vector* based at the origin and ending at the point $\mathbf{c}(t)$ at time $t$. Position vectors are often denoted by $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, and so the curve is described using the notation $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$ in place of $\mathbf{c}(t)$. By definition, the line integral is given by

$$\int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \frac{d\mathbf{r}}{dt}\, dt.$$

Formally canceling the $dt$'s, and using the parametrization independence to replace the limits of integration with the geometric curve $C$, we arrive at the notation $\int_C \mathbf{F} \cdot d\mathbf{r}$.

**EXAMPLE 11** Consider $C$, the perimeter of the unit square in $\mathbb{R}^2$, oriented in the counterclockwise sense (see Figure 7.2.12). Evaluate the line integral

$$\int_C x^2\, dx + xy\, dy.$$

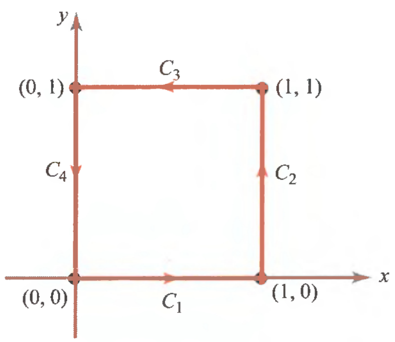



**Figure 7.2.12** The perimeter of the unit square, parametrized in four pieces.

SOLUTION We evaluate the integral using a convenient parametrization of $C$ that induces the given orientation. For example:

$$\mathbf{c}\colon [0, 4] \to \mathbb{R}^2, \qquad t \mapsto \begin{cases} (t, 0) & 0 \le t \le 1 \\ (1, t-1) & 1 \le t \le 2 \\ (3-t, 1) & 2 \le t \le 3 \\ (0, 4-t) & 3 \le t \le 4. \end{cases}$$

Then

$$\begin{aligned} \int_C x^2\, dx + xy\, dy &= \int_0^1 (t^2 + 0)\, dt + \int_1^2 [0 + (t-1)]\, dt \\ &\quad + \int_2^3 [-(3-t)^2 + 0]\, dt + \int_3^4 (0+0)\, dt \\ &= \frac{1}{3} + \frac{1}{2} + \left(-\frac{1}{3}\right) + 0 = \frac{1}{2}. \end{aligned}$$

Now let us reevaluate this line integral, using formula (4) and parametrizing the $C_i$ separately. Notice that $C = C_1 + C_2 + C_3 + C_4$, where $C_i$ are the oriented curves pictured in Figure 7.2.12. These can be parametrized as follows:

$$\begin{aligned}
C_1&: \mathbf{c}_1(t) = (t, 0), 0 \le t \le 1\\
C_2&: \mathbf{c}_2(t) = (1, t), 0 \le t \le 1\\
C_3&: \mathbf{c}_3(t) = (1-t, 1), 0 \le t \le 1\\
C_4&: \mathbf{c}_4(t) = (0, 1-t), 0 \le t \le 1,
\end{aligned}$$

and so

$$\begin{aligned}
\int_{C_1} x^2\,dx + xy\,dy &= \int_0^1 t^2\,dt = \frac{1}{3}\\
\int_{C_2} x^2\,dx + xy\,dy &= \int_0^1 t\,dt = \frac{1}{2}\\
\int_{C_3} x^2\,dx + xy\,dy &= \int_0^1 -(1-t)^2\,dt = -\frac{1}{3}\\
\int_{C_4} x^2\,dx + xy\,dy &= \int_0^1 0\,dt = 0.
\end{aligned}$$

Thus, again,

$$\int_C x^2\,dx + xy\,dy = \frac{1}{3} + \frac{1}{2} - \frac{1}{3} + 0 = \frac{1}{2}. \quad \blacktriangle$$

**EXAMPLE 12** An interesting application of the line integral is the mathematical formulation of Ampère's law, which relates electric currents to their magnetic effects.[6] Suppose $\mathbf{H}$ denotes a magnetic field in $\mathbb{R}^3$, and let $C$ be a closed oriented curve in $\mathbb{R}^3$. In appropriate physical units, Ampère's law states that

$$\int_C \mathbf{H} \cdot d\mathbf{s} = I,$$

where $I$ is the net current that passes through any surface bounded by $C$ (see Figure 7.2.13). ▲

Finally, let us mention that the line integral has another important physical meaning, specifically, the interpretation of $\int_C \mathbf{V} \cdot d\mathbf{s}$ as *circulation*, where $\mathbf{V}$ is the velocity

[6] The discovery that electric currents produce magnetic effects was made by Haas Christian Oersted circa 1820. See any elementary physics text for discussions of the physical basis of these ideas.

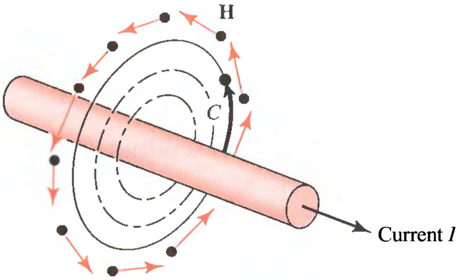



**Figure 7.2.13** The magnetic field $\mathbf{H}$ surrounding a wire carrying a current $I$ satisfies Ampère's law: $\int_C \mathbf{H} \cdot d\mathbf{s} = I$.

field of a fluid, as we shall discuss in Section 8.2. Thus, a wide variety of physical concepts, from the notion of work to electromagnetic fields and the motions of fluids, can be analyzed with the help of line integrals.

## EXERCISES

**1.** Let $\mathbf{F}(x, y, z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$. Evaluate the integral of $\mathbf{F}$ along each of the following paths:

(a) $\mathbf{c}(t) = (t, t, t), \quad 0 \le t \le 1$
(b) $\mathbf{c}(t) = (\cos t, \sin t, 0), \quad 0 \le t \le 2\pi$
(c) $\mathbf{c}(t) = (\sin t, 0, \cos t), \quad 0 \le t \le 2\pi$
(d) $\mathbf{c}(t) = (t^2, 3t, 2t^3), \quad -1 \le t \le 2$

**2.** Evaluate each of the following line integrals:

(a) $\int_{\mathbf{c}} x\,dy - y\,dx, \quad \mathbf{c}(t) = (\cos t, \sin t), \quad 0 \le t \le 2\pi$
(b) $\int_{\mathbf{c}} x\,dx + y\,dy, \quad \mathbf{c}(t) = (\cos \pi t, \sin \pi t), \quad 0 \le t \le 2$
(c) $\int_{\mathbf{c}} yz\,dx + xz\,dy + xy\,dz$, where $\mathbf{c}$ consists of straight-line segments joining $(1, 0, 0)$ to $(0, 1, 0)$ to $(0, 0, 1)$
(d) $\int_{\mathbf{c}} x^2\,dx - xy\,dy + dz$, where $\mathbf{c}$ is the parabola $z = x^2, y = 0$ from $(-1, 0, 1)$ to $(1, 0, 1)$.

**3.** Consider the force field $\mathbf{F}(x, y, z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$. Compute the work done in moving a particle along the parabola $y = x^2, z = 0$, from $x = -1$ to $x = 2$.

**4.** Let $\mathbf{c}$ be a smooth path.

(a) Suppose $\mathbf{F}$ is perpendicular to $\mathbf{c}'(t)$ at the point $\mathbf{c}(t)$. Show that

$$\int_{\mathbf{c}} \mathbf{F} \cdot d\mathbf{s} = 0.$$

(b) If $\mathbf{F}$ is parallel to $\mathbf{c}'(t)$ at $\mathbf{c}(t)$, show that

$$\int_{\mathbf{c}} \mathbf{F} \cdot d\mathbf{s} = \int_{\mathbf{c}} \|\mathbf{F}\|\, ds.$$

[By parallel to $\mathbf{c}'(t)$ we mean that $\mathbf{F}(\mathbf{c}(t)) = \lambda(t)\mathbf{c}'(t)$, where $\lambda(t) > 0$.]

**5.** Suppose the path $\mathbf{c}$ has length $l$, and $\|\mathbf{F}\| \leq M$. Prove that

$$\left|\int_{\mathbf{c}} \mathbf{F} \cdot d\mathbf{s}\right| \leq Ml.$$

**6.** Evaluate $\int_{\mathbf{c}} \mathbf{F} \cdot d\mathbf{s}$ where $\mathbf{F}(x, y, z) = y\mathbf{i} + 2x\mathbf{j} + y\mathbf{k}$ and the path $\mathbf{c}$ is defined by $\mathbf{c}(t) = t\mathbf{i} + t^2\mathbf{j} + t^3\mathbf{k}$, $0 \leq t \leq 1$.

**7.** Evaluate

$$\int_{\mathbf{c}} y\,dx + (3y^3 - x)\,dy + z\,dz$$

for each of the paths $\mathbf{c}(t) = (t, t^n, 0)$, $0 \leq t \leq 1$, where $n = 1, 2, 3, \ldots$.

**8.** This exercise refers to Example 12. Let $L$ be a very long wire, a planar section of which (with the plane perpendicular to the wire) is shown in Figure 7.2.14. Suppose this plane is the $xy$ plane. Experiments show that $\mathbf{H}$ is tangent to every circle in the $xy$ plane whose center is the axis of $L$, and that the magnitude of $\mathbf{H}$ is constant on every such circle $C$. Thus, $\mathbf{H} = H\mathbf{T}$, where $\mathbf{T}$ is a unit tangent vector to $C$ and $H$ is some scalar. Using this information, show that $H = I/2\pi r$, where $r$ is the radius of circle $C$ and $I$ is the current flowing in the wire.

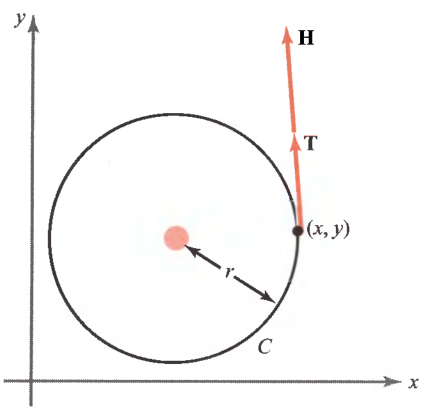



**Figure 7.2.14** A planar section of a long wire and a curve $C$ about the wire.

**9.** The image of the path $t \mapsto (\cos^3 t, \sin^3 t)$, $0 \leq t \leq 2\pi$ in the plane is shown in Figure 7.2.15. Evaluate the integral of the vector field $\mathbf{F}(x, y) = x\mathbf{i} + y\mathbf{j}$ around this curve.

**10.** Suppose $\mathbf{c}_1$ and $\mathbf{c}_2$ are two paths with the same endpoints and $\mathbf{F}$ is a vector field. Show that $\int_{\mathbf{c}_1} \mathbf{F} \cdot d\mathbf{s} = \int_{\mathbf{c}_2} \mathbf{F} \cdot d\mathbf{s}$ is equivalent to $\int_{C} \mathbf{F} \cdot d\mathbf{s} = 0$, where $C$ is the closed curve obtained by first moving along $\mathbf{c}_1$ and then moving along $\mathbf{c}_2$ in the opposite direction.

**11.** Let $\mathbf{c}(t)$ be a path and $\mathbf{T}$ the unit tangent vector. What is $\int_{\mathbf{c}} \mathbf{T} \cdot d\mathbf{s}$?

**12.** Let $\mathbf{F} = (z^3 + 2xy)\mathbf{i} + x^2\mathbf{j} + 3xz^2\mathbf{k}$. Show that the integral of $\mathbf{F}$ around the circumference of the unit square with vertices $(\pm 1, \pm 1)$ is zero.

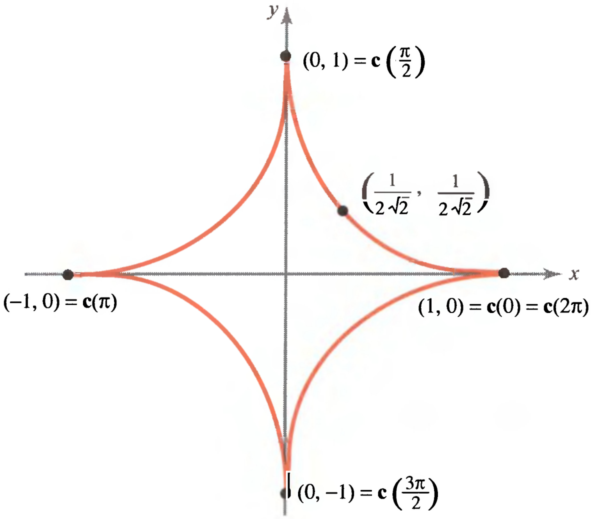



**Figure 7.2.15** The hypocycloid $\mathbf{c}(t) = (\cos^3 t, \sin^3 t)$ (Exercise 9).

**13.** Using the path in Exercise 9, observe that a $C^1$ map $\mathbf{c}\colon [a, b] \to \mathbb{R}^3$ can have an image that does not "look smooth." Do you think this could happen if $\mathbf{c}'(t)$ were always nonzero?

**14.** What is the value of the integral of a gradient field around a closed curve $C$?

**15.** Evaluate the line integral

$$\int_C 2xyz\,dx + x^2z\,dy + x^2y\,dz,$$

where $C$ is an oriented simple curve connecting $(1, 1, 1)$ to $(1, 2, 4)$.

**16.** Suppose $\nabla f(x, y, z) = 2xyze^{x^2}\mathbf{i} + ze^{x^2}\mathbf{j} + ye^{x^2}\mathbf{k}$. If $f(0, 0, 0) = 5$, find $f(1, 1, 2)$.

**17.** Consider the gravitational force field (with $G = m = M = 1$) defined [for $(x, y, z) \neq (0, 0, 0)$] by

$$\mathbf{F}(x, y, z) = -\frac{1}{(x^2 + y^2 + z^2)^{3/2}}(x\mathbf{i} + y\mathbf{j} + z\mathbf{k}).$$

Show that the work done by the gravitational force as a particle moves from $(x_1, y_1, z_1)$ to $(x_2, y_2, z_2)$ along any path depends only on the radii $R_1 = \sqrt{x_1^2 + y_1^2 + z_1^2}$ and $R_2 = \sqrt{x_2^2 + y_2^2 + z_2^2}$.

**18.** A cyclist rides up a mountain along the path shown in Figure 7.2.16. She makes one complete revolution around the mountain in reaching the top, while her vertical rate of climb is constant. Throughout the trip she exerts a force described by the vector field

$$\mathbf{F}(x, y, z) = y\mathbf{i} + x\mathbf{j} + \mathbf{k}.$$

What is the work done by the cyclist in traveling from A to B? What is unrealistic about this model of a cyclist?

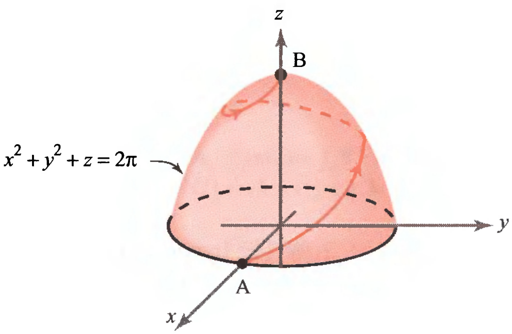



Figure 7.2.16 How much work is done in cycling up this mountain?

**19.** Let $\mathbf{c}$: $[a, b] \to \mathbb{R}^3$ be a path such that $\mathbf{c}'(t) \neq \mathbf{0}$. Recall from Section 4.1 that when this condition holds, $\mathbf{c}$ is said to be ***regular***. Let the function $f$ be defined by the formula $f(x) = \int_a^x \|\mathbf{c}'(t)\|\, dt$.

(a) What is $df/dx$?

(b) Using the answer to part (a), prove that $f$: $[a, b] \to [0, L]$, where $L$ is the length of $\mathbf{c}$, has a differentiable inverse $g$: $[0, L] \to [a, b]$ satisfying $f \circ g(s) = s$, $g \circ f(x) = x$. (You may use the one-variable inverse function theorem stated at the beginning of Section 3.5.)

(c) Compute $dg/ds$.

(d) Recall that a path $s \mapsto \mathbf{b}(s)$ is said to be of unit speed, or parametrized by arc length, if $\|\mathbf{b}'(s)\| = 1$. Show that the reparametrization of $\mathbf{c}$ given by $\mathbf{b}(s) = \mathbf{c} \circ g(s)$ is of unit speed. Conclude that any regular path can be reparametrized by arc length. (Thus, for example, the Frenet formulas in Exercise 17 of Section 4.2 can be applied to the reparametrization $\mathbf{b}$.)

**20.** Along a "thermodynamic path" $C$ in $(V, T, P)$ space,

(i) The heat gained is $\int_C \Lambda_V\, dV + K_V\, dT$, where $\Lambda_V$, $K_V$ are functions of $(V, T, P)$, depending on the particular physical system.

(ii) The work done is $\int_C P\, dV$.

For a van der Waals gas, we have

$$P(V, T) = \frac{RT}{V-b} - \frac{a}{V^2}, \qquad J\Lambda_V = \frac{RT}{V-b}, \qquad \text{and} \qquad K_V = \text{constant},$$

where $R$, $b$, $a$, and $J$ are known constants. Initially the gas is at a temperature $T_0$ and volume $V_0$.

(a) An ***adiabatic*** process is a thermodynamic motion $(V(t), T(t), P(t))$ for which

$$\frac{dT}{dV} = \frac{dT/dt}{dV/dt} = -\frac{\Lambda_V}{K_V}.$$

If the van der Waals gas undergoes an adiabatic process in which the volume doubles to $2V_0$, compute

(1) the heat gained;
(2) the work done; and
(3) the final volume, temperature, and pressure.

(b) After the process indicated in part (a), the gas is cooled (or heated) at constant volume until the original temperature $T_0$ is achieved. Compute

(1) the heat gained;
(2) the work done; and
(3) the final volume, temperature, and pressure.

(c) After the process indicated in part (b), the gas is compressed until the gas returns to its original volume $V_0$. The temperature is held constant throughout the process. Compute

(1) the heat gained;
(2) the work done; and
(3) the final volume, temperature, and pressure.

(d) For the cyclic process described in parts (a), (b), (c), compute

(1) the total heat gained; and
(2) the total work done.

# 7.3 Parametrized Surfaces

In Sections 7.1 and 7.2, we studied integrals of scalar and vector functions along curves. Now we turn to integrals over surfaces and begin by studying the geometry of surfaces themselves.

## Graphs Are Too Restrictive

We are already used to one kind of surface, namely, the graph of a function $f(x, y)$. Graphs were extensively studied in Chapter 2, and we know how to compute their tangent planes. However, it would be unduly limiting to restrict ourselves to this case. For example, many surfaces arise as level surfaces of functions. Suppose our surface $S$ is the set of points $(x, y, z)$ where $x - z + z^3 = 0$. Here $S$ is a sheet that (relative to the $xy$ plane) doubles back on itself (see Figure 7.3.1). Obviously, we want to

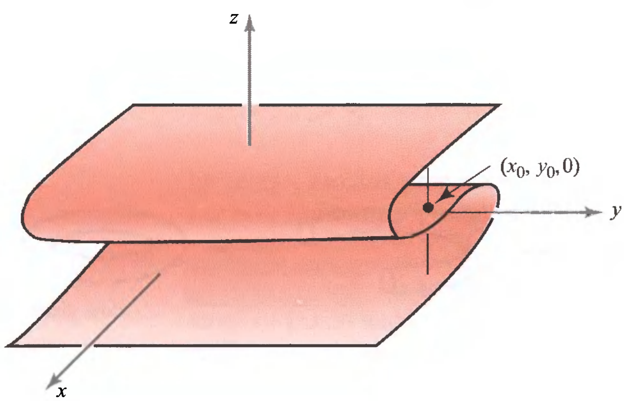



Figure 7.3.1 A surface that is not the graph of a function $z = f(x, y)$.

call $S$ a surface, because it is just a plane with a wrinkle. However, $S$ is *not* the graph of some function $z = f(x, y)$, because this means that for each $(x_0, y_0) \in \mathbb{R}^2$ there must be *one* $z_0$ with $(x_0, y_0, z_0) \in S$. As Figure 7.3.1 illustrates, this condition is violated.

Another example is the torus, or surface of a doughnut, which is depicted in Figure 7.3.2. Anyone would call a torus a surface; yet, by the same reasoning as before, a torus cannot be the graph of a differentiable function of two variables. These observations encourage us to extend our definition of a surface.

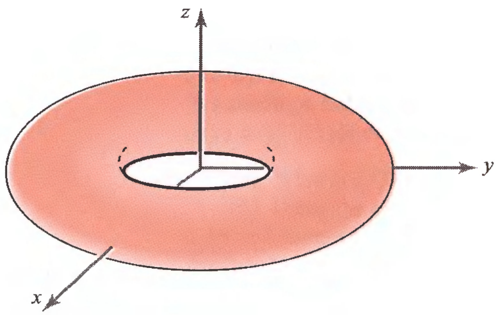



Figure 7.3.2 The torus is not the graph of a function of the form $z = f(x, y)$.

The motivation for the extended definition that follows is partly that a surface can be thought of as being obtained from the plane by "rolling," "bending," and "pushing." For example, to get a torus, we take a portion of the plane and roll it (see Figure 7.3.3), then take the two "ends" and bring them together until they meet (Figure 7.3.4).

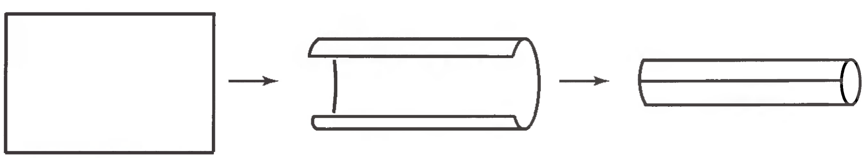


Figure 7.3.3 The first step in obtaining a torus from a rectangle is to make a cylinder.


Figure 7.3.4 Bend the cylinder and glue the ends to get a torus.

## Parametrized Surfaces as Mappings

In our study of differential calculus we dealt with mappings $f: A \subset \mathbb{R}^n \to \mathbb{R}^m$. Taking $n = 2$ and $m = 3$ corresponds to the case of a two-dimensional surface in 3-space. With surfaces, just as with curves, we want to distinguish a map (a parametrization) from its image (a geometric object). This leads us to the following definition.

**DEFINITION: Parametrized Surfaces** A ***parametrization of a surface*** is a function $\mathbf{\Phi}: D \subset \mathbb{R}^2 \to \mathbb{R}^3$, where $D$ is some domain in $\mathbb{R}^2$. The ***surface*** $S$ corresponding to the function $\mathbf{\Phi}$ is its image: $S = \mathbf{\Phi}(D)$. We can write

$$\mathbf{\Phi}(u, v) = (x(u, v), y(u, v), z(u, v)).$$

If $\mathbf{\Phi}$ is differentiable or is of class $C^1$ [which is the same as saying that $x(u, v)$, $y(u, v)$, and $z(u, v)$ are differentiable or $C^1$ functions of $(u, v)$], we call $S$ a ***differentiable or a $C^1$ surface***.

We can think of $\mathbf{\Phi}$ as twisting or bending the region $D$ in the plane to yield the surface $S$ (see Figure 7.3.5). Thus, each point $(u, v)$ in $D$ becomes a label for a point $(x(u, v), y(u, v), z(u, v))$ on $S$.

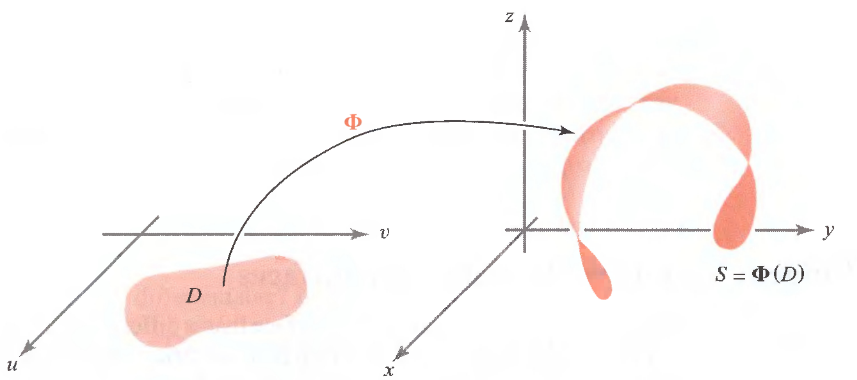



Figure 7.3.5 $\mathbf{\Phi}$ "twists" and "bends" $D$ onto the surface $S = \mathbf{\Phi}(D)$.

Of course, surfaces need not bend or twist at all. In fact, planes are flat, as shown in our first, and simplest, example.

**EXAMPLE 1** In Section 1.3 we studied the equation of a plane $P$. We did so in terms of graphs and level sets. Now we examine the same notion using a parametrization.

Let $P$ be a plane that is parallel to two vectors $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ and that passes through the tip of another vector $\boldsymbol{\gamma}$, as in Figure 7.3.6.

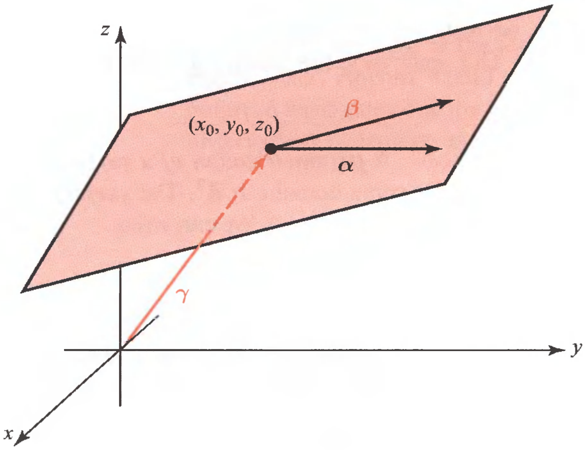



Figure 7.3.6 Describing a plane parametrically.

Our goal in this example is to find a parametrization of this plane. Notice that the vector $\boldsymbol{\alpha} \times \boldsymbol{\beta} = \mathbf{N}$, which we also write as $A\mathbf{i} + B\mathbf{j} + C\mathbf{k}$, is normal to $P$. If the tip of $\boldsymbol{\gamma}$ is the point $(x_0, y_0, z_0)$, then the equation of $P$ as a level set (as discussed in Section 1.3) is given by:

$$A(x - x_0) + B(y - y_0) + C(z - z_0) = 0.$$

However, the set of all points on the plane $P$ can also be described by the set of all vectors that are $\boldsymbol{\gamma}$ plus a linear combination of $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$. Using our preferred choice of real parameters $u$ and $v$, we arrive at the ***parametric equation of the plane*** $P$:

$$\boldsymbol{\Phi}(u, v) = \boldsymbol{\alpha} u + \boldsymbol{\beta} v + \boldsymbol{\gamma}. \quad \blacktriangle$$

## Tangent Vectors to Parametrized Surfaces

Suppose that $\boldsymbol{\Phi}$ is a parametrized surface that is differentiable at $(u_0, v_0) \in \mathbb{R}^2$. Fixing $u$ at $u_0$, we get a map $\mathbb{R} \to \mathbb{R}^3$ given by $t \mapsto \boldsymbol{\Phi}(u_0, t)$, whose image is a curve on the surface (Figure 7.3.7). From Chapters 2 and 4 we know that the vector tangent to this curve at the point $\boldsymbol{\Phi}(u_0, v_0)$, which we denote by $\mathbf{T}_v$, is given by

$$\mathbf{T}_v = \frac{\partial \boldsymbol{\Phi}}{\partial v} = \frac{\partial x}{\partial v}(u_0, v_0)\mathbf{i} + \frac{\partial y}{\partial v}(u_0, v_0)\mathbf{j} + \frac{\partial z}{\partial v}(u_0, v_0)\mathbf{k}.$$

Similarly, if we fix $v$ and consider the curve $t \mapsto \boldsymbol{\Phi}(t, v_0)$, we obtain the tangent vector to this curve at $\boldsymbol{\Phi}(u_0, v_0)$, given by

$$\mathbf{T}_u = \frac{\partial \boldsymbol{\Phi}}{\partial u} = \frac{\partial x}{\partial u}(u_0, v_0)\mathbf{i} + \frac{\partial y}{\partial u}(u_0, v_0)\mathbf{j} + \frac{\partial z}{\partial u}(u_0, v_0)\mathbf{k}.$$

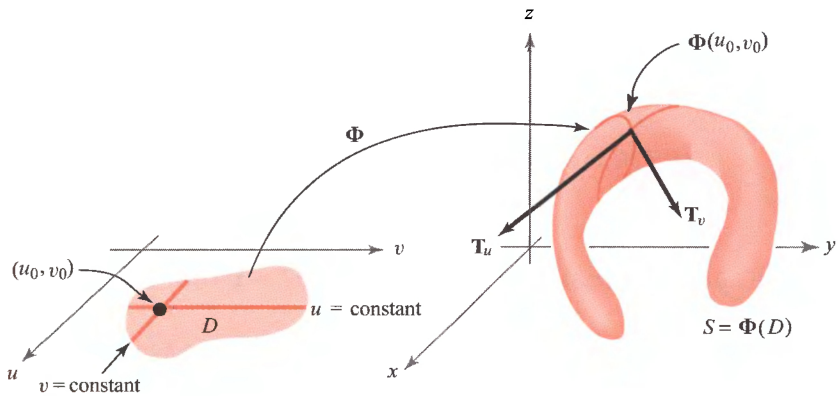



Figure 7.3.7 The tangent vectors $\mathbf{T}_u$ and $\mathbf{T}_v$ that are tangent to the curve on a surface $S$, and hence tangent to $S$.

## Regular Surfaces

Because the vectors $\mathbf{T}_u$ and $\mathbf{T}_v$ are tangent to two curves on the surface at a given point, the vector $\mathbf{T}_u \times \mathbf{T}_v$ ought to be normal to the surface at the same point.

We say that the surface $S$ is ***regular*** or ***smooth***[7] at $\mathbf{\Phi}(u_0, v_0)$, provided that $\mathbf{T}_u \times \mathbf{T}_v \neq \mathbf{0}$ at $(u_0, v_0)$. The surface is called ***regular*** if it is regular at all points $\mathbf{\Phi}(u_0, v_0) \in S$. The nonzero vector $\mathbf{T}_u \times \mathbf{T}_v$ is *normal* to $S$ (recall that the vector product of $\mathbf{T}_u$ and $\mathbf{T}_v$ is perpendicular to the plane spanned by $\mathbf{T}_u$ and $\mathbf{T}_v$); the fact that it is nonzero ensures that there will be a tangent plane. Intuitively, a smooth surface has no "corners."[8]

EXAMPLE 2 Consider the surface given by the equations

$$x = u\cos v, \qquad y = u\sin v, \qquad z = u, \qquad u \geq 0.$$

Is this surface differentiable? Is it regular?

SOLUTION These equations describe the surface $z = \sqrt{x^2 + y^2}$ (square the equations for $x, y$, and $z$ to check this), which is shown in Figure 7.3.8. This surface is a cone with a "point" at $(0, 0, 0)$; it is a differentiable surface because each component function is differentiable as a function of $u$ and $v$. However, *the surface*

[7] Strictly speaking, regularity depends on the parametrization $\mathbf{\Phi}$ and not just on its image $S$. Therefore, this terminology is somewhat imprecise; however, it is descriptive and should not cause confusion. (See Exercise 15.)

[8] In Section 3.5, we showed that level surfaces $f(x, y, z) = 0$ were in fact graphs of functions of two variables in some neighborhood of a point $(x_0, y_0, z_0)$ satisfying $\nabla f(x_0, y_0, z_0) \neq 0$. This united two concepts of a surface—graphs and level sets. Again, using the implicit function theorem, it is likewise possible to show that the image of a parametrized surface $\mathbf{\Phi}$ in the neighborhood of a point $(u_0, v_0)$ where $\mathbf{T}_u \times \mathbf{T}_v \neq \mathbf{0}$ is also the graph of a function of two variables. Thus, all definitions of a surface are consistent. (See Exercise 16.)

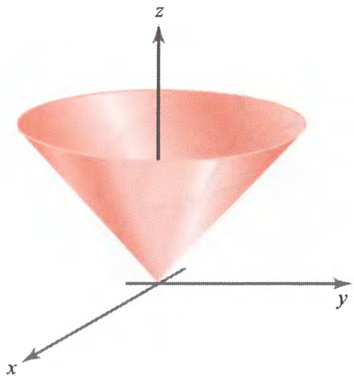



**Figure 7.3.8** The surface $z = \sqrt{x^2 + y^2}$ is a cone. It is not regular at its tip.

*is not regular at* (0, 0, 0). To see this, compute $\mathbf{T}_u$ and $\mathbf{T}_v$ at $(0, 0) \in \mathbb{R}^2$:

$$\mathbf{T}_u = \frac{\partial \mathbf{\Phi}}{\partial u} = \frac{\partial x}{\partial u}(0, 0)\mathbf{i} + \frac{\partial y}{\partial u}(0, 0)\mathbf{j} + \frac{\partial z}{\partial u}(0, 0)\mathbf{k} = (\cos 0)\mathbf{i} + (\sin 0)\mathbf{j} + \mathbf{k} = \mathbf{i} + \mathbf{k},$$

and similarly,

$$\mathbf{T}_v = \frac{\partial \mathbf{\Phi}}{\partial v} = 0(-\sin 0)\mathbf{i} + 0(\cos 0)\mathbf{j} + 0\mathbf{k} = \mathbf{0}.$$

Thus, $\mathbf{T}_u \times \mathbf{T}_v = \mathbf{0}$, and so, by definition, the surface is not regular at (0, 0, 0). ▲

## Tangent Plane to a Parametrized Surface

We can use the fact that $\mathbf{n} = \mathbf{T}_u \times \mathbf{T}_v$ is normal to the surface to both formally define the tangent plane and to compute it.

**DEFINITION: The Tangent Plane to a Surface** If a parametrized surface $\mathbf{\Phi}\colon D \subset \mathbb{R}^2 \to \mathbb{R}^3$ is *regular* at $\mathbf{\Phi}(u_0, v_0)$, that is, if $\mathbf{T}_u \times \mathbf{T}_v \neq \mathbf{0}$ at $(u_0, v_0)$, we define the ***tangent plane*** of the surface at $\mathbf{\Phi}(u_0, v_0)$ to be the plane determined by the vectors $\mathbf{T}_u$ and $\mathbf{T}_v$. Thus, $\mathbf{n} = \mathbf{T}_u \times \mathbf{T}_v$ is a normal vector, and an equation of the tangent plane at $(x_0, y_0, z_0)$ on the surface is given by

$$(x - x_0, y - y_0, z - z_0) \cdot \mathbf{n} = 0, \tag{1}$$

where $\mathbf{n}$ is evaluated at $(u_0, v_0)$; that is, the tangent plane is the set of $(x, y, z)$ satisfying (1). If $\mathbf{n} = (n_1, n_2, n_3) = n_1\mathbf{i} + n_2\mathbf{j} + n_3\mathbf{k}$, then formula (1) becomes

$$n_1(x - x_0) + n_2(y - y_0) + n_3(z - z_0) = 0. \tag{1$'$}$$

**EXAMPLE 3** Let $\mathbf{\Phi}\colon \mathbb{R}^2 \to \mathbb{R}^3$ be given by

$$x = u\cos v, \qquad y = u\sin v, \qquad z = u^2 + v^2.$$

Where does a tangent plane exist? Find the tangent plane at $\mathbf{\Phi}(1, 0)$.

SOLUTION We compute

$$\mathbf{T}_u = (\cos v)\mathbf{i} + (\sin v)\mathbf{j} + 2u\mathbf{k} \qquad \text{and} \qquad \mathbf{T}_v = -u(\sin v)\mathbf{i} + u(\cos v)\mathbf{j} + 2v\mathbf{k},$$

so the tangent plane at the point $\mathbf{\Phi}(u_0, v_0)$ is the set of vectors through $\mathbf{\Phi}(u_0, v_0)$ perpendicular to

$$(\mathbf{T}_u \times \mathbf{T}_v)(u_0, v_0) = (-2u_0^2\cos v_0 + 2v_0\sin v_0, -2u_0^2\sin v_0 - 2v_0\cos v_0, u_0)$$

if this vector is nonzero. Because $\mathbf{T}_u \times \mathbf{T}_v$ is equal to $\mathbf{0}$ at $(u_0, v_0) = (0, 0)$, we cannot find a tangent plane at $\mathbf{\Phi}(0, 0) = (0, 0, 0)$. However, we can find an equation of the tangent plane at all the other points, where $\mathbf{T}_u \times \mathbf{T}_v \neq \mathbf{0}$. At the point $\mathbf{\Phi}(1, 0) = (1, 0, 1)$,

$$\mathbf{n} = (\mathbf{T}_u \times \mathbf{T}_v)(1, 0) = (-2, 0, 1) = -2\mathbf{i} + \mathbf{k}.$$

Because we have the vector $\mathbf{n}$ normal to the surface and a point $(1, 0, 1)$ on the surface, we can use formula (1′) to obtain an equation of the tangent plane:

$$-2(x-1) + (z-1) = 0; \text{ that is, } z = 2x - 1. \quad \blacktriangle$$

**EXAMPLE 4** Suppose a surface $S$ is the graph of a differentiable function $g\colon \mathbb{R}^2 \to \mathbb{R}$. Write $S$ in parametric form and show that the surface is smooth at all points $(u_0, v_0, g(u_0, v_0)) \in \mathbb{R}^3$.

SOLUTION Write $S$ in parametric form as follows:

$$x = u, \qquad y = v, \qquad z = g(u, v),$$

which is the same as $z = g(x, y)$. Then at the point $(u_0, v_0)$,

$$\mathbf{T}_u = \mathbf{i} + \frac{\partial g}{\partial u}(u_0, v_0)\mathbf{k} \qquad \text{and} \qquad \mathbf{T}_v = \mathbf{j} + \frac{\partial g}{\partial v}(u_0, v_0)\mathbf{k},$$

and for $(u_0, v_0) \in \mathbb{R}^2$,

$$\mathbf{n} = \mathbf{T}_u \times \mathbf{T}_v = -\frac{\partial g}{\partial u}(u_0, v_0)\mathbf{i} - \frac{\partial g}{\partial v}(u_0, v_0)\mathbf{j} + \mathbf{k} \neq \mathbf{0}. \tag{2}$$

This is nonzero because the coefficient of $\mathbf{k}$ is 1; consequently, the parametrization $(u, v) \mapsto (u, v, g(u, v))$ is regular at all points. Moreover, the tangent plane at the point $(x_0, y_0, z_0) = (u_0, v_0, g(u_0, v_0))$ is given, by formula (1), as

$$(x - x_0, y - y_0, z - z_0) \cdot \left(-\frac{\partial g}{\partial u}, -\frac{\partial g}{\partial v}, 1\right) = 0,$$

where the partial derivatives are evaluated at $(u_0, v_0)$. Remembering that $x = u$ and $y = v$, we can write this as

$$z - z_0 = \left(\frac{\partial g}{\partial x}\right)(x - x_0) + \left(\frac{\partial g}{\partial y}\right)(y - y_0), \tag{3}$$

where $\partial g/\partial x$ and $\partial g/\partial y$ are evaluated at $(x_0, y_0)$. ▲

This example also shows that the definition of the tangent plane for parametrized surfaces agrees with the one for surfaces obtained as graphs, because equation (3) is the same formula we derived (in Chapter 2) for the plane tangent to $S$ at the point $(x_0, y_0, z_0) \in S$.

It is also useful to consider piecewise smooth surfaces, that is, surfaces composed of a certain number of images of smooth parametrized surfaces. For example, the surface of a cube in $\mathbb{R}^3$ is such a surface. These surfaces are considered in Section 7.4.

**EXAMPLE 5** Find a parametrization for the hyperboloid of one sheet:

$$x^2 + y^2 - z^2 = 1.$$

**SOLUTION** Because $x$ and $y$ appear in the combination $x^2 + y^2$, the surface is invariant under rotation about the $z$ axis, and so it is natural to write

$$x = r\cos\theta, \qquad y = r\sin\theta.$$

Then $x^2 + y^2 - z^2 = 1$ becomes $r^2 - z^2 = 1$. This we can conveniently parametrize by[9]

$$r = \cosh u, \qquad z = \sinh u.$$

Thus, a parametrization is

$$x = (\cosh u)(\cos\theta), \qquad y = (\cosh u)(\sin\theta), \qquad z = \sinh u,$$

where $0 \le \theta < 2\pi$, $-\infty < u < \infty$. ▲

[9]Recall from one-variable calculus that $\cosh u = (e^u + e^{-u})/2$ and $\sinh u = (e^u - e^{-u})/2$. One easily verifies from these definitions that $\cosh^2 u - \sinh^2 u = 1$.

## EXERCISES

*In Exercises 1 to 3, find an equation for the plane tangent to the given surface at the specified point.*

**1.** $x = 2u, \quad y = u^2 + v, \quad z = v^2, \quad$ *at* $(0, 1, 1)$

**2.** $x = u^2 - v^2, \quad y = u + v, \quad z = u^2 + 4v, \quad$ *at* $(-\frac{1}{4}, \frac{1}{2}, 2)$

**3.** $x = u^2, \quad y = u \sin e^v, \quad z = \frac{1}{3} u \cos e^v, \quad$ *at* $(13, -2, 1)$

**4.** At what points are the surfaces in Exercises 1 and 2 regular?

**5.** Find an expression for a unit vector normal to the surface

$$x = \cos v \sin u, \qquad y = \sin v \sin u, \qquad z = \cos u$$

at the image of a point $(u, v)$ for $u$ in $[0, \pi]$ and $v$ in $[0, 2\pi]$. Identify this surface.

**6.** Repeat Exercise 5 for the surface

$$x = 3\cos\theta \sin\phi, \qquad y = 2\sin\theta\sin\phi, \qquad z = \cos\phi$$

for $\theta$ in $[0, 2\pi]$ and $\phi$ in $[0, \pi]$.

**7.** Repeat Exercise 5 for the surface

$$x = \sin v, \qquad y = u, \qquad z = \cos v$$

for $0 \le v \le 2\pi$ and $-1 \le u \le 3$.

**8.** Repeat Exercise 5 for the surface

$$x = (2 - \cos v)\cos u, \qquad y = (2 - \cos v)\sin u, \qquad z = \sin v$$

for $-\pi \le u \le \pi, -\pi \le v \le \pi$. Is this surface regular?

**9.** (a) Develop a formula for the plane tangent to the surface $x = h(y, z)$.
(b) Obtain a similar formula for $y = k(x, z)$.

**10.** Find the equation of the plane tangent to the surface $x = u^2, y = v^2, z = u^2 + v^2$ at the point $u = 1, v = 1$.

**11.** Find a parametrization of the surface $z = 3x^2 + 8xy$ and use it to find the tangent plane at $x = 1, y = 0, z = 3$. Compare your answer with that using graphs.

**12.** Find a parametrization of the surface $x^3 + 3xy + z^2 = 2, z > 0$, and use it to find the tangent plane at the point $x = 1, y = 1/3, z = 0$. Compare your answer with that using level sets.

**13.** Consider the surface in $\mathbb{R}^3$ parametrized by

$$\Phi(r, \theta) = (r\cos\theta, r\sin\theta, \theta), \qquad 0 \le r \le 1 \qquad \text{and} \qquad 0 \le \theta \le 4\pi.$$

(a) Sketch and describe the surface.
(b) Find an expression for a unit normal to the surface.
(c) Find an equation for the plane tangent to the surface at the point $(x_0, y_0, z_0)$.
(d) If $(x_0, y_0, z_0)$ is a point on the surface, show that the horizontal line segment of unit length from the $z$ axis through $(x_0, y_0, z_0)$ is contained in the surface and in the plane tangent to the surface at $(x_0, y_0, z_0)$.

**14.** Given a sphere of radius 2 centered at the origin, find the equation for the plane that is tangent to it at the point $(1, 1, \sqrt{2})$ by considering the sphere as:

(a) a surface parametrized by $\mathbf{\Phi}(\theta, \phi) = (2\cos\theta \sin\phi, 2\sin\theta \sin\phi, 2\cos\phi)$;
(b) a level surface of $f(x, y, z) = x^2 + y^2 + z^2$; and
(c) the graph of $g(x, y) = \sqrt{4 - x^2 - y^2}$.

**15.** (a) Find a parametrization for the hyperboloid $x^2 + y^2 - z^2 = 25$.
(b) Find an expression for a unit normal to this surface.
(c) Find an equation for the plane tangent to the surface at $(x_0, y_0, 0)$, where $x_0^2 + y_0^2 = 25$.
(d) Show that the lines $(x_0, y_0, 0) + t(-y_0, x_0, 5)$ and $(x_0, y_0, 0) + t(y_0, -x_0, 5)$ lie in the surface *and* in the tangent plane found in part (c).

**16.** A parametrized surface is described by a differentiable function $\mathbf{\Phi}\colon \mathbb{R}^2 \to \mathbb{R}^3$. According to Chapter 2, the derivative should give a linear approximation that yields a representation of the tangent plane. This exercise demonstrates that this is indeed the case.

(a) Assuming $\mathbf{T}_u \times \mathbf{T}_v \neq \mathbf{0}$, show that the range of the linear transformation $\mathbf{D\Phi}(u_0, v_0)$ is the plane spanned by $\mathbf{T}_u$ and $\mathbf{T}_v$. [Here $\mathbf{T}_u$ and $\mathbf{T}_v$ are evaluated at $(u_0, v_0)$.]
(b) Show that $\mathbf{w} \perp (\mathbf{T}_u \times \mathbf{T}_v)$ if and only if $\mathbf{w}$ is in the range of $\mathbf{D\Phi}(u_0, v_0)$.
(c) Show that the tangent plane as defined in this section is the same as the "parametrized plane"

$$(u, v) \mapsto \mathbf{\Phi}(u_0, v_0) + \mathbf{D\Phi}(u_0, v_0)\begin{bmatrix} u - u_0 \\ v - v_0 \end{bmatrix}.$$

**17.** Consider the surfaces $\mathbf{\Phi}_1(u, v) = (u, v, 0)$ and $\mathbf{\Phi}_2(u, v) = (u^3, v^3, 0)$.

(a) Show that the image of $\mathbf{\Phi}_1$ and of $\mathbf{\Phi}_2$ is the $xy$ plane.
(b) Show that $\mathbf{\Phi}_1$ describes a regular surface, yet $\mathbf{\Phi}_2$ does not. Conclude that the notion of regularity of a surface $S$ depends on the existence of at least one regular parametrization for $S$.
(c) Prove that the tangent plane of $S$ is well defined independently of the regular (one-to-one) parametrization (you will need to use the inverse function theorem from Section 3.5).
(d) After these remarks, do you think you can find a regular parametrization of the cone of Figure 7.3.7?

**18.** Let $\mathbf{\Phi}$ be a regular surface at $(u_0, v_0)$; that is, $\mathbf{\Phi}$ is of class $C^1$ and $\mathbf{T}_u \times \mathbf{T}_v \neq \mathbf{0}$ at $(u_0, v_0)$.

(a) Use the implicit function theorem (Section 3.5) to show that the image of $\mathbf{\Phi}$ *near* $(u_0, v_0)$ is the graph of a $C^1$ function of two variables. If the $z$ component of $\mathbf{T}_u \times \mathbf{T}_v$ is nonzero, we can write it as $z = f(x, y)$.
(b) Show that the tangent plane at $\mathbf{\Phi}(u_0, v_0)$ defined by the plane spanned by $\mathbf{T}_u$ and $\mathbf{T}_v$ coincides with the tangent plane of the graph of $z = f(x, y)$ at this point.

# 7.4 Area of a Surface

Before proceeding to general surface integrals, let us first consider the problem of computing the area of a surface, just as we considered the problem of finding the arc length of a curve before discussing path integrals.

## Definition of Surface Area

In Section 7.3, we defined a parametrized surface $S$ to be the *image* of a function $\mathbf{\Phi}: D \subset \mathbb{R}^2 \to \mathbb{R}^3$, written as $\mathbf{\Phi}(u, v) = (x(u, v), y(u, v), z(u, v))$. The map $\mathbf{\Phi}$ was called the parametrization of $S$ and $S$ was said to be regular at $\mathbf{\Phi}(u, v) \in S$ provided that $\mathbf{T}_u \times \mathbf{T}_v \neq \mathbf{0}$, where

$$\mathbf{T}_u = \frac{\partial x}{\partial u}(u, v)\mathbf{i} + \frac{\partial y}{\partial u}(u, v)\mathbf{j} + \frac{\partial z}{\partial u}(u, v)\mathbf{k}$$

and

$$\mathbf{T}_v = \frac{\partial x}{\partial v}(u, v)\mathbf{i} + \frac{\partial y}{\partial v}(u, v)\mathbf{j} + \frac{\partial z}{\partial v}(u, v)\mathbf{k}.$$

Recall that a regular surface (loosely speaking) is one that has no corners or breaks.

In the rest of this chapter and in the next one, we shall consider only piecewise regular surfaces that are unions of images of parametrized surfaces $\mathbf{\Phi}_i: D_i \to \mathbb{R}^3$ for which:

(i) $D_i$ is an elementary region in the plane;

(ii) $\mathbf{\Phi}_i$ is of class $C^1$ and one-to-one, except possibly on the boundary of $D_i$; and

(iii) $S_i$, the image of $\mathbf{\Phi}_i$ is regular, except possibly at a finite number of points.

**DEFINITION: Area of a Parametrized Surface** We define the ***surface area***[10] $A(S)$ of a parametrized surface by

$$A(S) = \iint_D \|\mathbf{T}_u \times \mathbf{T}_v\|\, du\, dv \tag{1}$$

where $\|\mathbf{T}_u \times \mathbf{T}_v\|$ is the norm of $\mathbf{T}_u \times \mathbf{T}_v$. If $S$ is a union of surfaces $S_i$, its area is the sum of the areas of the $S_i$.

[10] As we have not yet discussed the independence of parametrization, it may seem that $A(S)$ depends on the parametrization $\mathbf{\Phi}$. We shall discuss independence of parametrization in Section 7.6; the use of this notation here should not cause confusion.

As the reader can easily verify, we have

$$\|\mathbf{T}_u \times \mathbf{T}_v\| = \sqrt{\left[\frac{\partial(x,y)}{\partial(u,v)}\right]^2 + \left[\frac{\partial(y,z)}{\partial(u,v)}\right]^2 + \left[\frac{\partial(x,z)}{\partial(u,v)}\right]^2}, \tag{2}$$

where

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix},$$

and so on. Thus, formula (1) becomes

$$A(S) = \iint_D \sqrt{\left[\frac{\partial(x,y)}{\partial(u,v)}\right]^2 + \left[\frac{\partial(y,z)}{\partial(u,v)}\right]^2 + \left[\frac{\partial(x,z)}{\partial(u,v)}\right]^2}\, du\, dv. \tag{3}$$

## Justification of the Area Formula

We can justify the definition of surface area by analyzing the integral $\iint_D \|\mathbf{T}_u \times \mathbf{T}_v\|\, du\, dv$ in terms of Riemann sums. For simplicity, suppose $D$ is a rectangle; consider the $n$th regular partition of $D$, and let $R_{ij}$ be the $ij$th rectangle in the partition, with vertices $(u_i, v_j), (u_{i+1}, v_j), (u_i, v_{j+1})$, and $(u_{i+1}, v_{j+1}), 0 \le i \le n-1, 0 \le j \le n-1$. Denote the values of $\mathbf{T}_u$ and $\mathbf{T}_v$ at $(u_i, v_j)$ by $\mathbf{T}_{u_i}$ and $\mathbf{T}_{v_j}$. We can think of the vectors $\Delta u \mathbf{T}_{u_i}$ and $\Delta v \mathbf{T}_{v_j}$ as tangent to the surface at $\mathbf{\Phi}(u_i, v_j) = (x_{ij}, y_{ij}, z_{ij})$, where $\Delta u = u_{i+1} - u_i$, $\Delta v = v_{j+1} - v_j$. Then these vectors form a parallelogram $P_{ij}$ that lies in the plane tangent to the surface at $(x_{ij}, y_{ij}, z_{ij})$ (see Figure 7.4.1).

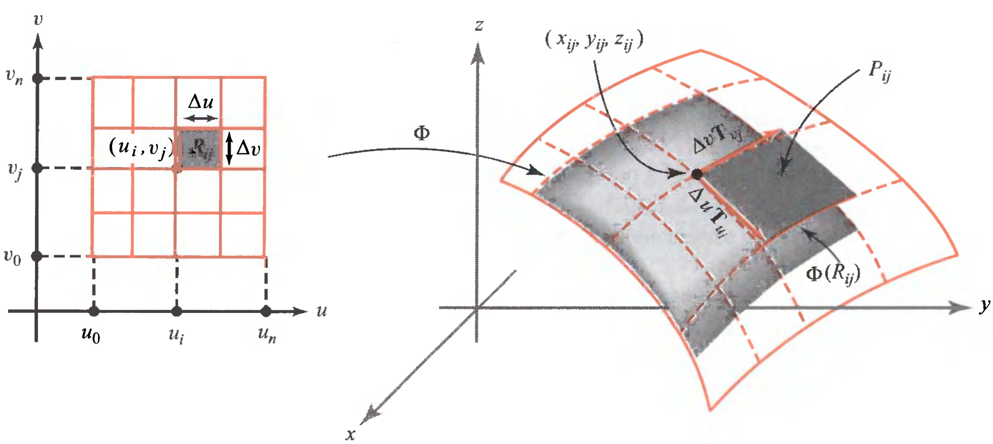



Figure 7.4.1 $\|\mathbf{T}_{u_i} \times \mathbf{T}_{v_j}\| \Delta u\, \Delta v$ is equal to the area of a parallelogram that approximates the area of a patch on a surface $S = \mathbf{\Phi}(D)$.

We thus have a "patchwork cover" of the surface by the $P_{ij}$. For $n$ large, the area of $P_{ij}$ is a good approximation to the area of $\mathbf{\Phi}(R_{ij})$. Because the area of the parallelogram spanned by two vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ is $\|\mathbf{v}_1 \times \mathbf{v}_2\|$ (see Chapter 1), we see that

$$A(P_{ij}) = \|\Delta u \mathbf{T}_{u_i} \times \Delta v \mathbf{T}_{v_j}\| = \|\mathbf{T}_{u_i} \times \mathbf{T}_{v_j}\| \Delta u\, \Delta v.$$

Therefore, the area of the patchwork cover is

$$A_n = \sum_{i=0}^{n-1}\sum_{j=0}^{n-1} A(P_{ij}) = \sum_{i=0}^{n-1}\sum_{j=0}^{n-1} \|\mathbf{T}_{u_i} \times \mathbf{T}_{v_j}\|\, \Delta u\, \Delta v.$$

As $n \to \infty$, the sums $A_n$ converge to the integral

$$\iint_D \|\mathbf{T}_u \times \mathbf{T}_v\|\, du\, dv.$$

Because $A_n$ should approximate the surface area better and better as $n \to \infty$, we are led to formula (1) as a reasonable definition of $A(S)$.

**EXAMPLE 1** Let $D$ be the region determined by $0 \le \theta \le 2\pi$, $0 \le r \le 1$ and let the function $\mathbf{\Phi}\colon D \to \mathbb{R}^3$, defined by

$$x = r\cos\theta, \qquad y = r\sin\theta, \qquad z = r$$

be a parametrization of a cone $S$ (see Figure 7.3.8). Find its surface area.

SOLUTION In formula (3),

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r,$$

$$\frac{\partial(y,z)}{\partial(r,\theta)} = \begin{vmatrix} \sin\theta & r\cos\theta \\ 1 & 0 \end{vmatrix} = -r\cos\theta,$$

and

$$\frac{\partial(x,z)}{\partial(r,\theta)} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ 1 & 0 \end{vmatrix} = r\sin\theta,$$

so the area integrand is

$$\|\mathbf{T}_r \times \mathbf{T}_\theta\| = \sqrt{r^2 + r^2\cos^2\theta + r^2\sin^2\theta} = r\sqrt{2}.$$

Clearly, $\|\mathbf{T}_r \times \mathbf{T}_\theta\|$ vanishes for $r = 0$, but $\mathbf{\Phi}(0, \theta) = (0, 0, 0)$ for any $\theta$. Thus, $(0, 0, 0)$ is the only point where the surface is not regular. We have

$$\iint_D \|\mathbf{T}_r \times \mathbf{T}_\theta\| \, dr \, d\theta = \int_0^{2\pi} \int_0^1 \sqrt{2} r \, dr \, d\theta = \int_0^{2\pi} \frac{1}{2}\sqrt{2} \, d\theta = \sqrt{2}\pi.$$

To confirm that this is the area of $\mathbf{\Phi}(D)$, we must verify that $\mathbf{\Phi}$ is one-to-one (for points not on the boundary of $D$). Let $D^0$ be the set of $(r, \theta)$ with $0 < r < 1$ and $0 < \theta < 2\pi$. Hence, $D^0$ is $D$ without its boundary. To see that $\mathbf{\Phi}\colon D^0 \to \mathbb{R}^3$ is one-to-one, assume that $\mathbf{\Phi}(r, \theta) = \mathbf{\Phi}(r', \theta')$ for $(r, \theta)$ and $(r', \theta') \in D^0$. Then

$$r \cos\theta = r' \cos\theta', \qquad r \sin\theta = r' \sin\theta', \qquad r = r'.$$

From these equations it follows that $\cos\theta = \cos\theta'$ and $\sin\theta = \sin\theta'$. Thus, either $\theta = \theta'$ or $\theta = \theta' + 2\pi n$. But the second case is impossible for $n$ a nonzero integer, because both $\theta$ and $\theta'$ belong to the open interval $(0, 2\pi)$, and thus cannot be more than $2\pi$ radians apart. This proves that off the boundary, $\mathbf{\Phi}$ is one-to-one. (Is $\mathbf{\Phi}\colon D \to \mathbb{R}^3$ one-to-one?) In future examples, we shall not usually verify that the parametrization is one-to-one when it is intuitively clear. ▲

**EXAMPLE 2** A ***helicoid*** is defined by $\mathbf{\Phi}\colon D \to \mathbb{R}^3$, where

$$x = r\cos\theta, \qquad y = r\sin\theta, \qquad z = \theta$$

and $D$ is the region where $0 \le \theta \le 2\pi$ and $0 \le r \le 1$ (Figure 7.4.2). Find its area.

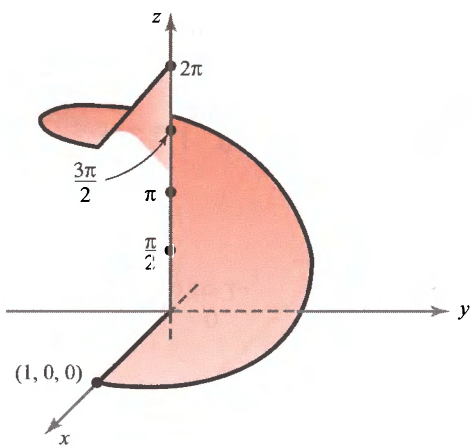



Figure 7.4.2 The helicoid $x = r\cos\theta, y = r\sin\theta, z = \theta$.

SOLUTION We compute $\partial(x, y)/\partial(r, \theta) = r$ as in Example 1, and

$$\frac{\partial(y, z)}{\partial(r, \theta)} = \begin{vmatrix} \sin\theta & r\cos\theta \\ 0 & 1 \end{vmatrix} = \sin\theta,$$

$$\frac{\partial(x, z)}{\partial(r, \theta)} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ 0 & 1 \end{vmatrix} = \cos\theta.$$

The area integrand is therefore $\sqrt{r^2+1}$, which never vanishes, so the surface is regular. The area of the helicoid is

$$\iint_D \|\mathbf{T}_r \times \mathbf{T}_\theta\|\, dr\, d\theta = \int_0^{2\pi}\int_0^1 \sqrt{r^2+1}\, dr\, d\theta = 2\pi \int_0^1 \sqrt{r^2+1}\, dr.$$

After a little computation (using the table of integrals), we find that this integral is equal to

$$\pi[\sqrt{2} + \log(1+\sqrt{2})]. \quad \blacktriangle$$

## Surface Area of a Graph

A surface $S$ given in the form $z = g(x, y)$, where $(x, y) \in D$, admits the parametrization

$$x = u, \qquad y = v, \qquad z = g(u, v)$$

for $(u, v) \in D$. When $g$ is of class $C^1$, this parametrization is smooth, and the formula for surface area reduces to

$$A(S) = \iint_D \left(\sqrt{\left(\frac{\partial g}{\partial x}\right)^2 + \left(\frac{\partial g}{\partial y}\right)^2 + 1}\right) dA, \tag{4}$$

after applying the formulas

$$\mathbf{T}_u = \mathbf{i} + \frac{\partial g}{\partial u}\mathbf{k}, \qquad \mathbf{T}_v = \mathbf{j} + \frac{\partial g}{\partial v}\mathbf{k},$$

and

$$\mathbf{T}_u \times \mathbf{T}_v = -\frac{\partial g}{\partial u}\mathbf{i} - \frac{\partial g}{\partial v}\mathbf{j} + \mathbf{k} = -\frac{\partial g}{\partial x}\mathbf{i} - \frac{\partial g}{\partial y}\mathbf{j} + \mathbf{k},$$

as noted in Example 4 of Section 7.3.

## Surfaces of Revolution

In most books on one-variable calculus, it is shown that the lateral surface area generated by revolving the graph of a function $y = f(x)$ about the $x$ axis is given by

$$A = 2\pi \int_a^b \left(|f(x)|\sqrt{1 + [f'(x)]^2}\right) dx. \tag{5}$$

If the graph is revolved about the $y$ axis, the surface area is

$$A = 2\pi \int_a^b \left(|x|\sqrt{1 + [f'(x)]^2}\right) dx. \tag{6}$$

We shall derive formula (5) by using the methods just developed; one can obtain formula (6) in a similar fashion (Exercise 10).

To derive formula (5) from formula (3), we must give a parametrization of $S$. Define the parametrization by

$$x = u, \qquad y = f(u)\cos v, \qquad z = f(u)\sin v$$

over the region $D$ given by

$$a \le u \le b, \qquad 0 \le v \le 2\pi.$$

This is indeed a parametrization of $S$, because for fixed $u$, the point

$$(u, f(u)\cos v, f(u)\sin v)$$

traces out a circle of radius $|f(u)|$ with the center $(u, 0, 0)$ (Figure 7.4.3).

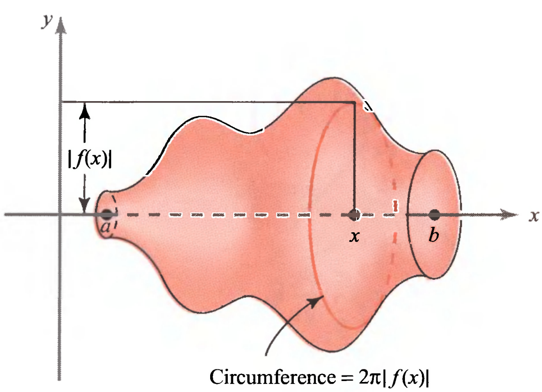



**Figure 7.4.3** The curve $y = f(x)$ rotated about the $x$ axis.

We calculate

$$\frac{\partial(x, y)}{\partial(u, v)} = -f(u)\sin v, \qquad \frac{\partial(y, z)}{\partial(u, v)} = f(u)f'(u), \qquad \frac{\partial(x, z)}{\partial(u, v)} = f(u)\cos v,$$

and so by formula (3)

$$\begin{aligned}
A(S) &= \iint_D \sqrt{\left[\frac{\partial(x,y)}{\partial(u,v)}\right]^2 + \left[\frac{\partial(y,z)}{\partial(u,v)}\right]^2 + \left[\frac{\partial(x,z)}{\partial(u,v)}\right]^2}\, du\, dv \\
&= \iint_D \sqrt{[f(u)]^2 \sin^2 v + [f(u)]^2[f'(u)]^2 + [f(u)]^2 \cos^2 v}\, du\, dv \\
&= \iint_D |f(u)|\sqrt{1+[f'(u)]^2}\, du\, dv \\
&= \int_a^b \int_0^{2\pi} |f(u)|\sqrt{1+[f'(u)]^2}\, dv\, du \\
&= 2\pi \int_a^b |f(u)|\sqrt{1+[f'(u)]^2}\, du,
\end{aligned}$$

which is formula (5).

If $S$ is the surface of revolution, then $2\pi|f(x)|$ is the circumference of the vertical cross section to $S$ at the point $x$ (Figure 7.4.3). Observe that we can write

$$2\pi \int_a^b |f(x)|\sqrt{1+[f'(x)]^2}\, dx = \int_{\mathbf{c}} 2\pi|f(x)|\, ds,$$

where the expression on the right is the path integral of $2\pi|f(x)|$ along the path given by $\mathbf{c}$: $[a,b] \to \mathbb{R}^2$, $t \mapsto (t, f(t))$. Therefore, *the lateral surface of a solid of revolution is obtained by integrating the cross-sectional circumference along the path that is the graph of the given function.*

## Historical Note

The most famous mathematician in ancient times was Archimedes. In addition to being an extraordinarily gifted mathematician, he was also an engineering genius on a scale never before seen and was greatly admired by his contemporaries and by later writers for his insights into mechanics. It was these talents that helped the people of the city of Syracuse in 214 B.C. to defend their city against the onslaught of the Roman legions under their commander Marcellus.

When the Romans besieged the city, they encountered an enemy whom Archimedes had supplied—totally unexpectedly—with powerful weapons, including artillery and burning mirrors, which, as legend has it, incinerated the Roman fleet.

The siege of Syracuse lasted two years, and the city finally fell as a result of acts of treason. In the aftermath of the assault, the old scientist was slain by a Roman soldier, even though the commander had asked his men to spare Archimedes' life. As the story goes, Archimedes was sitting in front of his house studying some geometric figures he had drawn in the sand. When a Roman soldier approached, Archimedes shouted out, "Don't disturb my figures!" The ruffian, feeling insulted, slew Archimedes.

To honor this great man, Marcellus erected a tomb for Archimedes on which, according to Archimedes's own wishes, were depicted a cone, a sphere, and a cylinder (Figure 7.4.4).

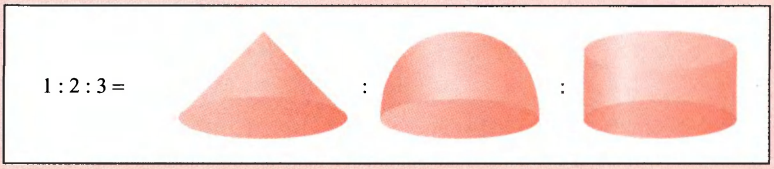



**Figure 7.4.4** Archimedes' theorem: The ratios of the volumes of a cone, a half ball, and a cylinder, all of the same height and radius, are 1:2:3.

Archimedes was incredibly proud of his calculation of the volume and surface area of the sphere, which justifiably were seen as truly outstanding accomplishments for their time. As in his works on centers of gravity, for which he provided no clear definition, Archimedes was able to compute the surface area of the sphere without having a clear definition of precisely what it was. However, as with many mathematical works, one knows the answer long before a proof or even the correct definition can be found.

The problem of properly defining surface areas is a difficult one. To Archimedes' credit, a careful theory of surface areas was not achieved until the twentieth century, after a long development that began in the seventeenth century with the discovery of calculus.

Christiaan Huygens (1629–1695) was the first person since Archimedes to give results on the areas of special surfaces beyond the sphere, and he obtained the areas of portions of surfaces of revolution, such as the paraboloid and hyperboloid.

The brilliant and prolific mathematician Leonhard Euler (1707–1783) presented the first fundamental work on the theory of surfaces in 1760 with *Recherches sur la courbure des surfaces*. However, it was in 1728, in a paper on shortest paths on surfaces, that Euler defined a surface as a graph $z = f(x,y)$. Euler was interested in studying the curvature of surfaces, and in 1771 he introduced the notion of the parametric surfaces that are described in this section.

After the rapid development of calculus in the early eighteenth century, formulas for the lengths of curves and areas of surfaces were developed. Although we do not know when all the area formulas presented in this section first appeared, they were certainly common by the end of the eighteenth century. The underlying concepts of the length of a curve and the area of a surface were understood intuitively before this time, and the use of formulas from calculus to compute areas was considered a great achievement.

Augustin-Louis Cauchy (1789–1857) was the first to take the step of defining the quantities of length and surface areas by integrals as we have presented in this book. The question of defining surface area independent of integrals was taken up somewhat later, but this posed many difficult problems that were not properly resolved until this century.

We end this section by describing the fascinating classic area problem of Plateau, which has enjoyed a long history in mathematics. The Belgian physicist Joseph Plateau (1801–1883) carried out many experiments from 1830 to 1869 on surface tension and capillary phenomena, experiments that had enormous impact at the time and were repeated by notable nineteenth-century physicists, such as Michael Faraday (1791–1867). The corresponding collection of mathematical problems relating to soap films was named in 1904 after Plateau by the great French mathematician Henri Lebesgue (1875–1941).

If a wire is dipped into a soap or glycerine solution, then one usually withdraws a soap film spanning the wire. Some examples are given in Figure 7.4.5, although readers might like to perform the experiment for themselves. Plateau raised the mathematical question: For a given boundary (wire), how does one prove the existence of such a surface (soap film) and how many surfaces can there be? The underlying physical principle is that nature tends to minimize area; that is, the surface that forms should be a surface of least area among all possible surfaces that have the given curve as their boundary. This again is another example of the action principle of Maupertuis and Leibniz (c.f. Section 3.3).

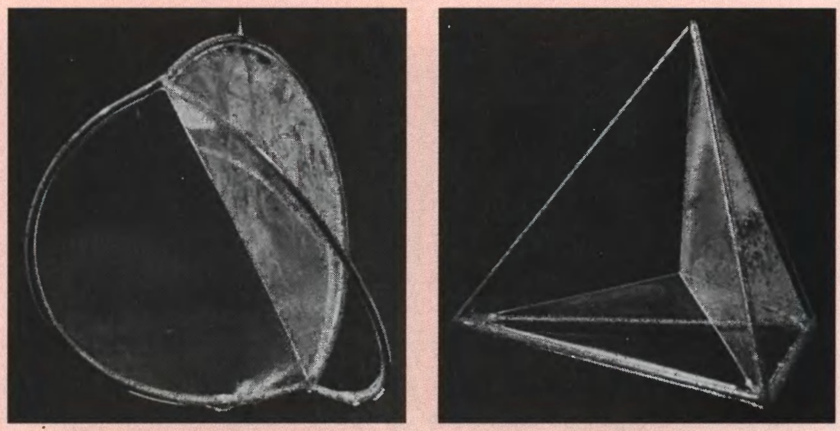


**Figure 7.4.5** Two soap films spanning wires.

For soap film surfaces that are disklike, the problem can be formulated in the following way. Let $D \subset \mathbb{R}^2$ be the unit disk defined to be the set $\{(x, y) \mid x^2 + y^2 \leq 1\}$ and let $\partial D$ be its boundary. Furthermore, suppose that the image $\Gamma$ of $\mathbf{c}\colon [0, 2\pi] \to \mathbb{R}^3$ is a simple closed curve, $\Gamma$ representing a wire in $\mathbb{R}^3$.

Let $\mathcal{S}$ be the set of all maps $\mathbf{\Phi}\colon D \to \mathbb{R}^3$ such that $\mathbf{\Phi}(\partial D) = \Gamma$, $\mathbf{\Phi}$ is of class $C^1$, and $\mathbf{\Phi}$ is one-to-one on $\partial D$. Each $\mathbf{\Phi} \in \mathcal{S}$ represents a parametric $C^1$ "disklike" surface "spanning" the wire $\Gamma$.

The soap films in Figure 7.4.5 are not disklike, but represent a system of multiple disklike surfaces. Figure 7.4.6 shows a contour that bounds two disklike surfaces and one nondisklike surface.


**Figure 7.4.6** Soap film surfaces; (b) and (c) are disklike surfaces, but (a) is not.

For each $\mathbf{\Phi} \in \mathcal{S}$, consider the area of the image surface, namely, $A(\mathbf{\Phi}) = \iint_D \|\mathbf{T}_u \times \mathbf{T}_v\| \, du \, dv$. This area is a function that assigns to each parametric surface its area. Plateau asked whether $A$ has a minimum on $\mathcal{S}$; that is, does there exist a $\mathbf{\Phi}_0$ such that $A(\mathbf{\Phi}_0) \leq A(\mathbf{\Phi})$ for all $\mathbf{\Phi} \in \mathcal{S}$? Unfortunately, the methods of this book are not adequate to solve this problem. We can tackle questions of finding minima of real-valued functions of several variables, but in no way can the set $\mathcal{S}$ be thought of as a region in $\mathbb{R}^n$ for *any n*!

In his own study of surfaces of least area, Weierstrass showed that if a minimum

$$\mathbf{\Phi}_0(u, v) = (x(u, v), y(u, v), z(u, v))$$

existed at all, it would have to satisfy (after suitable normalizations) the partial differential equations

(i) $\nabla^2 \mathbf{\Phi}_0 = 0$

(ii) $\dfrac{\partial \mathbf{\Phi}_0}{\partial u} \cdot \dfrac{\partial \mathbf{\Phi}_0}{\partial v} = 0$

(iii) $\left\| \dfrac{\partial \mathbf{\Phi}_0}{\partial u} \right\| = \left\| \dfrac{\partial \mathbf{\Phi}_0}{\partial v} \right\|$

where $\|\mathbf{w}\|$ denotes the "norm"or length of the vector $\mathbf{w}$. This example illustrates the intimate connections between problems of maxima and minima (the calculus of variations) and the subject of partial differential equations.

For well over 70 years, mathematicians such as Riemann, Weierstrass, H. A. Schwarz, Darboux, and Lebesgue puzzled over the challenge posed by Plateau. In 1931 the question was finally settled when Jesse Douglas showed that such a $\mathbf{\Phi}_0$ existed. However, many questions about soap films remain unsolved, and this area of research is still active today.[11]

## EXERCISES

**1.** Find the surface area of the unit sphere $S$ represented parametrically by $\mathbf{\Phi}\colon D \to S \subset \mathbb{R}^3$, where $D$ is the rectangle $0 \le \theta \le 2\pi$, $0 \le \phi \le \pi$ and $\mathbf{\Phi}$ is given by the equations

$$x = \cos\theta \sin\phi, \qquad y = \sin\theta \sin\phi, \qquad z = \cos\phi.$$

Note that we can represent the entire sphere parametrically, but we cannot represent it in the form $z = f(x, y)$.

**2.** In Exercise 1, what happens if we allow $\phi$ to vary from $-\pi/2$ to $\pi/2$? From 0 to $2\pi$? Why do we obtain different answers?

**3.** Find the area of the helicoid in Example 2 if the domain $D$ is $0 \le r \le 1$ and $0 \le \theta \le 3\pi$.

**4.** The torus $T$ can be represented parametrically by the function $\mathbf{\Phi}\colon D \to \mathbb{R}^3$, where $\mathbf{\Phi}$ is given by the coordinate functions $x = (R + \cos\phi)\cos\theta$, $y = (R + \cos\phi)\sin\theta$, $z = \sin\phi$; $D$ is the rectangle $[0, 2\pi] \times [0, 2\pi]$, that is, $0 \le \theta \le 2\pi$, $0 \le \phi \le 2\pi$; and $R > 1$ is fixed (see Figure 7.4.7). Show that $A(T) = (2\pi)^2 R$, first by using formula (3) and then by using formula (6).

**5.** Let $\mathbf{\Phi}(u, v) = (u - v, u + v, uv)$ and let $D$ be the unit disk in the $uv$ plane. Find the area of $\mathbf{\Phi}(D)$.

**6.** Find the area of the portion of the unit sphere that is cut out by the cone $z \ge \sqrt{x^2 + y^2}$ (see Exercise 1).

**7.** Show that the surface $x = 1/\sqrt{y^2 + z^2}$, where $1 \le x < \infty$, can be filled but not painted!

**8.** Find a parametrization of the surface $x^2 - y^2 = 1$, where $x > 0$, $-1 \le y \le 1$ and $0 \le z \le 1$. Use your answer to express the area of the surface as an integral.

[11] For more information on this fascinating subject, the reader may consult *The Parsimonious Universe: Shape and Form in the Natural World*, by S. Hildebrandt and A. Tromba, Springer-Verlag, New York/Heidelberg, 1995.

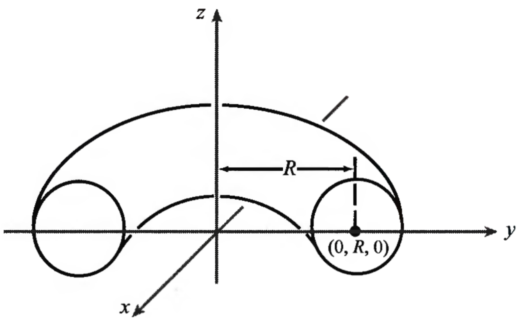



**Figure 7.4.7** A cross section of a torus.

**9.** Represent the ellipsoid $E$:

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$

parametrically and write out the integral for its surface area $A(E)$. (Do not evaluate the integral.)

**10.** Let the curve $y = f(x)$, $a \le x \le b$, be rotated about the $y$ axis. Show that the area of the surface swept out is given by equation (6); that is,

$$A = 2\pi \int_a^b |x|\sqrt{1+[f'(x)]^2}\,dx.$$

Interpret the formula geometrically using arc length and slant height.

**11.** Find the area of the surface obtained by rotating the curve $y = x^2$, $0 \le x \le 1$, about the $y$ axis.

**12.** Use formula (4) to compute the surface area of the cone in Example 1.

**13.** Find the area of the surface defined by $x + y + z = 1$, $x^2 + 2y^2 \le 1$.

**14.** Show that for the vectors $\mathbf{T}_u$ and $\mathbf{T}_v$, we have the formula

$$\|\mathbf{T}_u \times \mathbf{T}_v\| = \sqrt{\left[\frac{\partial(x,y)}{\partial(u,v)}\right]^2+\left[\frac{\partial(y,z)}{\partial(u,v)}\right]^2+\left[\frac{\partial(x,z)}{\partial(u,v)}\right]^2}.$$

**15.** Compute the area of the surface given by

$$x = r\cos\theta,\qquad y = 2r\cos\theta,\qquad z=\theta,\qquad 0\le r\le 1,\qquad 0\le\theta\le 2\pi.$$

Sketch.

**16.** Prove *Pappus' theorem*: Let $\mathbf{c}$: $[a, b] \to \mathbb{R}^2$ be a $C^1$ path whose image lies in the right half plane and is a simple closed curve. The area of the lateral surface generated by rotating the image of **c** about the $y$ axis is equal to $2\pi\bar{x}l(\mathbf{c})$, where $\bar{x}$ is the average value of the $x$ coordinates of points on **c** and $l(\mathbf{c})$ is the length of **c**. (See Exercises 8 to 11, Section 7.1, for a discussion of average values.)

**17.** The cylinder $x^2 + y^2 = x$ divides the unit sphere $S$ into two regions $S_1$ and $S_2$, where $S_1$ is inside the cylinder and $S_2$ outside. Find the ratio of areas $A(S_2)/A(S_1)$.

**18.** Suppose a surface $S$ that is the graph of a function $z = f(x, y)$, where $(x, y) \in D \subset \mathbb{R}^2$ can also be described as the set of $(x, y, z) \in \mathbb{R}^3$ with $F(x, y, z) = 0$ (a level surface). Derive a formula for $A(S)$ that involves only $F$.

**19.** Calculate the area of the frustum shown in Figure 7.4.8 using (a) geometry alone and, second, (b) a surface area formula.

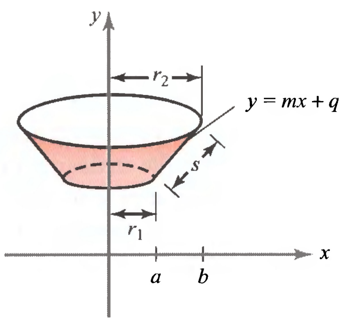



Figure 7.4.8 A line segment revolved around the $y$ axis becomes a frustum of a cone.

**20.** A cylindrical hole of radius 1 is bored through a solid ball of radius 2 to form a ring coupler, as shown in Figure 7.4.9. Find the volume and outer surface area of this coupler.

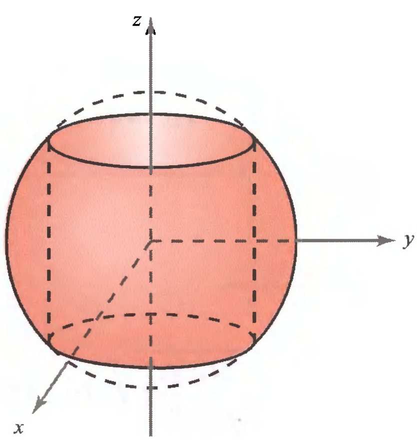



Figure 7.4.9 Find the outer surface area and volume of the shaded region.

**21.** Find the area of the graph of the function $f(x, y) = \frac{2}{3}(x^{3/2} + y^{3/2})$ that lies over the domain $[0, 1] \times [0, 1]$.

**22.** Express the surface area of the following graphs over the indicated region $D$ as a double integral. Do not evaluate.

(a) $(x + 2y)^2$; $D = [-1, 2] \times [0, 2]$
(b) $xy + x/(y + 1)$; $D = [1, 4] \times [1, 2]$
(c) $xy^3 e^{x^2 y^2}$; $D =$ unit circle centered at the origin
(d) $y^3 \cos^2 x$; $D =$ triangle with vertices $(-1, 1)$, $(0, 2)$, and $(1, 1)$

**23.** Show that the surface area of the upper hemisphere of radius $R$, $z = \sqrt{R^2 - x^2 - y^2}$, can be computed by formula (4), evaluated as an improper integral.

## 7.5 Integrals of Scalar Functions Over Surfaces

Now we are ready to define the integral of a *scalar* function $f$ *over a surface* $S$. This concept is a natural generalization of the area of a surface, which corresponds to the integral over $S$ of the scalar function $f(x, y, z) = 1$. This is quite analogous to considering the path integral as a generalization of arc length. In the next section we shall deal with the integral of a *vector* function $\mathbf{F}$ over a surface. These concepts will play a crucial role in the vector analysis treated in the final chapter.

Let us start with a surface $S$ parametrized by a mapping $\mathbf{\Phi}\colon D \to S \subset \mathbb{R}^3$, where $D$ is an elementary region, which we write as $\mathbf{\Phi}(u, v) = (x(u, v), y(u, v), z(u, v))$.

**DEFINITION: The Integral of a Scalar Function Over a Surface** If $f(x, y, z)$ is a real-valued continuous function defined on a parametrized surface $S$, we define the ***integral of*** $f$ ***over*** $S$ to be

$$\iint_S f(x, y, z)\, dS = \iint_S f\, dS = \iint_D f(\mathbf{\Phi}(u, v))\|\mathbf{T}_u \times \mathbf{T}_v\|\, du\, dv. \qquad (1)$$

Written out, equation (1) becomes

$$\iint_S f\, dS = \iint_D f(x(u, v), y(u, v), z(u, v)) \sqrt{\left[\frac{\partial(x, y)}{\partial(u, v)}\right]^2 + \left[\frac{\partial(y, z)}{\partial(u, v)}\right]^2 + \left[\frac{\partial(x, z)}{\partial(u, v)}\right]^2}\, du\, dv. \qquad (2)$$

Thus, if $f$ is identically 1, we recover the area formula (3) of Section 7.4. Like surface area, the surface integral is independent of the particular parametrization used. This will be discussed in Section 7.6.

We can gain some intuitive knowledge about this integral by considering it as a limit of sums. Let $D$ be a rectangle partitioned into $n^2$ rectangles $R_{ij}$ with areas $\Delta u\, \Delta v$. Let $S_{ij} = \mathbf{\Phi}(R_{ij})$ *be the portion of the surface* $\mathbf{\Phi}(D)$ corresponding to $R_{ij}$ (see Figure 7.5.1), and let $A(S_{ij})$ be the area of this portion of the surface. For large $n$, $f$ will be approximately constant on $S_{ij}$, and we form the sum

$$S_n = \sum_{i=0}^{n-1}\sum_{j=0}^{n-1} f(\mathbf{\Phi}(u_i, v_j))A(S_{ij}), \tag{3}$$

where $(u_i, v_j) \in R_{ij}$. From Section 7.4 we have a formula for $A(S_{ij})$:

$$A(S_{ij}) = \iint_{R_{ij}} \|\mathbf{T}_u \times \mathbf{T}_v\|\, du\, dv,$$

which, by the mean-value theorem for integrals, equals $\|\mathbf{T}_{u_i^*} \times \mathbf{T}_{v_j^*}\|\, \Delta u\, \Delta v$ for some point $(u_i^*, v_j^*)$ in $R_{ij}$. Hence, our sum becomes

$$S_n = \sum_{i=0}^{n-1}\sum_{j=0}^{n-1} f(\mathbf{\Phi}(u_i, v_j))\|\mathbf{T}_{u_i^*} \times \mathbf{T}_{v_j^*}\|\Delta u\, \Delta v,$$

which is an approximating sum for the last integral in formula (1). Therefore,

$$\operatorname*{limit}_{n\to\infty} S_n = \iint_S f\, dS.$$

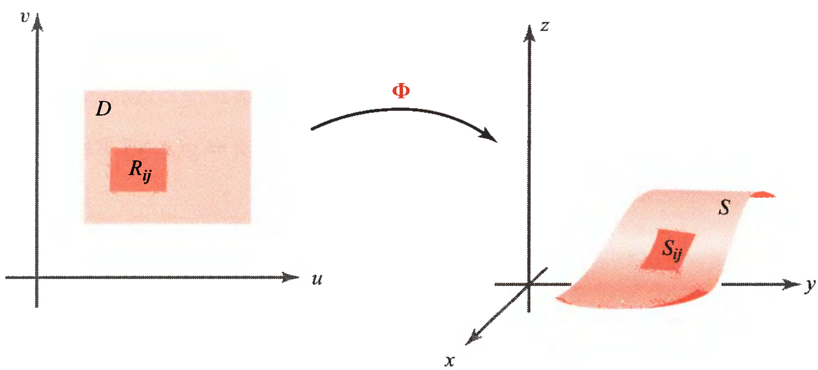



**Figure 7.5.1** $\mathbf{\Phi}$ takes a portion $R_{ij}$ of $D$ to a portion of $S$.

Note that each term in the sum in formula (3) is the value of $f$ at some point $\mathbf{\Phi}(u_i, v_j)$ times the area of $S_{ij}$. Compare this with the Riemann-sum interpretation of the path integral in Section 7.1.

If $S$ is a union of parametrized surfaces $S_i$, $i = 1, \dots, N$, that do not intersect except possibly along curves defining their boundaries, then the integral of $f$ over $S$ is defined by

$$\iint_S f\, dS = \sum_{i=1}^{N} \iint_{S_i} f\, dS,$$

as we should expect. For example, the integral over the surface of a cube may be expressed as the sum of the integrals over the six sides.

**EXAMPLE 1** Suppose a helicoid is described as in Example 2, Section 7.4, and let $f$ be given by $f(x, y, z) = \sqrt{x^2 + y^2 + 1}$. Find $\iint_S f\, dS$.

SOLUTION As in Examples 1 and 2 of Section 7.4,

$$\frac{\partial(x, y)}{\partial(r, \theta)} = r, \qquad \frac{\partial(y, z)}{\partial(r, \theta)} = \sin\theta, \qquad \frac{\partial(x, z)}{\partial(r, \theta)} = \cos\theta.$$

Also, $f(r\cos\theta, r\sin\theta, \theta) = \sqrt{r^2 + 1}$. Therefore,

$$\begin{aligned}\iint_S f(x, y, z)\, dS &= \iint_D f(\mathbf{\Phi}(r, \theta)) \|\mathbf{T}_r \times \mathbf{T}_\theta\|\, dr\, d\theta \\ &= \int_0^{2\pi} \int_0^1 \sqrt{r^2 + 1}\sqrt{r^2 + 1}\, dr\, d\theta = \int_0^{2\pi} \frac{4}{3}\, d\theta = \frac{8}{3}\pi. \quad \blacktriangle\end{aligned}$$

## Surface Integrals Over Graphs

Suppose $S$ is the graph of a $C^1$ function $z = g(x, y)$. Recall from Section 7.4 that we can parametrize $S$ by

$$x = u, \qquad y = v, \qquad z = g(u, v),$$

and that in this case

$$\|\mathbf{T}_u \times \mathbf{T}_v\| = \sqrt{1 + \left(\frac{\partial g}{\partial u}\right)^2 + \left(\frac{\partial g}{\partial v}\right)^2},$$

so

$$\iint_S f(x,y,z)\,dS = \iint_D f(x,y,g(x,y))\sqrt{1+\left(\frac{\partial g}{\partial x}\right)^2+\left(\frac{\partial g}{\partial y}\right)^2}\,dx\,dy. \quad (4)$$

**EXAMPLE 2** Let $S$ be the surface defined by $z = x^2 + y$, where $D$ is the region $0 \le x \le 1$, $-1 \le y \le 1$. Evaluate $\iint_S x\,dS$.

**SOLUTION** If we let $z = g(x,y) = x^2 + y$, formula (4) gives

$$\begin{aligned}\iint_S x\,dS &= \iint_D x\sqrt{1+\left(\frac{\partial g}{\partial x}\right)^2+\left(\frac{\partial g}{\partial y}\right)^2}\,dx\,dy = \int_{-1}^{1}\int_0^1 x\sqrt{1+4x^2+1}\,dx\,dy\\ &= \frac{1}{8}\int_{-1}^{1}\left[\int_0^1 (2+4x^2)^{1/2}(8x\,dx)\right]dy = \frac{2}{3}\cdot\frac{1}{8}\int_{-1}^{1}[(2+4x^2)^{3/2}]|_0^1\,dy\\ &= \frac{1}{12}\int_{-1}^{1}(6^{3/2}-2^{3/2})\,dy = \frac{1}{6}(6^{3/2}-2^{3/2}) = \sqrt{6}-\frac{\sqrt{2}}{3}\\ &= \sqrt{2}\left(\sqrt{3}-\frac{1}{3}\right). \quad \blacktriangle\end{aligned}$$

**EXAMPLE 3** Evaluate $\iint_S z^2\,dS$, where $S$ is the unit sphere $x^2+y^2+z^2=1$.

**SOLUTION** For this problem, it is convenient to use spherical coordinates and to represent the sphere parametrically by the equations $x = \cos\theta\sin\phi$, $y = \sin\theta\sin\phi$, $z = \cos\phi$, over the region $D$ in the $\theta\phi$ plane given by the inequalities $0 \le \phi \le \pi$, $0 \le \theta \le 2\pi$. From equation (1) we get

$$\iint_S z^2\,dS = \iint_D (\cos\phi)^2\|\mathbf{T}_\theta \times \mathbf{T}_\phi\|\,d\theta\,d\phi.$$

A little computation [use formula (2) of Section 7.4; see Exercise 6] shows that

$$\|\mathbf{T}_\theta \times \mathbf{T}_\phi\| = \sin\phi.$$

(Note that for $0 \le \phi \le \pi$, we have $\sin\phi \ge 0$). Thus,

$$\begin{aligned}\iint_S z^2\,dS &= \int_0^{2\pi}\int_0^{\pi}\cos^2\phi\sin\phi\,d\phi\,d\theta\\ &= \frac{1}{3}\int_0^{2\pi}[-\cos^3\phi]_0^{\pi}\,d\theta = \frac{2}{3}\int_0^{2\pi}d\theta = \frac{4\pi}{3}. \quad \blacktriangle\end{aligned}$$

This example also shows that on a sphere of radius $R$,

$$\iint_S f\, ds = \int_0^{2\pi} \int_0^{\pi} f(\phi, \theta) R^2 \sin\phi \, d\phi \, d\theta,$$

or, for short, the *area element on the sphere* is given by

$$dS = R^2 \sin\phi \, d\phi \, d\theta.$$

## Integrals Over Graphs

We now develop another formula for surface integrals when the surface can be represented as a graph. To do so, we let $S$ be the graph of $z = g(x, y)$ and consider formula (4). We claim that

$$\iint_S f(x, y, z)\, dS = \iint_D \frac{f(x, y, g(x, y))}{\cos\theta}\, dx\, dy, \tag{5}$$

where $\theta$ is the angle the normal to the surface makes with the unit vector $\mathbf{k}$ at the point $(x, y, g(x, y))$ (see Figure 7.5.2). Describing the surface by the equation $\phi(x, y, z) = z - g(x, y) = 0$, a normal vector $\mathbf{N}$ is $\nabla\phi$; that is,

$$\mathbf{N} = -\frac{\partial g}{\partial x}\mathbf{i} - \frac{\partial g}{\partial y}\mathbf{j} + \mathbf{k} \tag{6}$$

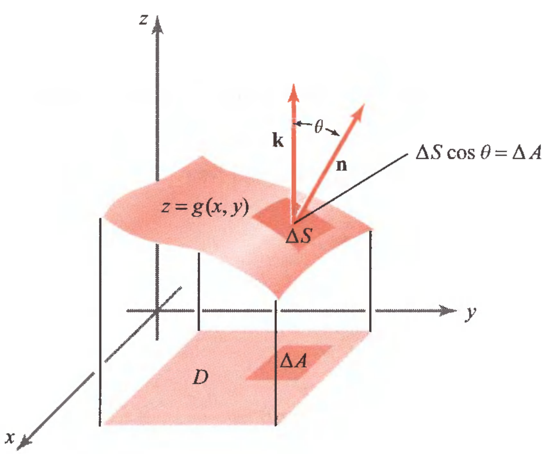



**Figure 7.5.2** The area of a patch of area $\Delta S$ over a patch $\Delta A$ is $\Delta S = \Delta A/\cos\theta$, where $\theta$ is the angle the unit normal $\mathbf{n}$ makes with $\mathbf{k}$.

[see Example 4 of Section 7.3, or recall that the normal to a surface with equation $g(x, y, z) =$ constant is given by $\nabla g$]. Thus,

$$\cos\theta = \frac{\mathbf{N}\cdot\mathbf{k}}{\|\mathbf{N}\|} = \frac{1}{\sqrt{(\partial g/\partial x)^2 + (\partial g/\partial y)^2 + 1}}.$$

Substitution of this formula into equation (4) gives equation (5). Note that $\cos\theta = \mathbf{n}\cdot\mathbf{k}$, where $\mathbf{n} = \mathbf{N}/\|\mathbf{N}\|$ is the unit normal. Thus, we can write

$$dS = \frac{dx\,dy}{\mathbf{n}\cdot\mathbf{k}}$$

The result is, in fact, obvious geometrically, for if a small rectangle in the $xy$ plane has area $\Delta A$, then the area of the portion above it on the surface is $\Delta S = \Delta A/\cos\theta$ (Figure 7.5.2). This intuitive approach can help us to remember formula (5) and to apply it in problems.

**EXAMPLE 4** Compute $\iint_S x\,dS$, where $S$ is the triangle with vertices (1, 0, 0), (0, 1, 0), (0, 0, 1) (see Figure 7.5.3).

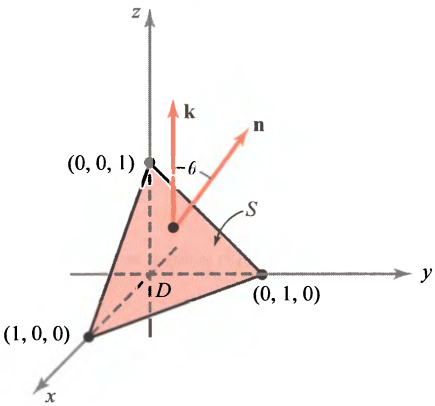



**Figure 7.5.3** In computing a specific surface integral, one finds a formula for the unit normal $\mathbf{n}$ and computes the angle $\theta$ in preparation for formula (5).

**SOLUTION** This surface is the plane described by the equation $x + y + z = 1$. Because the surface is a plane, the angle $\theta$ is constant and a unit normal vector is $\mathbf{n} = (1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3})$. Thus, $\cos\theta = \mathbf{n}\cdot\mathbf{k} = 1/\sqrt{3}$, and by equation (5),

$$\iint_S x\,dS = \sqrt{3}\iint_D x\,dx\,dy,$$

where $D$ is the domain in the $xy$ plane. But

$$\sqrt{3}\iint_D x\,dx\,dy = \sqrt{3}\int_0^1\int_0^{1-x} x\,dy\,dx = \sqrt{3}\int_0^1 x(1-x)\,dx = \frac{\sqrt{3}}{6}. \quad \blacktriangle$$

Integrals of functions over surfaces are useful for computing the mass of a surface when the mass density function $m$ is known. The total mass of a surface with mass density (per unit area) $m$ is given by

$$M(S) = \iint_S m(x, y, z)\, dS. \tag{7}$$

**EXAMPLE 5** Let $\mathbf{\Phi}: D \to \mathbb{R}^3$ be the parametrization of the helicoid $S = \mathbf{\Phi}(D)$ of Example 2 of Section 7.4. Recall that $\mathbf{\Phi}(r, \theta) = (r\cos\theta, r\sin\theta, \theta)$, where $0 \le \theta \le 2\pi$, and $0 \le r \le 1$. Suppose $S$ has a mass density at $(x, y, z) \in S$ equal to twice the distance of $(x, y, z)$ from the central axis (see Figure 7.4.2), that is, $m(x, y, z) = 2\sqrt{x^2 + y^2} = 2r$, in the cylindrical coordinate system. Find the total mass of the surface.

SOLUTION Applying formula (7),

$$M(S) = \iint_S 2\sqrt{x^2 + y^2}\, dS = \iint_D 2r\, dS = \iint_D 2r \|\mathbf{T}_r \times \mathbf{T}_\theta\|\, dr\, d\theta.$$

From Example 2 of Section 7.4, we see that $\|\mathbf{T}_r \times \mathbf{T}_\theta\| = \sqrt{1 + r^2}$. Thus,

$$\begin{aligned} M(S) &= \iint_D 2r\sqrt{1 + r^2}\, dr\, d\theta = \int_0^{2\pi} \int_0^1 2r\sqrt{1 + r^2}\, dr\, d\theta \\ &= \int_0^{2\pi} \left[\frac{2}{3}(1 + r^2)^{3/2}\right]_0^1 d\theta = \int_0^{2\pi} \frac{2}{3}(2^{3/2} - 1)\, d\theta = \frac{4\pi}{3}(2^{3/2} - 1). \quad \blacktriangle \end{aligned}$$

## EXERCISES

1. Compute $\iint_S xy\, dS$, where $S$ is the surface of the tetrahedron with sides $z = 0$, $y = 0$, $x + z = 1$, and $x = y$.

2. Evaluate $\iint_S xyz\, dS$, where $S$ is the triangle with vertices $(1, 0, 0)$, $(0, 2, 0)$, and $(0, 1, 1)$.

3. Evaluate $\iint_S z\, dS$, where $S$ is the upper hemisphere of radius $a$, that is, the set of $(x, y, z)$ with $z = \sqrt{a^2 - x^2 - y^2}$.

4. Evaluate $\iint_S (x + y + z)\, dS$, where $S$ is the boundary of the unit ball $B$; that is, $S$ is the set of $(x, y, z)$ with $x^2 + y^2 + z^2 = 1$. (HINT: Use the symmetry of the problem.)

5. (a) Compute the area of the portion of the cone $x^2 + y^2 = z^2$ with $z \ge 0$ that is inside the sphere $x^2 + y^2 + z^2 = 2Rz$, where $R$ is a positive constant.
   (b) What is the area of that portion of the sphere that is inside the cone?

6. Verify that in spherical coordinates, on a sphere of radius $R$,

$$\|\mathbf{T}_\phi \times \mathbf{T}_\theta\|\, d\phi\, d\theta = R^2 \sin\phi\, d\phi\, d\theta.$$

7. Evaluate $\iint_S z\, dS$, where $S$ is the surface $z = x^2 + y^2$, $x^2 + y^2 \le 1$.

**8.** Evaluate the surface integral $\iint_S z^2\, dS$, where $S$ is the boundary of the cube $C = [-1, 1] \times [-1, 1] \times [-1, 1]$. (HINT: Do each face separately and add the results.)

**9.** Find the mass of a spherical surface $S$ of radius $R$ such that at each point $(x, y, z) \in S$ the mass density is equal to the distance of $(x, y, z)$ to some fixed point $(x_0, y_0, z_0) \in S$.

**10.** A metallic surface $S$ is in the shape of a hemisphere $z = \sqrt{R^2 - x^2 - y^2}$, where $(x, y)$ satisfies $0 \leq x^2 + y^2 \leq R^2$. The mass density at $(x, y, z) \in S$ is given by $m(x, y, z) = x^2 + y^2$. Find the total mass of $S$.

**11.** Let $S$ be the sphere of radius $R$.

(a) Argue by symmetry that

$$\iint_S x^2\, dS = \iint_S y^2\, dS = \iint_S z^2\, dS.$$

(b) Use this fact and some clever thinking to evaluate, with very little computation, the integral

$$\iint_S x^2\, dS.$$

(c) Does this help in Exercise 10?

**12.** (a) Use Riemann sums to justify the formula

$$\frac{1}{A(S)} \iint_S f(x, y, z)\, dS$$

for the *average value* of $f$ over the surface $S$.

(b) In Example 3 of this section, show that the average of $f(x, y, z) = z^2$ over the sphere is $1/3$.

(c) Define the ***center of gravity*** $(\bar{x}, \bar{y}, \bar{z})$ of a surface $S$ to be such that $\bar{x}$, $\bar{y}$, and $\bar{z}$ are the average values of the $x$, $y$, and $z$ coordinates on $S$. Show that the center of gravity of the triangle in Example 4 of this section is $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$.

**13.** Find the $x$, $y$, and $z$ coordinates of the center of gravity of the octant of the solid sphere of radius $R$ and centered at the origin determined by $x \geq 0$, $y \geq 0$, $z \geq 0$. (HINT: Write this octant as a parametrized surface—see Example 3 of this section and Exercise 12.)

**14.** Find the $z$ coordinate of the center of gravity (the average $z$ coordinate) of the surface of a hemisphere ($z \leq 0$) with radius $r$ (see Exercise 12). Argue by symmetry that the average $x$ and $y$ coordinates are both zero.

**15.** Let $\Phi: D \subset \mathbb{R}^2 \to \mathbb{R}^3$ be a parametrization of a surface $S$ defined by

$$x = x(u, v), \qquad y = y(u, v), \qquad z = z(u, v).$$

(a) Let

$$\frac{\partial \mathbf{\Phi}}{\partial u} = \left(\frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u}\right) \quad \text{and} \quad \frac{\partial \mathbf{\Phi}}{\partial v} = \left(\frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v}\right),$$

that is, $\partial \mathbf{\Phi}/\partial u = \mathbf{T}_u$ and $\partial \mathbf{\Phi}/\partial v = \mathbf{T}_v$, and set

$$E = \left\|\frac{\partial \mathbf{\Phi}}{\partial u}\right\|^2, \qquad F = \frac{\partial \mathbf{\Phi}}{\partial u} \cdot \frac{\partial \mathbf{\Phi}}{\partial v}, \qquad G = \left\|\frac{\partial \mathbf{\Phi}}{\partial v}\right\|^2.$$

Show that

$$\sqrt{EG - F^2} = \|\mathbf{T}_u \times \mathbf{T}_v\|,$$

and that the surface area of $S$ is

$$A(S) = \iint_D \sqrt{EG - F^2}\, du\, dv.$$

In this notation, how can we express $\iint_S f\, dS$ for a general function of $f$?

(b) What does the formula for $A(S)$ become if the vectors $\partial \mathbf{\Phi}/\partial u$ and $\partial \mathbf{\Phi}/\partial v$ are orthogonal?

(c) Use parts (a) and (b) to compute the surface area of a sphere of radius $a$.

**16.** ***Dirichlet's functional*** for a parametrized surface $\mathbf{\Phi}\colon D \to \mathbb{R}^3$ is defined by[12]

$$J(\mathbf{\Phi}) = \frac{1}{2}\iint_D \left(\left\|\frac{\partial \mathbf{\Phi}}{\partial u}\right\|^2 + \left\|\frac{\partial \mathbf{\Phi}}{\partial v}\right\|^2\right) du\, dv.$$

Use Exercise 15 to argue that the area $A(\mathbf{\Phi}) \le J(\mathbf{\Phi})$ and equality holds if

$$\text{(a)}\ \left\|\frac{\partial \mathbf{\Phi}}{\partial u}\right\|^2 = \left\|\frac{\partial \mathbf{\Phi}}{\partial v}\right\|^2 \quad \text{and} \quad \text{(b)}\ \frac{\partial \mathbf{\Phi}}{\partial u} \cdot \frac{\partial \mathbf{\Phi}}{\partial v} = 0.$$

Compare these equations with Exercise 15 and the remarks at the end of Section 7.4. A parametrization $\mathbf{\Phi}$ that satisfies conditions (a) and (b) is said to be ***conformal***.

**17.** Let $D \subset \mathbb{R}^2$ and $\mathbf{\Phi}\colon D \to \mathbb{R}^2$ be a smooth function $\mathbf{\Phi}(u, v) = (x(u, v), y(u, v))$ satisfying conditions (a) and (b) of Exercise 16 and assume that

$$\det \begin{bmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\[2ex] \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{bmatrix} > 0.$$

[12]Dirichlet's functional played a major role in the mathematics of the nineteenth century. The mathematician Georg Friedrich Bernhard Riemann (1826–1866) used it to develop his complex function theory and to give a proof of the famous Riemann mapping theorem. Today it is still used extensively as a tool in the study of partial differential equations.

Show that $x$ and $y$ satisfy the ***Cauchy–Riemann equations*** $\partial x/\partial u = \partial y/\partial v$, $\partial x/\partial v = -\partial y/\partial u$. Conclude that $\nabla^2\mathbf{\Phi} = 0$ (i.e., each component of $\mathbf{\Phi}$ is harmonic).

**18.** Let $S$ be a sphere of radius $r$ and $\mathbf{p}$ be a point inside or outside the sphere (but not on it). Show that

$$\iint_S \frac{1}{\|\mathbf{x}-\mathbf{p}\|}\, dS = \begin{cases} 4\pi r & \text{if } \ \mathbf{p} \text{ is inside } S \\ 4\pi r^2/d & \text{if } \ \mathbf{p} \text{ is outside } S, \end{cases}$$

where $d$ is the distance from $\mathbf{p}$ to the center of the sphere and the integration is over the sphere.

**19.** Find the surface area of that part of the cylinder $x^2 + z^2 = a^2$ that is inside the cylinder $x^2 + y^2 = 2ay$ and also in the positive octant ($x \geq 0, y \geq 0, z \geq 0$). Assume $a > 0$.

**20.** Let a surface $S$ be defined implicitly by $F(x, y, z) = 0$ for $(x, y)$ in a domain $D$ of $\mathbb{R}^2$. Show that

$$\iint_S \left|\frac{\partial F}{\partial z}\right| dS = \iint_D \sqrt{\left(\frac{\partial F}{\partial x}\right)^2 + \left(\frac{\partial F}{\partial y}\right)^2 + \left(\frac{\partial F}{\partial z}\right)^2}\, dx\, dy.$$

Compare with Exercise 18 of Section 7.4.

# 7.6 Surface Integrals of Vector Fields

The goal of this section is to develop the notion of the integral of a vector field over a surface. Recall that the definition of the line integral of a vector field was motivated by the fundamental physical notion of *work*. Similarly, there is a basic physical notion of *flux* that motivates the definition of the surface integral of a vector field.

For example, if the vector field is the velocity field of a fluid (perhaps the velocity field of a flowing river), and one puts an imagined mathematical surface into the fluid, one can ask: "What is the rate at which fluid is crossing the given surface (measured in, say, cubic meters per second)?" The answer is given by the surface integral of the fluid velocity vector field over the surface.

We shall come back to the physical interpretation shortly and reconcile it with the formal definition that we give first.

## Definition of the Surface Integral

We now define the integral of a vector field, denoted $\mathbf{F}$ over a surface $S$. We first give the definition and later in this section give its physical interpretation. This can also be used as a *motivation* for the definition if the reader so desires. Also, we shall start with a parametrized surface $\mathbf{\Phi}$ and later study the question of independence of parametrization.

**DEFINITION: The Surface Integral of Vector Fields** Let $\mathbf{F}$ be a vector field defined on $S$, the image of a parametrized surface $\boldsymbol{\Phi}$. The ***surface integral*** of $\mathbf{F}$ over $\boldsymbol{\Phi}$, denoted by

$$\iint_{\boldsymbol{\Phi}} \mathbf{F} \cdot d\mathbf{S},$$

is defined by (see Figure 7.6.1))

$$\iint_{\boldsymbol{\Phi}} \mathbf{F} \cdot d\mathbf{S} = \iint_{D} \mathbf{F} \cdot (\mathbf{T}_u \times \mathbf{T}_v)\, du\, dv.$$

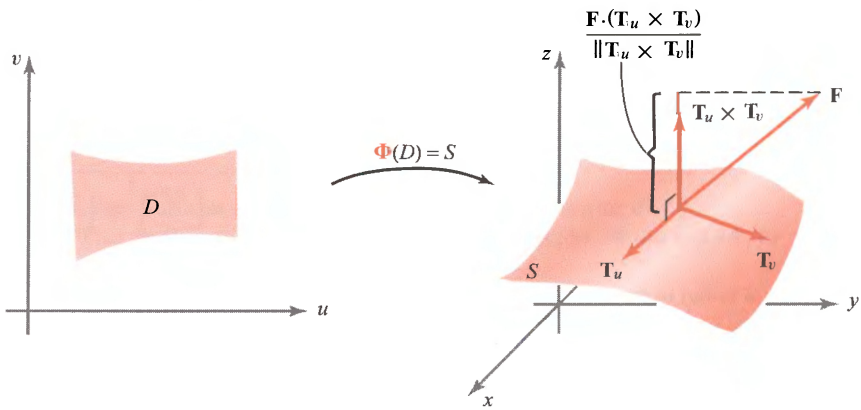



Figure 7.6.1 The geometric significance of $\mathbf{F} \cdot (\mathbf{T}_u \times \mathbf{T}_v)$.

**EXAMPLE 1** Let $D$ be the rectangle in the $\theta\phi$ plane defined by

$$0 \le \theta \le 2\pi, \qquad 0 \le \phi \le \pi,$$

and let the surface $S$ be defined by the parametrization $\boldsymbol{\Phi}\colon D \to \mathbb{R}^3$ given by

$$x = \cos\theta \sin\phi, \qquad y = \sin\theta \sin\phi, \qquad z = \cos\phi.$$

(Thus, $\theta$ and $\phi$ are the angles of spherical coordinates, and $S$ is the unit sphere parametrized by $\boldsymbol{\Phi}$.) Let $\mathbf{r}$ be the position vector $\mathbf{r}(x, y, z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$. Compute $\iint_{\boldsymbol{\Phi}} \mathbf{r} \cdot d\mathbf{S}$.

SOLUTION First we find

$$\mathbf{T}_\theta = (-\sin\phi \sin\theta)\mathbf{i} + (\sin\phi \cos\theta)\mathbf{j}$$

$$\mathbf{T}_\phi = (\cos\theta \cos\phi)\mathbf{i} + (\sin\theta \cos\phi)\mathbf{j} - (\sin\phi)\mathbf{k},$$

and hence

$$\mathbf{T}_\theta \times \mathbf{T}_\phi = (-\sin^2\phi \cos\theta)\mathbf{i} - (\sin^2\phi \sin\theta)\mathbf{j} - (\sin\phi \cos\phi)\mathbf{k}.$$

Then we evaluate

$$\begin{aligned}\mathbf{r}\cdot(\mathbf{T}_\theta\times\mathbf{T}_\phi) &= (x\mathbf{i}+y\mathbf{j}+z\mathbf{k})\cdot(\mathbf{T}_\theta\times\mathbf{T}_\phi)\\ &= [(\cos\theta\sin\phi)\mathbf{i}+(\sin\theta\sin\phi)\mathbf{j}+(\cos\phi)\mathbf{k}]\\ &\quad\cdot(-\sin\phi)[(\sin\phi\cos\theta)\mathbf{i}+(\sin\phi\sin\theta)\mathbf{j}+(\cos\phi)\mathbf{k}]\\ &= (-\sin\phi)(\sin^2\phi\cos^2\theta+\sin^2\phi\sin^2\theta+\cos^2\phi) = -\sin\phi.\end{aligned}$$

Thus,

$$\iint_{\Phi}\mathbf{r}\cdot d\mathbf{S} = \iint_D -\sin\phi\, d\phi\, d\theta = \int_0^{2\pi}(-2)\,d\theta = -4\pi. \qquad \blacktriangle$$

## Orientation

An analogy can be drawn between the surface integral $\iint_{\Phi}\mathbf{F}\cdot d\mathbf{S}$ and the line integral $\int_{\mathbf{c}}\mathbf{F}\cdot d\mathbf{s}$. Recall that the line integral is an oriented integral. We needed the notion of orientation of a curve to extend the definition of $\int_{\mathbf{c}}\mathbf{F}\cdot d\mathbf{s}$ to line integrals $\int_C\mathbf{F}\cdot d\mathbf{s}$ over oriented curves. We extend the definition of $\iint_{\Phi}\mathbf{F}\cdot d\mathbf{S}$ to *oriented surfaces* in a similar fashion; that is, given a surface $S$ parametrized by a mapping $\mathbf{\Phi}$, we want to define $\iint_S\mathbf{F}\cdot d\mathbf{S} = \iint_{\Phi}\mathbf{F}\cdot d\mathbf{S}$ and show that it is independent of the parametrization, except possibly for the sign. To accomplish this, we need the notion of orientation of a surface.

**DEFINITION: Oriented Surfaces** An ***oriented surface*** is a two-sided surface with one side specified as the ***outside*** or ***positive side***; we call the other side the ***inside*** or ***negative side***.[13] At each point $(x, y, z)\in S$ there are two unit normal vectors $\mathbf{n}_1$ and $\mathbf{n}_2$, where $\mathbf{n}_1 = -\mathbf{n}_2$ (see Figure 7.6.2). Each of these two normals can be associated with one side of the surface. Thus, to specify a side of a surface $S$, at each point we choose a unit normal vector $\mathbf{n}$ that points away from the positive side of $S$ at that point.

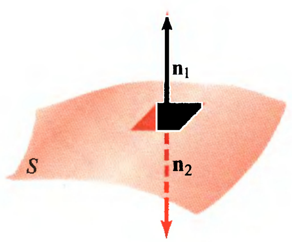



Figure 7.6.2 The two possible unit normals to a surface at a point.

[13] We use the term "side" in an intuitive sense. This concept can be developed rigorously, but this will not be done here. Also, the choice of the side to be named the "outside" is often dictated by the surface itself, as, for example, is the case with a sphere. In other cases, the naming is somewhat arbitrary (see the piece of surface depicted in Figure 7.6.2, for instance).

This definition assumes that our surface does have two sides. In fact, this is necessary, because there are examples of surfaces with only one side! The first known example of such a surface was the Möbius strip (named after the German mathematician and astronomer A. F. Möbius, who, along with the mathematician J. B. Listing, discovered it in 1858). Pictures of such a surface are given in Figures 7.6.3 and 7.6.4. At each point of $M$ there are two unit normals, $\mathbf{n}_1$ and $\mathbf{n}_2$. However, $\mathbf{n}_1$ does not determine a unique side of $M$, and neither does $\mathbf{n}_2$. To see this intuitively, we can slide $\mathbf{n}_2$ around the closed curve $C$ (Figure 7.6.3). When $\mathbf{n}_2$ returns to a fixed point $p$ on $C$ it will coincide with $\mathbf{n}_1$, showing that both $\mathbf{n}_1$ and $\mathbf{n}_2$ point away from the same side of $M$ and, consequently, that $M$ has only one side.

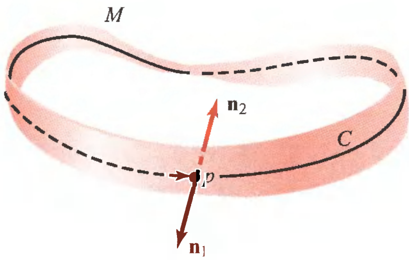



**Figure 7.6.3** The Möbius strip: Slide $\mathbf{n}_2$ around $C$ once; when $\mathbf{n}_2$ returns to its initial point, it will coincide with $\mathbf{n}_1 = -\mathbf{n}_2$.

Figure 7.6.4 is a Möbius strip as drawn by the well-known twentieth-century mathematician and artist M. C. Escher. It depicts ants crawling along the Möbius band. After one trip around the band (without crossing an edge) they end up on the "opposite side" of the surface.

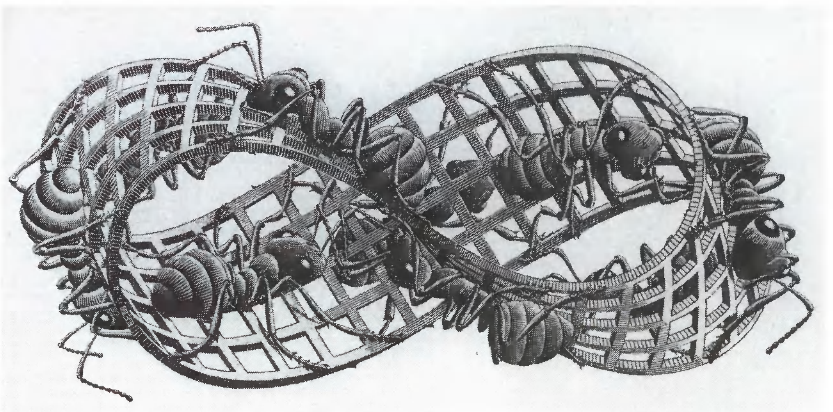


**Figure 7.6.4** Ants walking on a Möbius strip.

Let $\mathbf{\Phi}\colon D \to \mathbb{R}^3$ be a parametrization of an oriented surface $S$ and suppose $S$ is regular at $\mathbf{\Phi}(u_0, v_0)$, $(u_0, v_0) \in D$; thus, the vector $(\mathbf{T}_{u_0} \times \mathbf{T}_{v_0})/\|\mathbf{T}_{u_0} \times \mathbf{T}_{v_0}\|$ is defined. If $\mathbf{n}(\mathbf{\Phi}(u_0, v_0))$ denotes the unit normal to $S$ at $\mathbf{\Phi}(u_0, v_0)$, it follows that

$$(\mathbf{T}_{u_0} \times \mathbf{T}_{v_0})/\|\mathbf{T}_{u_0} \times \mathbf{T}_{v_0}\| = \pm\mathbf{n}(\mathbf{\Phi}(u_0, v_0)).$$

The parametrization $\mathbf{\Phi}$ is said to be ***orientation-preserving*** if we have the + sign; that is, if $(\mathbf{T}_u \times \mathbf{T}_v)/\|\mathbf{T}_u \times \mathbf{T}_v\| = \mathbf{n}(\mathbf{\Phi}(u, v))$ at all $(u, v) \in D$ for which $S$ is smooth at $\mathbf{\Phi}(u, v)$. In other words, $\mathbf{\Phi}$ is orientation-preserving if the vector $\mathbf{T}_u \times \mathbf{T}_v$ points to the outside of the surface. If $\mathbf{T}_u \times \mathbf{T}_v$ points to the inside of the surface at all points $(u, v) \in D$ for which $S$ is regular at $\mathbf{\Phi}(u, v)$, then $\mathbf{\Phi}$ is said to be ***orientation-reversing***. Using the preceding notation, this condition corresponds to the choice $(\mathbf{T}_u \times \mathbf{T}_v)/\|\mathbf{T}_u \times \mathbf{T}_v\| = -\mathbf{n}(\mathbf{\Phi}(u, v))$.

It follows from this discussion that the Möbius band $M$ cannot be parametrized by a single parametrization for which $\mathbf{n} = \mathbf{T}_u \times \mathbf{T}_v \neq \mathbf{0}$ and $\mathbf{n}$ is continuous over the whole surface[14] (if there were such a parameterization, then $M$ would indeed have two sides, one determined by $\mathbf{n}$ and one determined by $-\mathbf{n}$). The sphere in Example 1 can be parametrized by a single parametrization, but not by one that is everywhere one-to-one—see the discussion at the beginning of Section 7.4.

*Thus, any one-to-one parametrized surface for which $\mathbf{T}_u \times \mathbf{T}_v$ never vanishes can be considered as an oriented surface with a positive side determined by the direction of $\mathbf{T}_u \times \mathbf{T}_v$.*

**EXAMPLE 2** We can give the unit sphere $x^2 + y^2 + z^2 = 1$ in $\mathbb{R}^3$ (Figure 7.6.5) an orientation by selecting the unit vector $\mathbf{n}(x, y, z) = \mathbf{r}$, where $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, which points to the outside of the surface. This choice corresponds to our intuitive notion of outside for the sphere.

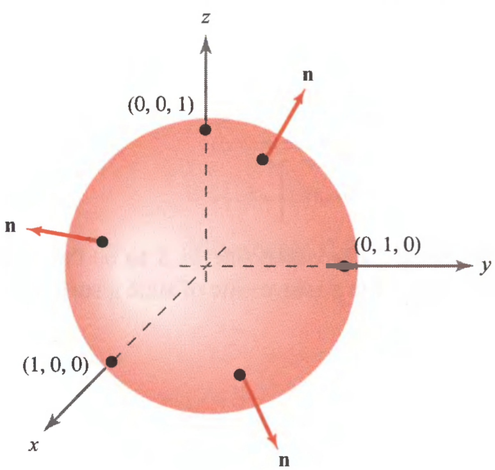



**Figure 7.6.5** The unit sphere oriented by its outward normal $\mathbf{n}$.

Now that the sphere $S$ is an oriented surface, consider the parametrization $\mathbf{\Phi}$ of $S$ given in Example 1. The cross product of the tangent vectors $\mathbf{T}_\theta$ and $\mathbf{T}_\phi$—that is,

[14] There is a single parametrization obtained by cutting a strip of paper, twisting it, and gluing the ends, but it produces a discontinuous $\mathbf{n}$ on the surface.

a normal to $S$ is given by

$$(-\sin\phi)[(\cos\theta\sin\phi)\mathbf{i} + (\sin\theta\sin\phi)\mathbf{j} + (\cos\phi)\mathbf{k}] = -\mathbf{r}\sin\phi.$$

Because $-\sin\phi \le 0$ for $0 \le \phi \le \pi$, this normal vector points inward from the sphere. Thus, the given parametrization $\mathbf{\Phi}$ is *orientation-reversing*. By swapping the order of $\theta$ and $\phi$, we would get an orientation-preserving parametrization. ▲

## Orientation and the Vector Surface Element of a Sphere

Consider the sphere of radius $R$, namely, $x^2 + y^2 + z^2 = R^2$. It is standard practice to orient the sphere with the *outward unit normal.* In terms of the position vector $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, the outward unit normal is given by

$$\mathbf{n} = \frac{\mathbf{r}}{R}.$$

The order of spherical coordinates that goes along with this orientation, as is evident from Example 2, is given by the order $(\phi, \theta)$. The computation in Example 2 shows that the surface-area element is then given by

$$d\mathbf{S} = \mathbf{n}\cdot(\mathbf{T}_\phi \times \mathbf{T}_\theta)\,d\phi\,d\theta = \mathbf{r}R\sin\phi\,d\phi\,d\theta = \mathbf{n}R^2\sin\phi\,d\phi\,d\theta.$$

## The Orientation of Graphs

The next example discusses the orientation conventions for graphs. We shall compute the area element on graphs later in this section.

**EXAMPLE 3** Let $S$ be a surface described by $z = g(x, y)$. As in equation (6), Section 7.5, there are two unit normal vectors to $S$ at $(x_0, y_0, g(x_0, y_0))$, namely, $\pm\mathbf{n}$, where

$$\mathbf{n} = \frac{-\dfrac{\partial g}{\partial x}(x_0, y_0)\mathbf{i} - \dfrac{\partial g}{\partial y}(x_0, y_0)\mathbf{j} + \mathbf{k}}{\sqrt{\left[\dfrac{\partial g}{\partial x}(x_0, y_0)\right]^2 + \left[\dfrac{\partial g}{\partial y}(x_0, y_0)\right]^2 + 1}}.$$

We can orient all such surfaces by taking the positive side of $S$ to be the side away from which $\mathbf{n}$ points (Figure 7.6.6). Thus, the positive side of such a surface is

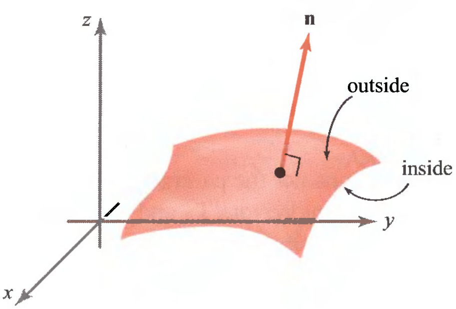



**Figure 7.6.6** $\mathbf{n}$ points away from the outside of the surface.

determined by the unit normal $\mathbf{n}$ with positive $\mathbf{k}$ component—that is, it is *upward-pointing*. If we parametrize this surface by $\mathbf{\Phi}(u, v) = (u, v, g(u, v))$, then $\mathbf{\Phi}$ will be orientation-preserving. ▲

## Independence of Parametrization

We now state without proof a theorem showing that the integral over an oriented surface is independent of the parametrization. The proof of this theorem is analogous to that of Theorem 1 (Section 7.2); the heart of the proof is again the change of variables formula—this time applied to double integrals.

**THEOREM 4: Independence of Surface Integrals on Parametrizations** Let $S$ be an oriented surface and let $\mathbf{\Phi}_1$ and $\mathbf{\Phi}_2$ be two regular orientation-preserving parametrizations, with $\mathbf{F}$ a continuous vector field defined on $S$. Then

$$\iint_{\mathbf{\Phi}_1} \mathbf{F} \cdot d\mathbf{S} = \iint_{\mathbf{\Phi}_2} \mathbf{F} \cdot d\mathbf{S}.$$

If $\mathbf{\Phi}_1$ is orientation-preserving and $\mathbf{\Phi}_2$ orientation-reversing, then

$$\iint_{\mathbf{\Phi}_1} \mathbf{F} \cdot d\mathbf{S} = -\iint_{\mathbf{\Phi}_2} \mathbf{F} \cdot d\mathbf{S}.$$

If $f$ is a real-valued continuous function defined on $S$, and if $\mathbf{\Phi}_1$ and $\mathbf{\Phi}_2$ are parametrizations of $S$, then

$$\iint_{\mathbf{\Phi}_1} f\,dS = \iint_{\mathbf{\Phi}_2} f\,dS.$$

Note that if $f = 1$, we obtain

$$A(S) = \iint_{\mathbf{\Phi}_1} dS = \iint_{\mathbf{\Phi}_2} dS,$$

thus showing that area is independent of parametrization.

We can therefore unambiguously use the notation

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_{\mathbf{\Phi}} \mathbf{F} \cdot d\mathbf{S}$$

(or a sum of such integrals, if $S$ is a union of parametrized surfaces that intersect only along their boundary curves) where $\mathbf{\Phi}$ is an orientation-preserving parametrization.

Theorem 4 guarantees that the value of the integral does not depend on the selection of $\mathbf{\Phi}$.

## Relation with Scalar Integrals

Recall from formula (1) of Section 7.2 that a line integral $\int_{\mathbf{c}} \mathbf{F} \cdot d\mathbf{s}$ can be thought of as the path integral of the tangential component of $\mathbf{F}$ along $\mathbf{c}$ (although for the case in which $\mathbf{c}$ intersects itself, the integral obtained is technically not a path integral). A similar situation holds for surface integrals, because we are assuming that the mappings $\mathbf{\Phi}$ defining the surface $S$ are one-to-one except perhaps on the boundary of $D$, which can be ignored for the purposes of integration. Thus, in defining integrals over surfaces, we assume in this book that the surfaces are nonintersecting.

For an oriented smooth surface $S$ and an orientation-preserving parametrization $\mathbf{\Phi}$ of $S$, we can express $\iint_S \mathbf{F} \cdot d\mathbf{S}$ as an integral of a real-valued function $f$ over the surface. Let $\mathbf{n} = (\mathbf{T}_u \times \mathbf{T}_v)/\|\mathbf{T}_u \times \mathbf{T}_v\|$ be the unit normal pointing to the outside of $S$. Then

$$\begin{aligned}\iint_S \mathbf{F} \cdot d\mathbf{S} &= \iint_{\mathbf{\Phi}} \mathbf{F} \cdot d\mathbf{S} = \iint_D \mathbf{F} \cdot (\mathbf{T}_u \times \mathbf{T}_v)\, du\, dv \\ &= \iint_D \mathbf{F} \cdot \left(\frac{\mathbf{T}_u \times \mathbf{T}_v}{\|\mathbf{T}_u \times \mathbf{T}_v\|}\right) \|\mathbf{T}_u \times \mathbf{T}_v\|\, du\, dv \\ &= \iint_D (\mathbf{F} \cdot \mathbf{n}) \|\mathbf{T}_u \times \mathbf{T}_v\|\, du\, dv = \iint_S (\mathbf{F} \cdot \mathbf{n})\, dS = \iint_S f\, dS,\end{aligned}$$

where $f = \mathbf{F} \cdot \mathbf{n}$. We have thus proved the following theorem.

**THEOREM 5** $\iint_S \mathbf{F} \cdot d\mathbf{S}$, the surface integral of $\mathbf{F}$ over $S$, is equal to the integral of the normal component of $\mathbf{F}$ over the surface. In short,

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_S \mathbf{F} \cdot \mathbf{n}\, dS.$$

The observation in Theorem 5 can often save computational effort, as Example 4 demonstrates.

## The Physical Interpretation of Surface Integrals

The geometric and physical significance of the surface integral can be understood by expressing it as a limit of Riemann sums. For simplicity, we assume $D$ is a rectangle. Fix a parametrization $\mathbf{\Phi}$ of $S$ that preserves orientation and partition the region $D$ into $n^2$ pieces $D_{ij}$, $0 \le i \le n-1$, $0 \le j \le n-1$. We let $\Delta u$ denote the length of the horizontal side of $D_{ij}$ and $\Delta v$ denote the length of the vertical side of $D_{ij}$. Let $(u, v)$ be a point in $D_{ij}$, and $(x, y, z) = \mathbf{\Phi}(u, v)$ the corresponding point on the surface. We consider the parallelogram with sides $\Delta u\, \mathbf{T}_u$ and $\Delta v\, \mathbf{T}_v$ lying in the plane tangent to $S$ at $(x, y, z)$ and the parallelepiped formed by $\mathbf{F}$, $\Delta u\, \mathbf{T}_u$, and $\Delta v\, \mathbf{T}_v$. The volume of the parallelepiped is the absolute value of the triple product

$$\mathbf{F}\cdot(\Delta u\,\mathbf{T}_u \times \Delta v\,\mathbf{T}_v) = \mathbf{F}\cdot(\mathbf{T}_u \times \mathbf{T}_v)\,\Delta u\,\Delta v.$$

The vector $\mathbf{T}_u \times \mathbf{T}_v$ is normal to the surface at $(x, y, z)$ and points away from the outside of the surface. Thus, the number $\mathbf{F}\cdot(\mathbf{T}_u \times \mathbf{T}_v)$ is positive when the parallelepiped lies on the outside of the surface (Figure 7.6.7).

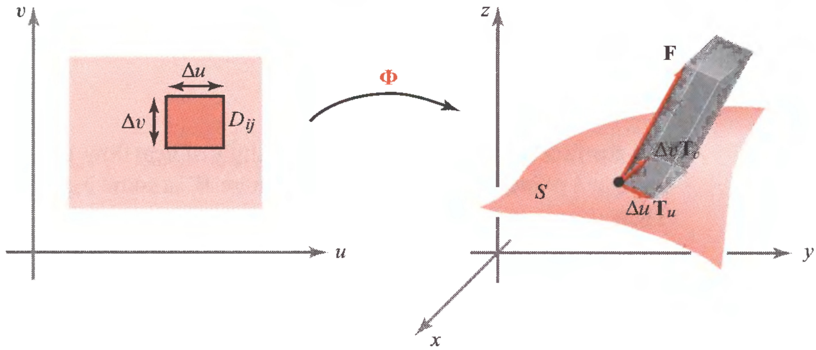



**Figure 7.6.7** $\mathbf{F}\cdot(\mathbf{T}_u \times \mathbf{T}_v) > 0$ when the parallelpiped formed by $\Delta v\,\mathbf{T}_v$, $\Delta u\,\mathbf{T}_u$, and $\mathbf{F}$ lies to the "outside" of the surface $S$.

In general, the parallelepiped lies on that side of the surface away from which $\mathbf{F}$ is pointing. If we think of $\mathbf{F}$ as the velocity field of a fluid, $\mathbf{F}(x, y, z)$ is pointing in the direction in which fluid is moving across the surface near $(x, y, z)$. Moreover, the number

$$|\mathbf{F}\cdot(\mathbf{T}_u\,\Delta u \times \mathbf{T}_v\,\Delta v)|$$

measures the amount of fluid that passes through the tangent parallelogram per unit time. Because the sign of $\mathbf{F}\cdot(\Delta u\,\mathbf{T}_u \times \Delta v\,\mathbf{T}_v)$ is positive if the vector $\mathbf{F}$ is pointing outward at $(x, y, z)$ and negative if $\mathbf{F}$ is pointing inward, $\sum_{i,j}\mathbf{F}\cdot(\mathbf{T}_u \times \mathbf{T}_v)\,\Delta u\,\Delta v$ is an approximate measure of the net quantity of fluid to flow outward across the surface per unit time. (Remember that "outward" or "inward" depends on our choice of parametrization. Figure 7.6.8 illustrates $\mathbf{F}$ directed outward and inward, given $\mathbf{T}_u$ and $\mathbf{T}_v$.) Hence, *the integral* $\iint_S \mathbf{F}\cdot d\mathbf{S}$ *is the net quantity of fluid to flow across the surface per unit time, that is, the rate of fluid flow*. This integral is also called the ***flux*** of $\mathbf{F}$ across the surface.

In the case where $\mathbf{F}$ represents an electric or a magnetic field, $\iint_S \mathbf{F}\cdot d\mathbf{S}$ is also commonly known as the flux. The reader may be familiar with physical laws (such as *Faraday's law*) that relate flux of a vector field to a circulation (or current) in a bounding loop. This is the historical and physical basis of Stokes's theorem, which we will discuss in Section 8.2. The corresponding principle in fluid mechanics is called *Kelvin's circulation theorem*.

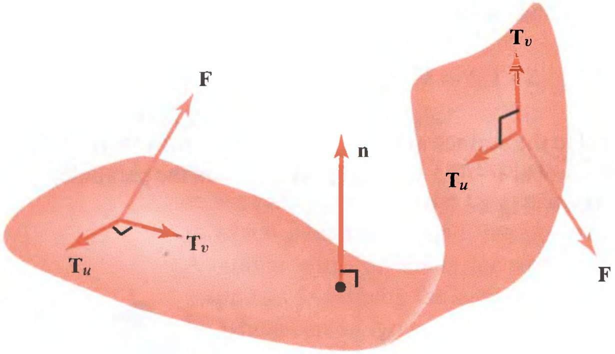



Figure 7.6.8 When $\mathbf{F}\cdot(\mathbf{T}_u \times \mathbf{T}_v) > 0$ (left), $\mathbf{F}$ points outward; when $\mathbf{F}\cdot(\mathbf{T}_u \times \mathbf{T}_v) < 0$ (right), $\mathbf{F}$ points inward.

Surface integrals also apply to the study of heat flow. Let $T(x, y, z)$ be the temperature at a point $(x, y, z) \in W \subset \mathbb{R}^3$, where $W$ is some region and $T$ is a $C^1$ function. Then

$$\nabla T = \frac{\partial T}{\partial x}\mathbf{i} + \frac{\partial T}{\partial y}\mathbf{j} + \frac{\partial T}{\partial z}\mathbf{k}$$

represents the temperature gradient, and heat "flows" with the vector field $-k\,\nabla T = \mathbf{F}$, where $k$ is a positive constant (see Section 8.5). Therefore, $\iint_S \mathbf{F}\cdot d\mathbf{S}$ is the total rate of heat flow or flux across the surface $S$.

EXAMPLE 4 Suppose a temperature function is given in $\mathbb{R}^3$ by the formula $T(x, y, z) = x^2 + y^2 + z^2$, and let $S$ be the unit sphere $x^2 + y^2 + z^2 = 1$ oriented with the outward normal (see Example 2). Find the heat flux across the surface $S$ if $k = 1$.

SOLUTION We have

$$\mathbf{F} = -\nabla T(x, y, z) = -2x\mathbf{i} - 2y\mathbf{j} - 2z\mathbf{k}.$$

On $S$, the vector $\mathbf{n}(x, y, z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ is the unit "outward" normal to $S$ at $(x, y, z)$, and $f(x, y, z) = \mathbf{F}\cdot\mathbf{n} = -2x^2 - 2y^2 - 2z^2 = -2$ is the normal component of $\mathbf{F}$. From Theorem 5 we can see that the surface integral of $\mathbf{F}$ is equal to the integral of its normal component $f = \mathbf{F}\cdot\mathbf{n}$ over $S$. Thus,

$$\iint_S \mathbf{F}\cdot d\mathbf{S} = \iint_S f\,dS = -2\iint_S dS = -2A(S) = -2(4\pi) = -8\pi.$$

The flux of heat is directed toward the center of the sphere (why toward?). Clearly, our observation that $\iint_S \mathbf{F}\cdot d\mathbf{S} = \iint_S f\,dS$ has saved us considerable computational time.

In this example, $\mathbf{F}(x, y, z) = -2x\mathbf{i} - 2y\mathbf{j} - 2z\mathbf{k}$ could also represent an electric field, in which case $\iint_S \mathbf{F}\cdot d\mathbf{S} = -8\pi$ would be the electric flux across $S$. ▲

**EXAMPLE 5** **Gauss' Law** There is an important physical law, due to the great mathematician and physicist K. F. Gauss, that relates the flux of an electric field $\mathbf{E}$ over a "closed" surface $S$ (for example, a sphere or an ellipsoid) to the net charge $Q$ enclosed by the surface, namely (in suitable units),

$$\iint_S \mathbf{E} \cdot d\mathbf{S} = Q \tag{1}$$

(see Figure 7.6.9). Gauss' law will be discussed in detail in Chapter 8. This law is analogous to Ampère's law (see Example 12, Section 7.2).

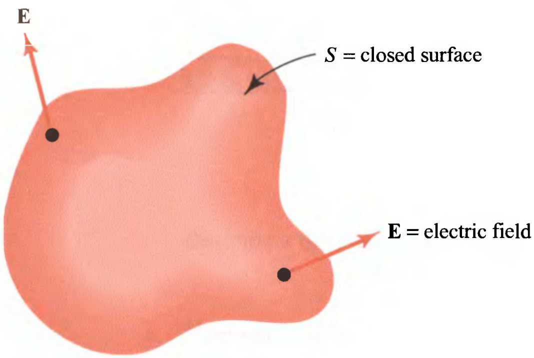



**Figure 7.6.9** Gauss' law: $\iint_S \mathbf{E} \cdot d\mathbf{S} = Q$, where $Q$ is the net charge inside $S$.

Suppose that $\mathbf{E} = E\mathbf{n}$; that is, $\mathbf{E}$ is a constant scalar multiple of the unit normal to $S$. Then Gauss' law, equation (1) in Example 5, becomes

$$\iint_S \mathbf{E} \cdot d\mathbf{S} = \iint_S E\, dS = E \iint_S dS = Q$$

because $E = \mathbf{E} \cdot \mathbf{n}$. Thus,

$$E = \frac{Q}{A(S)}. \tag{2}$$

In the case where $S$ is the sphere of radius $R$, equation (2) becomes

$$E = \frac{Q}{4\pi R^2} \tag{3}$$

(see Figure 7.6.10).

Now suppose that $\mathbf{E}$ arises from an isolated point charge, $Q$. From symmetry it is reasonable that $\mathbf{E} = E\mathbf{n}$, where $\mathbf{n}$ is the unit normal to any sphere centered at $Q$. Hence, equation (3) holds. Consider a second point charge, $Q_0$, located at a distance

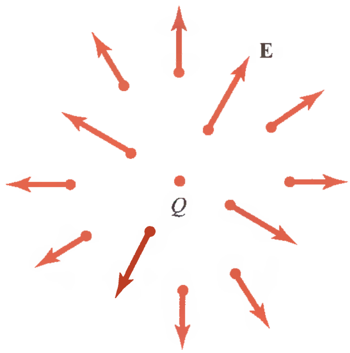



**Figure 7.6.10** The field $\mathbf{E}$ due to a point charge $Q$ is $\mathbf{E} = Q\mathbf{n}/4\pi R^2$.

$R$ from $Q$. The force $\mathbf{F}$ that acts on this second charge, $Q_0$, is given by

$$\mathbf{F} = \mathbf{E}Q_0 = EQ_0\mathbf{n} = \frac{QQ_0}{4\pi R^2}\mathbf{n}.$$

If $F$ is the magnitude of $\mathbf{F}$, we have

$$F = \frac{QQ_0}{4\pi R^2},$$

which is ***Coulomb's law*** for the force between two point charges.[15] ▲

## Surface Integrals Over Graphs

Finally, let us derive the surface-integral formulas for vector fields $\mathbf{F}$ over surfaces $S$ that are graphs of functions. Consider the surface $S$ described by $z = g(x, y)$, where $(x, y) \in D$, where $S$ is oriented with the upward pointing unit normal:

$$\mathbf{n} = \frac{-\dfrac{\partial g}{\partial x}\mathbf{i} - \dfrac{\partial g}{\partial y}\mathbf{j} + \mathbf{k}}{\sqrt{\left(\dfrac{\partial g}{\partial x}\right)^2 + \left(\dfrac{\partial g}{\partial y}\right)^2 + 1}}.$$

We have seen that we can parametrize $S$ by $\mathbf{\Phi}\colon D \to \mathbb{R}^3$ given by $\mathbf{\Phi}(x, y) = (x, y, g(x, y))$. In this case, $\iint_S \mathbf{F} \cdot d\mathbf{S}$ can be written in a particularly simple form. We have

$$\mathbf{T}_x = \mathbf{i} + \frac{\partial g}{\partial x}\mathbf{k}, \qquad \mathbf{T}_y = \mathbf{j} + \frac{\partial g}{\partial y}\mathbf{k}.$$

[15] Sometimes one sees the formula $F = (1/4\pi\varepsilon_0)QQ_0/R^2$. The extra constant $\varepsilon_0$ appears when MKS units are used for measuring charge. We are using CGS, or Gaussian, units.

Thus, $\mathbf{T}_x \times \mathbf{T}_y = -(\partial g/\partial x)\mathbf{i} - (\partial g/\partial y)\mathbf{j} + \mathbf{k}$. If $\mathbf{F} = F_1\mathbf{i} + F_2\mathbf{j} + F_3\mathbf{k}$ is a continuous vector field, then we get

**The Surface Integral of a Vector Field Over a Graph $S$**

$$\begin{aligned}\iint_S \mathbf{F}\cdot d\mathbf{S} &= \iint_D \mathbf{F}\cdot(\mathbf{T}_x \times \mathbf{T}_y)\,dx\,dy \\ &= \iint_D \left[F_1\left(-\frac{\partial g}{\partial x}\right) + F_2\left(-\frac{\partial g}{\partial y}\right) + F_3\right]dx\,dy.\end{aligned} \tag{4}$$

**EXAMPLE 6** The equations

$$z = 12, \qquad x^2 + y^2 \le 25$$

describe a disk of radius 5 lying in the plane $z = 12$. Suppose $\mathbf{r}$ is the vector field

$$\mathbf{r}(x, y, z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}.$$

Compute $\iint_S \mathbf{r}\cdot d\mathbf{S}$.

SOLUTION We shall do this in three ways. First, we have $\partial z/\partial x = \partial z/\partial y = 0$, because $z = 12$ is constant on the disk, so

$$\mathbf{r}(x, y, z)\cdot(\mathbf{T}_x \times \mathbf{T}_y) = \mathbf{r}(x, y, z)\cdot(\mathbf{i}\times\mathbf{j}) = \mathbf{r}(x, y, z)\cdot\mathbf{k} = z.$$

Using the original definition at the beginning of this section, the integral becomes

$$\iint_S \mathbf{r}\cdot d\mathbf{S} = \iint_D z\,dx\,dy = \iint_D 12\,dx\,dy = 12(\text{area of } D) = 300\pi.$$

A second solution: Because the disk is parallel to the $xy$ plane, the outward unit normal is $\mathbf{k}$. Hence, $\mathbf{n}(x, y, z) = \mathbf{k}$ and $\mathbf{r}\cdot\mathbf{n} = z$. However, $\|\mathbf{T}_x \times \mathbf{T}_y\| = \|\mathbf{k}\| = 1$, and so we know from the discussion preceding Theorem 5 that

$$\iint_S \mathbf{r}\cdot d\mathbf{S} = \iint_S \mathbf{r}\cdot\mathbf{n}\;dS = \iint_S z\;dS = \iint_D 12\,dx\,dy = 300\pi.$$

Third, we may solve this problem by using formula (4) directly, with $g(x, y) = 12$ and $D$ the disk $x^2 + y^2 \leq 25$:

$$\iint_S \mathbf{r} \cdot d\mathbf{S} = \iint_D (x \cdot 0 + y \cdot 0 + 12)\, dx\, dy = 12(\text{area of } D) = 300\pi. \quad \blacktriangle$$

## Summary: Formulas for Surface Integrals

1. **Parametrized Surface: $\Phi(u, v)$**

   (a) Integral of a scalar function $f$:

   $$\iint_S f\, dS = \iint_D f(\Phi(u, v))\|\mathbf{T}_u \times \mathbf{T}_v\|\, du\, dv$$

   (b) Scalar surface element:

   $$dS = \|\mathbf{T}_u \times \mathbf{T}_v\|\, du\, dv$$

   (c) Integral of a vector field $\mathbf{F}$:

   $$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_D \mathbf{F} \cdot (\mathbf{T}_u \times \mathbf{T}_v)\, du\, dv$$

   (d) Vector surface element:

   $$d\mathbf{S} = (\mathbf{T}_u \times \mathbf{T}_v)\, du\, dv = \mathbf{n}\, dS$$

2. **Graph**: $z = g(x, y)$

   (a) Integral of a scalar function $f$:

   $$\iint_S f\, dS = \iint_D \frac{f(x, y, g(x, y))}{\cos\theta}\, dx\, dy$$

   (b) Scalar surface element:

   $$dS = \frac{dx\, dy}{\cos\theta} = \sqrt{\left(\frac{\partial g}{\partial x}\right)^2 + \left(\frac{\partial g}{\partial y}\right)^2 + 1}\, dx\, dy,$$

   where $\cos\theta = \mathbf{n} \cdot \mathbf{k}$, and $\mathbf{n}$ is a unit normal vector to the surface.

(c) Integral of a vector field $\mathbf{F}$:

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_D \left( -F_1 \frac{\partial g}{\partial x} - F_2 \frac{\partial g}{\partial y} + F_3 \right) dx\, dy$$

(d) Vector surface element:

$$d\mathbf{S} = \mathbf{n} \cdot dS = \left( -\frac{\partial g}{\partial x}\mathbf{i} - \frac{\partial g}{\partial y}\mathbf{j} + \mathbf{k} \right) dx\, dy$$

3. **Sphere:** $x^2 + y^2 + z^2 = R^2$

(a) Scalar surface element:

$$dS = R^2 \sin\phi\, d\phi\, d\theta$$

(b) Vector surface element:

$$d\mathbf{S} = (x\mathbf{i} + y\mathbf{j} + z\mathbf{k}) R \sin\phi\, d\phi\, d\theta = \mathbf{r} R \sin\phi\, d\phi\, d\theta = \mathbf{n} R^2 \sin\phi\, d\phi\, d\theta$$

## EXERCISES

**1.** Let the temperature of a point in $\mathbb{R}^3$ be given by $T(x, y, z) = 3x^2 + 3z^2$. Compute the heat flux across the surface $x^2 + z^2 = 2$, $0 \le y \le 2$, if $k = 1$.

**2.** Compute the heat flux across the unit sphere $S$ if $T(x, y, z) = x$. Can you interpret your answer physically?

**3.** Let $S$ be the closed surface that consists of the hemisphere $x^2 + y^2 + z^2 = 1$, $z \ge 0$, and its base $x^2 + y^2 \le 1$, $z = 0$. Let $\mathbf{E}$ be the electric field defined by $\mathbf{E}(x, y, z) = 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}$. Find the electric flux across $S$. (Hint: Break $S$ into two pieces $S_1$ and $S_2$ and evaluate $\iint_{S_1} \mathbf{E} \cdot d\mathbf{S}$ and $\iint_{S_2} \mathbf{E} \cdot d\mathbf{S}$ separately.)

**4.** Let the velocity field of a fluid be described by $\mathbf{F} = \sqrt{y}\mathbf{i}$ (measured in meters per second). Compute how many cubic meters of fluid per second are crossing the surface $x^2 + z^2 = 1$, $0 \le y \le 1$, $0 \le x \le 1$.

**5.** Evaluate $\iint_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S}$, where $S$ is the surface $x^2 + y^2 + 3z^2 = 1$, $z \le 0$, and $\mathbf{F}$ is the vector field $\mathbf{F} = y\mathbf{i} - x\mathbf{j} + zx^3y^2\mathbf{k}$. (Let $\mathbf{n}$, the unit normal, be upward pointing.)

**6.** Evaluate $\iint_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S}$ where $\mathbf{F} = (x^2 + y - 4)\mathbf{i} + 3xy\mathbf{j} + (2xz + z^2)\mathbf{k}$ and $S$ is the surface $x^2 + y^2 + z^2 = 16$, $z \ge 0$. (Let $\mathbf{n}$, the unit normal, be upward pointing.)

**7.** Calculate the integral $\iint_S \mathbf{F} \cdot d\mathbf{S}$, where $S$ is the entire surface of the solid half ball $x^2 + y^2 + z^2 \leq 1, z \geq 0$, and $\mathbf{F} = (x + 3y^5)\mathbf{i} + (y + 10xz)\mathbf{j} + (z - xy)\mathbf{k}$. (Let $S$ be oriented by the outward pointing normal.)

**8.*** A restaurant is being built on the side of a mountain. The architect's plans are shown in Figure 7.6.11.

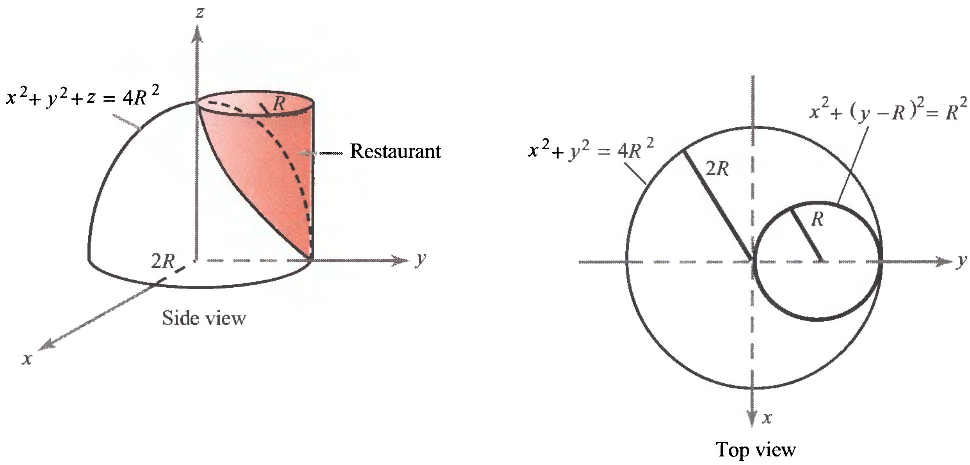



**Figure 7.6.11** Restaurant plans.

(a) The vertical curved wall of the restaurant is to be built of glass. What will be the surface area of this wall?

(b) To be large enough to be profitable, the consulting engineer informs the developer that the volume of the interior must exceed $\pi R^4/2$. For what $R$ does the proposed structure satisfy this requirement?

(c) During a typical summer day, the environs of the restaurant are subject to a temperature field given by

$$T(x, y, z) = 3x^2 + (y - R)^2 + 16z^2.$$

A heat flux density $\mathbf{V} = -k\,\nabla T$ ($k$ is a constant depending on the grade of insulation to be used) through all sides of the restaurant (including the top and the contact with the hill) produces a heat flux. What is this total heat flux? (Your answer will depend on $R$ and $k$.)

**9.** Find the flux of the vector field $\mathbf{V}(x, y, z) = 3xy^2\mathbf{i} + 3x^2y\mathbf{j} + z^3\mathbf{k}$ out of the unit sphere.

**10.** Evaluate the surface integral $\iint_S \mathbf{F} \cdot \mathbf{n}\, dA$, where $\mathbf{F}(x, y, z) = \mathbf{i} + \mathbf{j} + z(x^2 + y^2)^2\mathbf{k}$ and $S$ is the surface of the cylinder $x^2 + y^2 \leq 1, 0 \leq z \leq 1$.

---

*The solution to this problem may be somewhat time-consuming.

**11.** Let $S$ be the surface of the unit sphere. Let $\mathbf{F}$ be a vector field and $F_r$ its radial component. Prove that

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \int_{\theta=0}^{2\pi} \int_{\phi=0}^{\pi} F_r \sin\phi \, d\phi \, d\theta.$$

What is the corresponding formula for real-valued functions $f$?

**12.** Prove the following mean-value theorem for surface integrals: If $\mathbf{F}$ is a continuous vector field, then

$$\iint_S \mathbf{F} \cdot \mathbf{n} \, dS = [\mathbf{F}(\mathrm{Q}) \cdot \mathbf{n}(\mathrm{Q})] A(S)$$

for some point $\mathrm{Q} \in S$, where $A(S)$ is the area of $S$. [HINT: Prove it for real functions first, by reducing the problem to one of a double integral: Show that if $g \geq 0$, then

$$\iint_D fg \, dA = f(\mathrm{Q}) \iint_D g \, dA$$

for some $\mathrm{Q} \in D$ (do it by considering $(\iint_D fg \, dA)/(\iint_D g \, dA)$ and using the intermediate value theorem).]

**13.** Work out a formula like that in Exercise 11 for integration over the surface of a cylinder.

**14.** Let $S$ be a surface in $\mathbb{R}^3$ that is actually a subset $D$ of the $xy$ plane. Show that the integral of a scalar function $f(x, y, z)$ over $S$ reduces to the double integral of $f(x, y, z)$ over $D$. What does the surface integral of a vector field over $S$ become? (Make sure your answer is compatible with Example 6.)

**15.** Let the velocity field of a fluid be described by $\mathbf{F} = \mathbf{i} + x\mathbf{j} + z\mathbf{k}$ (measured in meters per second). Compute how many cubic meters of fluid per second are crossing the surface described by $x^2 + y^2 + z^2 = 1, z \geq 0$.

**16.** (a) A uniform fluid that flows vertically downward (heavy rain) is described by the vector field $\mathbf{F}(x, y, z) = (0, 0, -1)$. Find the total flux through the cone $z = (x^2 + y^2)^{1/2}$, $x^2 + y^2 \leq 1$.

(b) The rain is driven sideways by a strong wind so that it falls at a 45° angle, and it is described by $\mathbf{F}(x, y, z) = -(\sqrt{2}/2, 0, \sqrt{2}/2)$. Now what is the flux through the cone?

**17.** For $a > 0, b > 0, c > 0$, let $S$ be the upper half ellipsoid

$$S = \left\{ (x, y, z) \,\middle|\, \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1, z \geq 0 \right\},$$

with orientation determined by the upward normal. Compute $\iint_S \mathbf{F} \cdot d\mathbf{S}$ where $\mathbf{F}(x, y, z) = (x^3, 0, 0)$.

**18.** If $S$ is the upper hemisphere $\{(x, y, z) \mid x^2 + y^2 + z^2 = 1, z \geq 0\}$ oriented by the normal pointing out of the sphere, compute $\iint_S \mathbf{F} \cdot d\mathbf{S}$ for parts (a) and (b).

(a) $\mathbf{F}(x, y, z) = x\mathbf{i} + y\mathbf{j}$

(b) $\mathbf{F}(x, y, z) = y\mathbf{i} + x\mathbf{j}$

(c) For each of these vector fields, compute $\iint_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S}$ and $\int_C \mathbf{F} \cdot d\mathbf{s}$ where $C$ is the unit circle in the $xy$ plane traversed in the counterclockwise direction (as viewed from the positive $z$ axis). (Notice that $C$ is the boundary of $S$. The phenomenon illustrated here will be studied more thoroughly in the next chapter, using Stokes' theorem.)

# 7.7 Applications to Differential Geometry, Physics, and Forms of Life*

In the first half of the nineteenth century, the great German mathematician Karl Friedrich Gauss developed a theory of curved surfaces in $\mathbb{R}^3$. More than a century earlier, Isaac Newton had defined a measure of the curvature of a space curve, and Gauss was able to find extensions of this idea of curvature that would apply to surfaces. In so doing, Gauss made several remarkable discoveries.

## Curvature of Surfaces

For paths $\mathbf{c}\colon [a, b] \to \mathbb{R}^3$ that have unit speed—that is, $\|\mathbf{c}'(t)\| = 1$—the curvature $\kappa$ of the image curve $\kappa(\mathbf{c}(t))$ at the point $\mathbf{c}(t)$ is defined to be the length of the acceleration vector. That is, $\|\mathbf{c}''(t)\| = \kappa(\mathbf{c}(t))$. For paths $\mathbf{c}$ in space, the curvature is a true measure of the curvature of the geometric image curve $C$. As we saw at the end of Section 7.1, the "total curvature" $\int \kappa\, ds$ over $C$ has "topological" implications. The same, and even more, will hold for Gauss' definition of the total curvature of a surface. We begin with some definitions.

Let $\boldsymbol{\Phi}\colon D \to \mathbb{R}^3$ be a smooth parametrized surface. Then, as we know,

$$\mathbf{T}_u = \frac{\partial \boldsymbol{\Phi}}{\partial u} \quad \text{and} \quad \mathbf{T}_v = \frac{\partial \boldsymbol{\Phi}}{\partial v}$$

are tangent vectors to the image surface $S = \boldsymbol{\Phi}(D)$ at the point $\boldsymbol{\Phi}(u, v)$. We will also assume that there is a well-defined normal vector; that is, we assume the surface is regular: $\mathbf{T}_u \times \mathbf{T}_v \neq \mathbf{0}$.

Let

$$E = \left\| \frac{\partial \boldsymbol{\Phi}}{\partial u} \right\|^2, \qquad F = \frac{\partial \boldsymbol{\Phi}}{\partial u} \cdot \frac{\partial \boldsymbol{\Phi}}{\partial v}, \qquad G = \left\| \frac{\partial \boldsymbol{\Phi}}{\partial v} \right\|^2.$$

In Exercise 15 of Section 7.5, we saw that

$$\|\mathbf{T}_u \times \mathbf{T}_v\|^2 = EG - F^2.$$

*This section can be skipped on a first reading without loss of continuity.

For notational reasons, we denote $EG - F^2$ by $W$. Furthermore, we let

$$\mathbf{N} = \frac{\mathbf{T}_u \times \mathbf{T}_v}{\|\mathbf{T}_u \times \mathbf{T}_v\|} = \frac{\mathbf{T}_u \times \mathbf{T}_v}{\sqrt{W}}$$

denote the *unit* normal vector to the image surface at $p = \mathbf{\Phi}(u, v)$. Next we will define two new measures of the curvature of a surface at $p$—the "Gauss curvature," $K(p)$, and the "mean curvature," $H(p)$. Both of these curvatures have deep connections to the curvature of space curves, which illuminate the meaning of their definitions, but we do not explore these here.

To define these two curvatures, we first define three new functions $\ell, m, n$ on $S$ as follows:

$$\begin{aligned}
\ell(p) &= \mathbf{N}(u, v) \cdot \frac{\partial^2 \mathbf{\Phi}}{\partial u^2} = \mathbf{N}(u, v) \cdot \mathbf{\Phi}_{uu} \\
m(p) &= \mathbf{N}(u, v) \cdot \frac{\partial^2 \mathbf{\Phi}}{\partial u \partial v} = \mathbf{N}(u, v) \cdot \mathbf{\Phi}_{uv} \\
n(p) &= \mathbf{N}(u, v) \cdot \frac{\partial^2 \mathbf{\Phi}}{\partial v^2} = \mathbf{N}(u, v) \cdot \mathbf{\Phi}_{vv}.
\end{aligned} \tag{1}$$

The ***Gauss curvature*** $K(p)$ of $S$ at $p$ is given by

$$K(p) = \frac{\ell n - m^2}{W}, \tag{2}$$

and the ***mean curvature*** $H(p)$ of $S$ at $p$ is defined by[16]

$$H(p) = \frac{G\ell + En - 2Fm}{2W}, \tag{3}$$

where the right-hand sides of both expressions are calculated at the point $p = \mathbf{\Phi}(u, v)$.

**EXAMPLE 1** **Planes Have Zero Curvature** Let $\mathbf{\Phi}(u, v) = \boldsymbol{\alpha} u + \boldsymbol{\beta} v + \boldsymbol{\gamma}$, $(u, v) \in \mathbb{R}^2$, where $\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}$ are vectors in $\mathbb{R}^3$. According to Example 1 of Section 7.3, this determines a parametrized plane in $\mathbb{R}^3$. Show that at every point, both the Gauss and mean curvatures are zero, and hence $K$ and $H$ vanish identically.

SOLUTION Because $\mathbf{\Phi}_{uu} = \mathbf{\Phi}_{uv} = \mathbf{\Phi}_{vv} \equiv 0$, the functions $\ell, m, n$ vanish everywhere, and so do $H$ and $K$. Thus, a plane has "zero" curvature. Hence, at least in this example, we ought to be convinced that $H$ and $K$ actually do measure the *flatness* of the plane. Conversely, one can show that if $H$ and $K$ vanish identically, then $S$ is part of a plane (see Exercise 10). ▲

[16]Technically speaking, $K(p)$ and $H(p)$ could, in principle, depend on the parametrization $\mathbf{\Phi}$ of $S$, but one can show that they are, in fact, independent of $\mathbf{\Phi}$.

**EXAMPLE 2** **Curvature of a Hemisphere** Let

$$\mathbf{\Phi}(u, v) = (u, v, g(u, v)),$$

where $g(u, v) = \sqrt{R^2 - u^2 - v^2}$ is a parametrization of the "upper hemisphere" of radius $R$. Show that the Gauss curvature at every point is $1/R^2$ and the mean curvature is $1/R$.

SOLUTION We must first calculate the following quantities:

$$\mathbf{T}_u, \mathbf{T}_v, \mathbf{T}_u \times \mathbf{T}_v, \mathbf{\Phi}_{uu}, \mathbf{\Phi}_{vv}, \mathbf{\Phi}_{uv}, E, G, F, \ell, m, n.$$

First of all, we have

$$\mathbf{\Phi}_u = \mathbf{T}_u = \mathbf{i} - \frac{u}{\sqrt{R^2 - u^2 - v^2}}\mathbf{k}$$

$$\mathbf{\Phi}_v = \mathbf{T}_v = \mathbf{j} - \frac{v}{\sqrt{R^2 - v^2 - v^2}}\mathbf{k}.$$

From formula (2) in Section 7.3, we have

$$\begin{aligned}\mathbf{T}_u \times \mathbf{T}_v &= -\frac{\partial g}{\partial u}\mathbf{i} - \frac{\partial g}{\partial v}\mathbf{j} + \mathbf{k}\\ &= \frac{u}{\sqrt{R^2 - u^2 - v^2}}\mathbf{i} + \frac{v}{\sqrt{R^2 - u^2 - v^2}}\mathbf{j} + \mathbf{k}.\end{aligned}$$

Therefore,

$$E = \|\mathbf{\Phi}_u\|^2 = 1 + \frac{u^2}{R^2 - u^2 - v^2} = \frac{R^2 - v^2}{R^2 - u^2 - v^2}$$

$$G = \|\mathbf{\Phi}_v\|^2 = \frac{R^2 - u^2}{R^2 - u^2 - v^2}$$

$$F = \mathbf{\Phi}_u \cdot \mathbf{\Phi}_v = \frac{uv}{R^2 - u^2 - v^2}.$$

From Exercise 15 of Section 7.5, we know that

$$\begin{aligned}\|\mathbf{T}_u \times \mathbf{T}_v\|^2 = EG - F^2 &= \frac{(R^2 - v^2)(R^2 - u^2) - u^2v^2}{(R^2 - u^2 - v^2)^2}\\ &= \frac{R^4 - R^2u^2 - R^2v^2}{(R^2 - u^2 - v^2)^2} = \frac{R^2}{(R^2 - u^2 - v^2)} = W.\end{aligned}$$

Now a direct calculation shows that

$$\mathbf{\Phi}_{uu} = \frac{R^2 - v^2}{(R^2 - u^2 - v^2)^{3/2}}\mathbf{k}$$

$$\mathbf{\Phi}_{vv} = \frac{R^2 - u^2}{(R^2 - u^2 - v^2)^{3/2}}\mathbf{k}$$

$$\mathbf{\Phi}_{uv} = \frac{uv}{(R^2 - u^2 - v^2)^{3/2}}\mathbf{k}.$$

Furthermore,

$$\begin{aligned}
\mathbf{N} &= \frac{\mathbf{T}_u \times \mathbf{T}_v}{\|\mathbf{T}_u \times \mathbf{T}_v\|} = \frac{\mathbf{T}_u \times \mathbf{T}_v}{\sqrt{W}} \\
&= \frac{\sqrt{R^2 - u^2 - v^2}}{R} \cdot \left(\frac{u}{\sqrt{R^2 - u^2 - v^2}}\mathbf{i} + \frac{v}{\sqrt{R^2 - u^2 - v^2}}\mathbf{j} + \mathbf{k}\right) \\
&= \frac{1}{R}\left(u\mathbf{i} + v\mathbf{j} + \sqrt{R^2 - u^2 - v^2}\,\mathbf{k}\right).
\end{aligned}$$

Thus,

$$\ell = \mathbf{N} \cdot \mathbf{\Phi}_{uv} = \frac{1}{R}\left(\frac{R^2 - v^2}{R^2 - u^2 - v^2}\right)$$

$$n = \mathbf{N} \cdot \mathbf{\Phi}_{vv} = \frac{1}{R}\left(\frac{R^2 - u^2}{R^2 - u^2 - v^2}\right)$$

$$m = \mathbf{N} \cdot \mathbf{\Phi}_{uv} = \frac{1}{R}\left(\frac{uv}{R^2 - u^2 - v^2}\right).$$

Therefore,

$$\begin{aligned}
\ell n - m^2 &= \frac{1}{R^2}\left(\frac{(R^2 - v^2)(R^2 - u^2) - u^2v^2}{(R^2 - u^2 - v^2)^2}\right) \\
&= \frac{1}{R^2 - u^2 - v^2}.
\end{aligned}$$

Dividing this by $W$ yields $K = 1/R^2$. Thus, the Gauss curvature does not change from point to point on the hemisphere; that is, it is constant. This conforms to our intuition that the sphere is perfectly symmetrical and that its curvature is everywhere equal. Hence, the mean curvature should also be constant. This is verified by the following calculation:

$$
\begin{aligned}
H &= \frac{G\ell + En - 2Fm}{2W} \\
&= \frac{1}{2W}\left\{\left(\frac{R^2-u^2}{R^2-u^2-v^2}\right)\frac{1}{R}\left(\frac{R^2-v^2}{R^2-u^2-v^2}\right)\right. \\
&\quad \left. + \left(\frac{R^2-v^2}{R^2-u^2-v^2}\right)\frac{1}{R}\left(\frac{R^2-u^2}{R^2-u^2-v^2}\right) - 2\frac{u^2v^2}{(R^2-u^2-v^2)^2}\right\} \\
&= \frac{1}{W}\left\{\frac{R}{R^2-u^2-v^2}\right\} = \frac{1}{R}. \quad \blacktriangle
\end{aligned}
$$

## Surfaces of Constant Curvature

Surfaces of constant Gauss and mean curvature are of great interest to mathematicians. It was known in the nineteenth century that the only closed and bounded smooth surfaces with "no boundary" and with constant Gauss curvature were spheres. In the twentieth century, the Russian mathematician Alexandrov showed that the only closed and bounded smooth surfaces without a boundary that do not intersect themselves and that have constant mean curvature must also be spheres. Mathematicians believed that Alexandrov's result held even if the surface *was* allowed to intersect itself, but no one could find a proof. In 1984, Professor Henry Wente (Toledo, Ohio) startled the world by finding a self-intersecting torus of constant mean curvature.

Surfaces of constant mean curvature are physically relevant and occur throughout nature. Soap bubble formations have constant nonzero mean curvature (see Figure 7.7.1), and soap film formations (containing no air) have constant mean curvature zero (see Figures 7.7.2 and 7.7.3).

**Figure 7.7.1** Soap bubble formation; $H =$ constant.

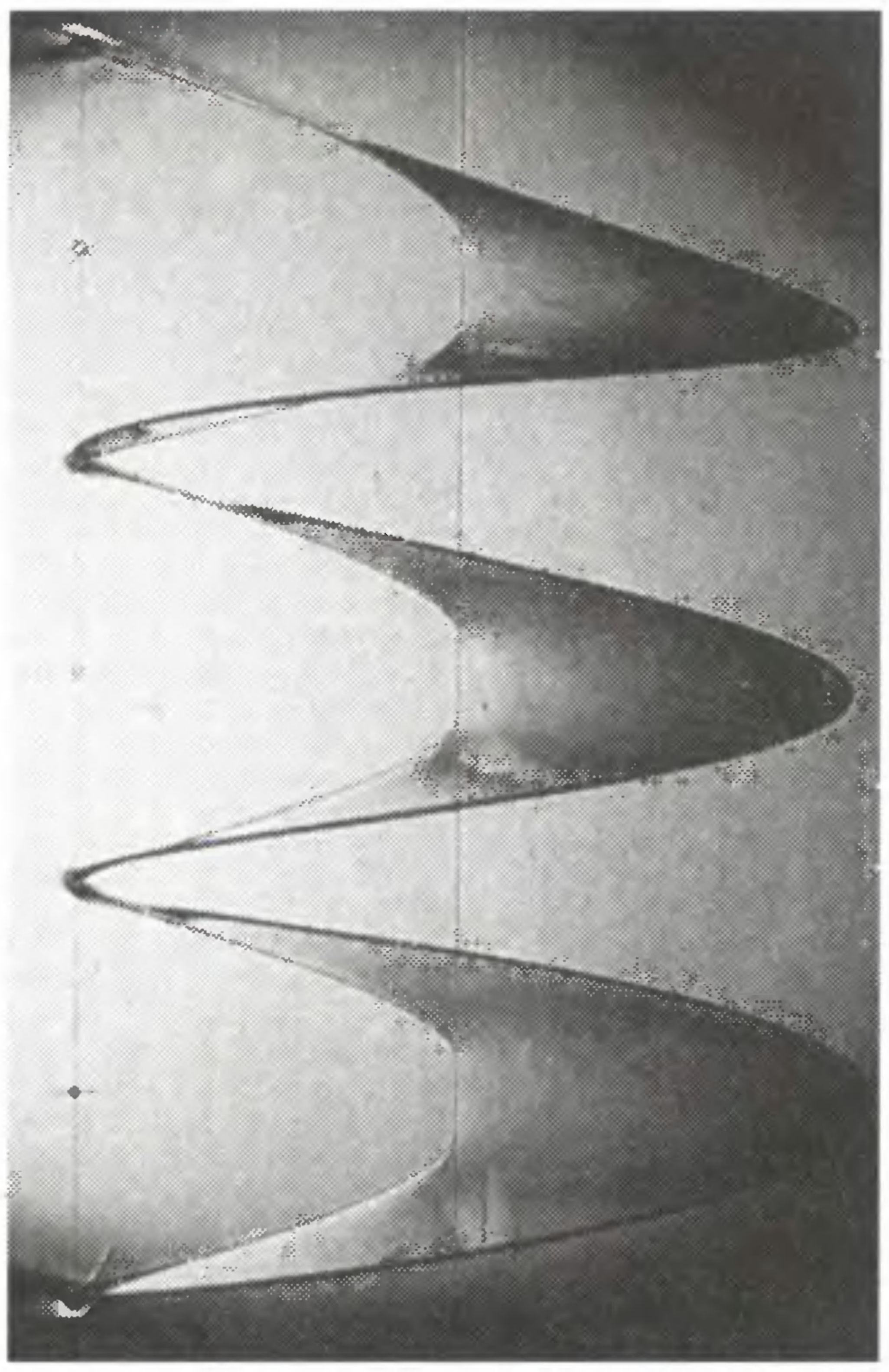


Figure 7.7.2 A helicoid, $H = 0$.

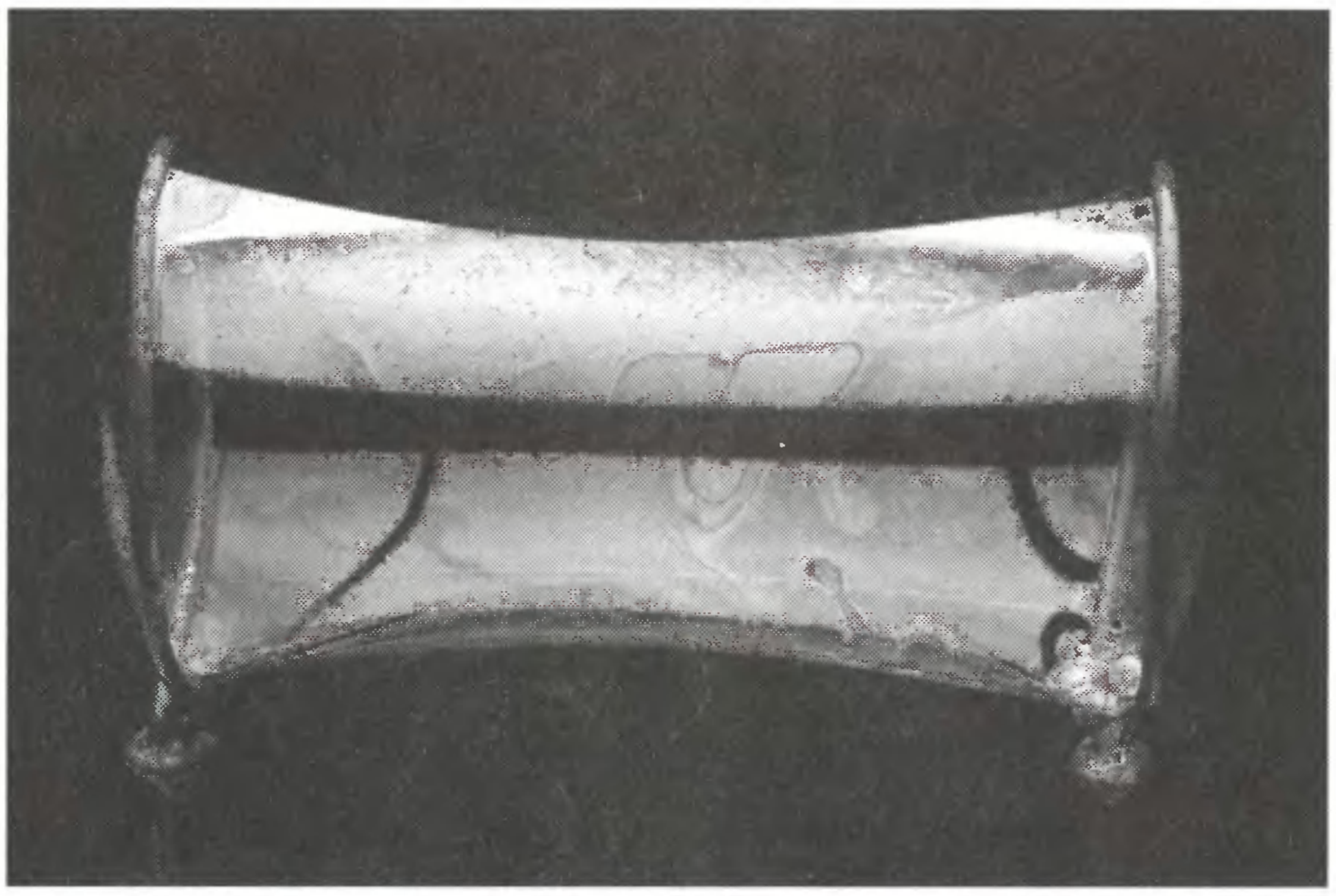


Figure 7.7.3 A soap film, $H = 0$, spanning two circular wires; this one is the catenoid.

In the early nineteenth century, the French mathematician Delaunay discovered all surfaces of revolution that have constant mean curvature. They are the cylinder, sphere, catenoid, unduloid, and nodoid. The catenoid exists as a soap film surface spanning two circular contours.

## Optimal Shapes in Nature

Throughout the ages, people have speculated on why things are shaped the way they are. Why are the earth and the stars "round" and not cubical? Why are life forms shaped the way they are?

In 1917, the British natural philosopher D'Arcy Thompson published a provocative work entitled *On Growth and Form*, in which he investigated the forces behind the creation of living forms in nature. He wrote:

> In an organism, great or small, it is not merely the nature of the motions of the living substance which we must interpret in terms of force (according to kinetics), but also the conformation of the organism itself, whose permanence or equilibrium is explained by the interaction or balance of forces, as described in statics.

Surprisingly, Thompson discovered *all* of Delaunay's surfaces in the form of unicellular organisms (see Figure 7.7.4). The constant mean curvature of these organisms can be explained by minimum principles similar to those described in the Historical Note in Section 3.3. In 1952, Watson and Crick determined that the structure of DNA is that of a double helix, a discovery that set the stage for the genetic revolution. We know from soap films, as in Figure 7.2.2, that nature likes helicoid forms, and nature tends to repeat patterns. A better understanding of the scientific principles underlying life may ultimately help mathematics play a more prominent role in this area of theoretical biology.

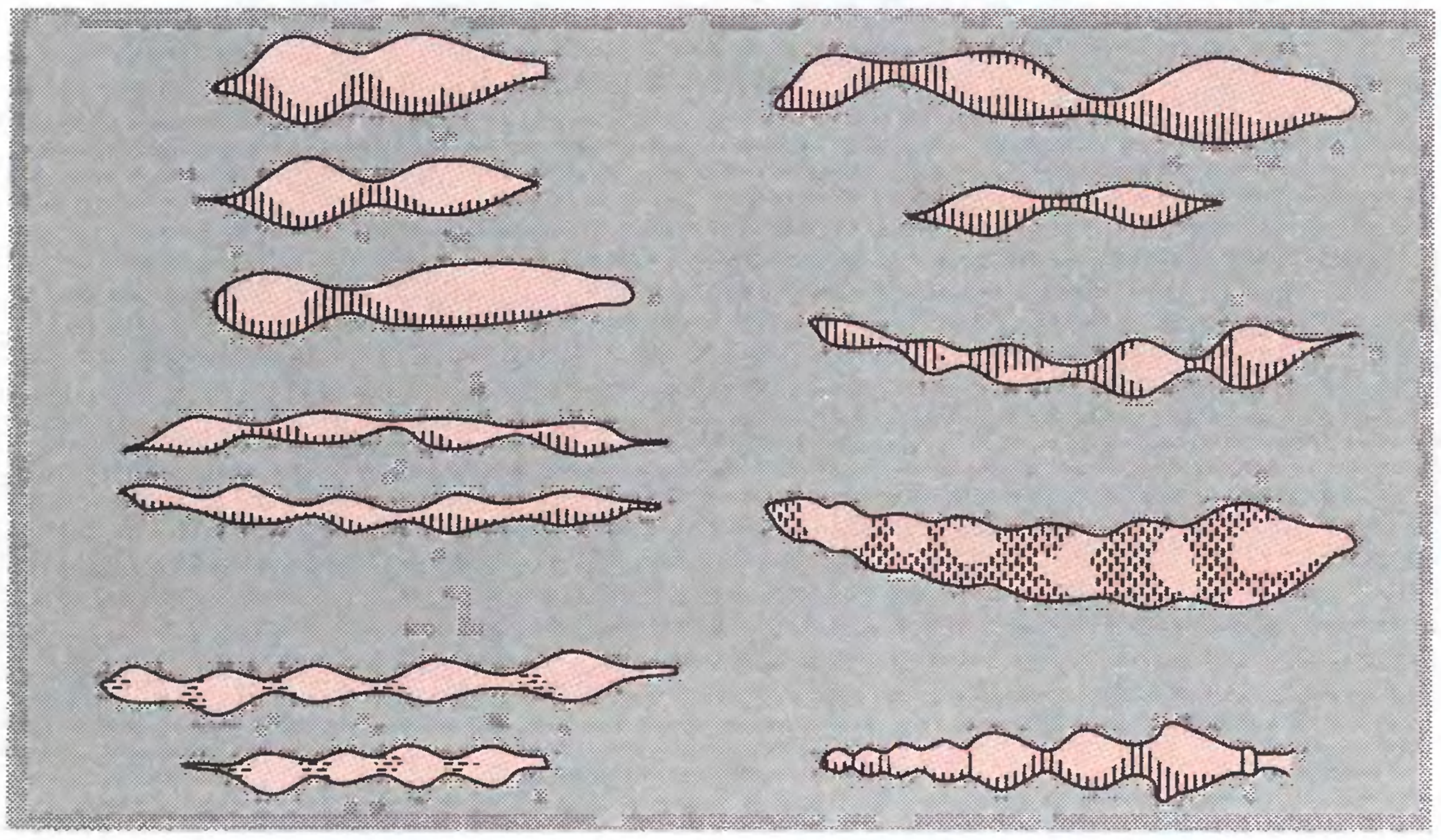


**Figure 7.7.4** Surfaces of revolution of constant mean curvature as unicellulars.

## Curvature and Physics

The theory of curved surfaces, initiated by Gauss, has had a profound effect on physics. Gauss realized that the Gauss curvature K of a surface depended only on the measure of distance *on the surface itself*; that is, curvature was *intrinsic to the surface.* This is not true of the mean curvature $H$. Thus, beings "living" on the surface would be able to tell that the surface was curving, without any reference to an "external" world. Gauss himself found this mathematical result to be so striking that he named it *theorema egregium*, or "remarkable theorem." Gauss' theory was generalized by his student Bernhard Riemann to $n$-dimensional surfaces for which one could describe a notion of curvature.

Recall that Newton created the idea of a gravitational force acting over vast galactic distances, pulling galaxies together as well as pushing them apart (see Figure 7.7.5). In the early 1900s, Albert Einstein used Riemann's ideas to develop the *general theory of relativity*, a theory of gravitation that eliminated the need to consider forces (as Newton did) acting over great distances. Einstein's theory explained the bending of light by the sun, black holes, the expansion of the universe, the formation of galaxies, and the Big Bang itself. For most applications, including the dynamics of our solar system, Newton's theory suffices and is commonly used today by NASA to plan space missions, as we saw in Section 4.1. But for

Figure 7.7.5 The Andromeda Galaxy. It will collide with the Milky Way in roughly 2 billion years.

cosmological applications on the grand scale, Einstein's theory replaced that of Isaac Newton, published in his *Principia* in 1687.

As a testament to his genius, and despite the astounding success of this theory, Newton was nevertheless disturbed by questions about *how* this gravitational force acted. He could give no other explanation than to say, "I have not been able to deduce from phenomena the reason for these properties of gravitation, and I do not invent hypotheses; for anything which cannot be deduced from phenomena should be called an hypothesis." Moreover, in a letter to his friend, Richard Bentley, Newton wrote:

> That gravity should be innate, inherent and essential to matter, so that one body may act upon another at a distance, through which their action may be conveyed from one to another, is to me so great an absurdity that I believe no man, who has in philosophical matters a competent faculty of thinking, can ever fall into it.

Newton coined the term *action at a distance* (which means "force acting at a distance") to describe the mysterious effect of gravitation over large distances. This effect is as difficult to understand today as it was in Newton's time.

Johann Bernoulli found it difficult to believe in the concept of a force that acts through a vacuum of space over distances of even hundreds of millions of miles. He viewed this force as a concept revolting to minds unaccustomed to accepting

**Figure 7.7.6** Albert Einstein (1879–1955) at his desk in the Patent Office, Bern, 1905.

any principle in physics, save those that are incontestable and evident. Additionally, Leibniz considered gravitation to be an incorporeal and inexplicable power, philosophically void.

It was perhaps Albert Einstein's greatest inspiration (see Figure 7.7.6) to replace Newton's model of gravitation with a model that would have thrilled the early Greeks—*a geometric model of gravitation*. In Einstein's theory, the concept of a force acting through great distances has been replaced by the *curvature* of a space–time[17] world. As the quote at the beginning of the chapter illustrates, W. K. Clifford had a premonition of events to come! In order to elucidate Einstein's scheme, we shall present an oversimplified model that conveys some of his basic ideas.

We represent space by a surface that we imagine as an originally flat trampoline (the vacuum state), which is at some point strongly deformed by the weight of a gigantic steel ball (the sun). A tiny steel ball rolling on the trampoline is our planet Earth (see Figure 7.7.7)

If we roll the small steel ball across the flat trampoline, it will travel in a straight-line path. However, if we now place the gigantic steel ball in the center, it will cause the trampoline to bend, or "curve," even "far away" from the large ball. If we then give our little ball a push, it will no longer travel in a straight line but in a curved

[17]Space–time is locally like $\mathbb{R}^4$ with three space coordinates and one time coordinate.

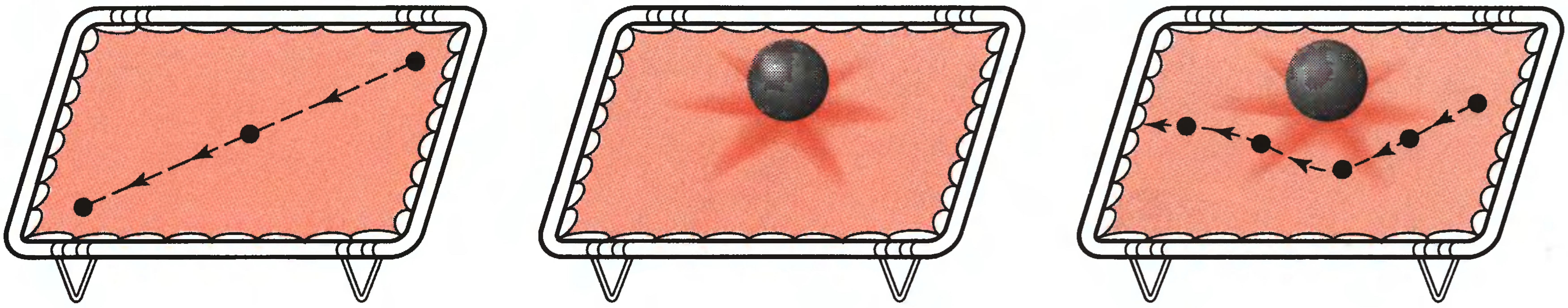



Figure 7.7.7 (a) A particle on a taut trampoline moves in a straight line. (b) A heavy steel ball distorts the trampoline. (c) A particle on the distorted trampoline follows a curved path.

path. The big ball affects the trajectory of the little ball by curving the space around it. With just the right push, the little ball might even orbit the big one for a while. This trampoline model explains how a large body could, by curving space, influence a small one over great distance.

Einstein stated that space–time is curved by matter and energy. In this curved space–time, even light rays are *bent* as they pass near massive objects like our sun. Thanks to Gauss and Riemann, the curvature of space–time requires no external "universe" in which it curves.

The equations that tell one how much space and time are curved by matter and energy are known as *Einstein's field equations.* A description of them is beyond the scope of this book, but the mathematical kernel from which these equations arise is not; this kernel is based on another remarkable result of the research of Gauss and Bonnet.

## Gauss–Bonnet Theorem

In Example 2, we computed the Gauss curvature $K$ of the sphere $x^2+y^2+z^2=R^2$ of radius $R$ and found it to be the constant $1/R^2$. The Gauss curvature $K$ is a scalar-valued function over the surface, and as such we can integrate it over the surface. We wish to consider a constant times this surface integral, namely,

$$\frac{1}{2\pi}\iint_S K\,dA.$$

For the sphere of radius $R$, this quantity becomes

$$\frac{1}{2\pi R^2}\iint_S dA=\frac{4\pi R^2}{2\pi R^2}=2.$$

What Gauss and Bonnet discovered was that if $S$ is *any* "sphere-like" closed surface (closed and bounded, but with no boundary, as in Figure 7.7.8), then

$$\frac{1}{2\pi}\iint_S K\,dA = 2$$

still holds.[18]

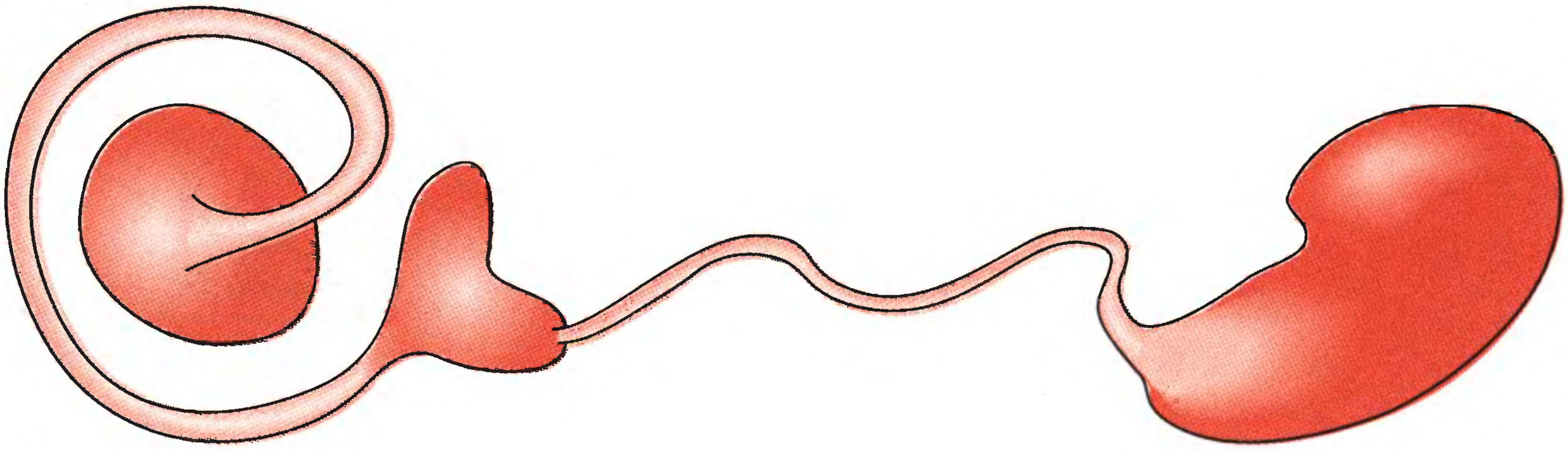



Figure 7.7.8 A deformed sphere. $1/2\pi \iint_S K\,dA = 2$.

Thus, the integral

$$\frac{1}{2\pi}\iint_S K\,dA$$

always equals the integer 2, and is therefore a *topological invariant* of the surface. That the integral of curvature should be an interesting quantity should be already clear from the discussion at the end of Section 7.1.

Now consider a torus, or doughnut. The torus can be considered as coming from the sphere by cutting out two disks and gluing in a handle (see Figure 7.7.9).

Moreover, we can continue this process adding $1, 2, 3, \ldots, g$ handles to the sphere. If $g$ handles are attached, we call the resulting surface a surface of genus $g$, as in Figure 7.7.10. Notice that the torus has genus 1.

If two surfaces have a different genus, they are topologically distinct, and thus cannot be obtained from one another by bending or stretching. Interestingly, even two surfaces with the same genus can sit in space in quite different and complex ways, as in Figure 7.7.11. Astonishingly, even though the integral (or total curvature) given by $(1/2\pi)\iint_S K\,dA$ depends on the genus, it does not depend on how the surface sits in space (and thus not on $K$).

[18]Roughly speaking, this means that $S$ can be obtained from the sphere by bending and stretching (like with a balloon) but not tearing (the balloon bursts!)

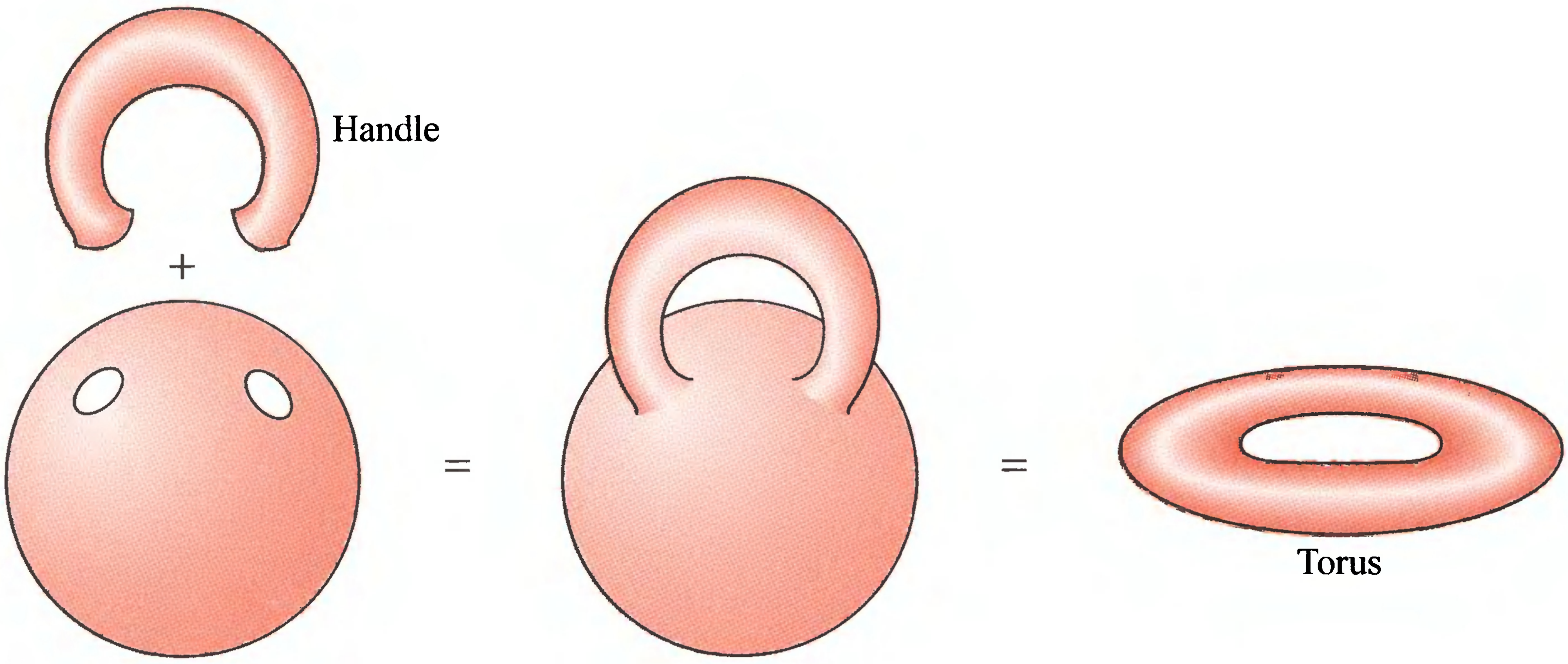



Figure 7.7.9 Gluing a handle to a sphere to obtain a torus.

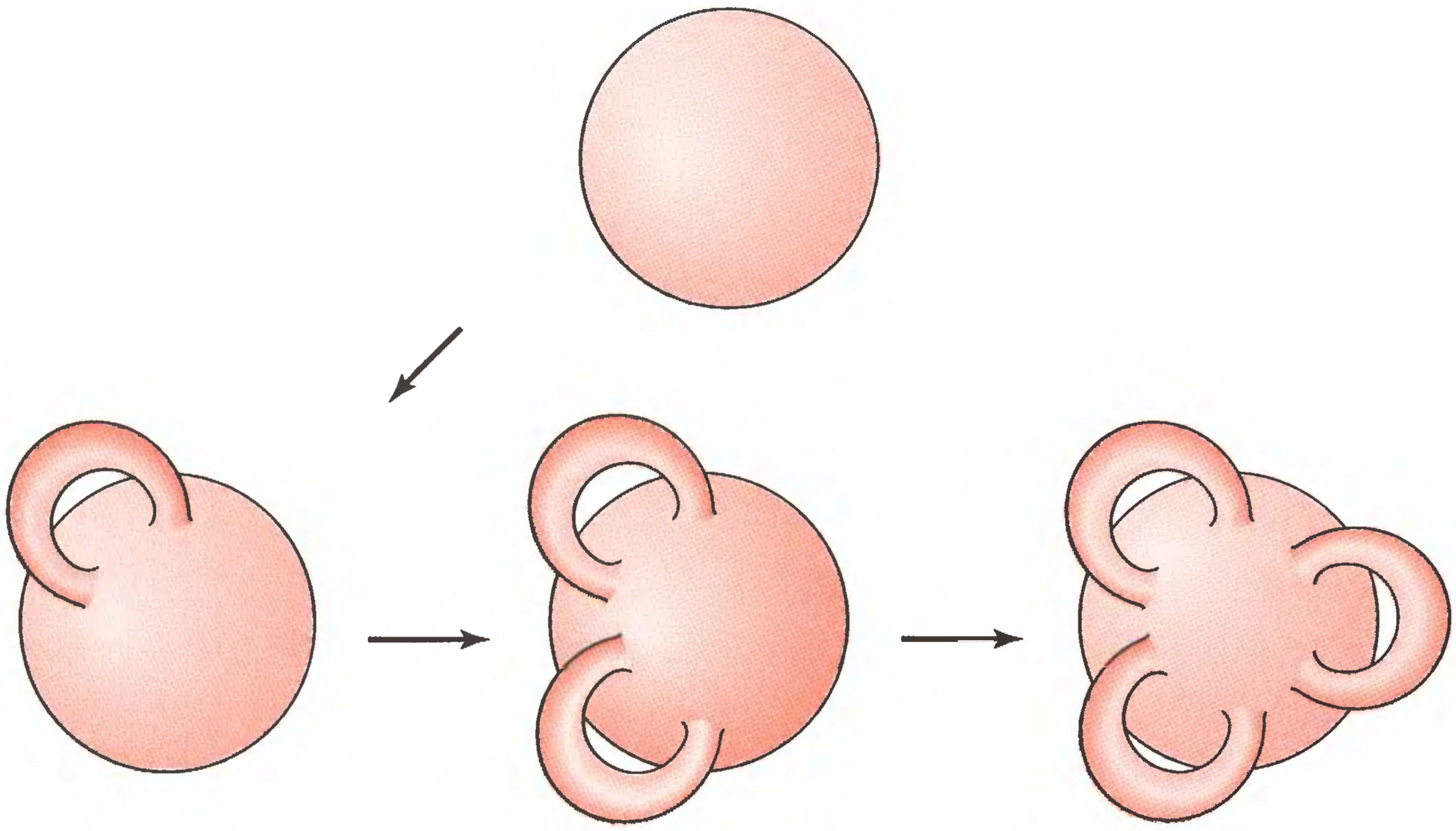


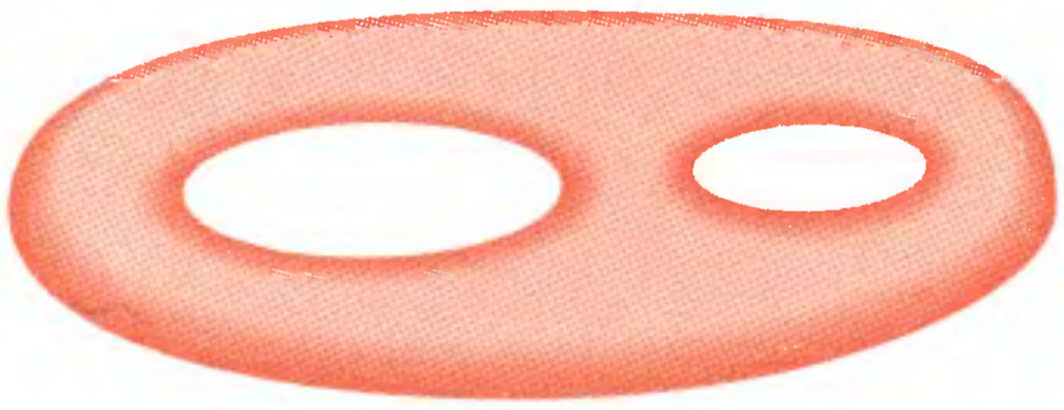
Figure 7.7.10 A sphere with 0, 1, 2, 3 handles attached.

Gauss and Bonnet proved that

$$\frac{1}{2\pi}\iint_S K\,dA = 2 - 2g.$$

Thus, for the sphere ($g = 0$), it is always 2 (already verified); for the torus, it is always 0 (see Exercise 8).

There is something even more remarkable connected to the theorem of Gauss–Bonnet, observed by the great German mathematician David Hilbert (Figure 7.7.12).


Simple double doughnut


Baker's pretzel

Figure 7.7.11 Two manifestations of a surface $S$ in $\mathbb{R}^3$ of genus 2.

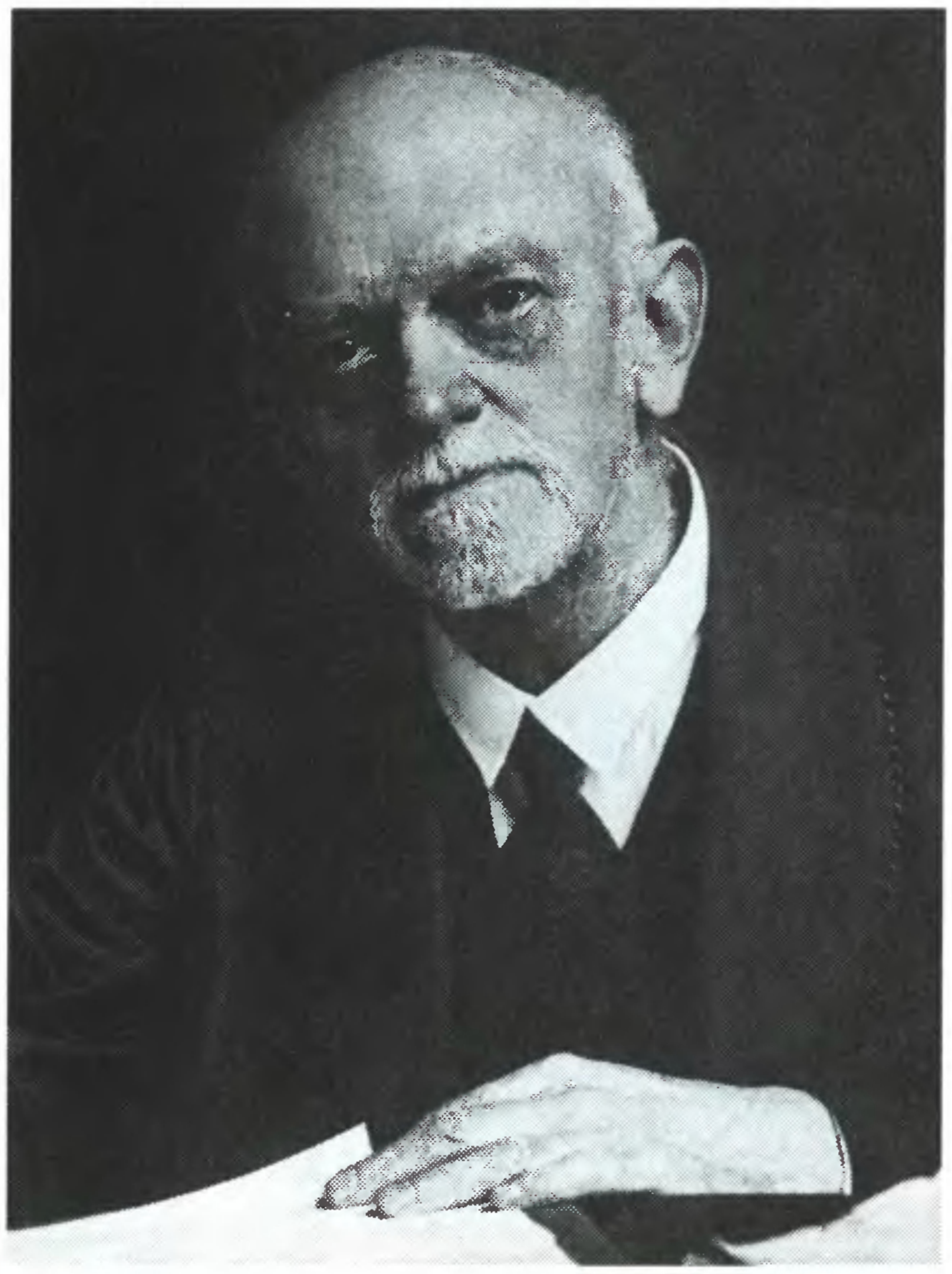


Figure 7.7.12 David Hilbert (1862–1943) was a leading mathematician of his time.

Hilbert observed that the Gauss–Bonnet theorem is, in effect, a two-dimensional version of Einstein's field equations. In the physics literature, this fact is known as *Hilbert's action principle* in general relativity.[19] Not surprisingly, similar geometric ideas are being employed by contemporary researchers in an effort to unify gravity and quantum mechanics—to "quantize" gravity, so to speak.

[19]See C. Misner, K. Thorne, and A. Wheeler, *Gravitation*, Freeman, New York, 1972.

## EXERCISES

1. The helicoid can be described by

$$\mathbf{\Phi}(u, v) = (u \cos v, u \sin v, bv), \text{ where } b \neq 0.$$

Show that $H = 0$ and that $K = -b^2/(b^2 + u^2)^2$. In Figures 7.7.1 and 7.7.5, we see that the helicoid is actually a soap film surface. Surfaces in which $H = 0$ are called ***minimal surfaces***.

2. Consider the saddle surface $z = xy$. Show that

$$K = \frac{-1}{(1 + x^2 + y^2)^2},$$

and that

$$H = \frac{-xy}{(1 + x^2 + y^2)^{3/2}}.$$

3. Show that $\mathbf{\Phi}(u, v) = (u, v, \log \cos v - \log \cos u)$ has mean curvature zero (and is thus a minimal surface; see Exercise 1).

4. Find the Gauss curvature of the elliptic paraboloid

$$z = \frac{x^2}{a^2} + \frac{y^2}{b^2}.$$

5. Find the Gauss curvature of the hyperbolic paraboloid

$$z = \frac{x^2}{a^2} - \frac{y^2}{b^2}.$$

6. Compute the Gauss curvature of the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

7. Show that Enneper's surface

$$\mathbf{\Phi}(u, v) = \left(u - \frac{u^3}{3} + uv^2, v - \frac{v^3}{3} + u^2 v, u^2 - v^2\right)$$

is a minimal surface ($H = 0$).

8. Consider the torus $T$ given in Exercise 4, Section 7.4. Compute its Gauss curvature and verify the theorem of Gauss–Bonnet. [Hint: Show that $\|T_\theta \times T_\phi\|^2 = (R + \cos \phi)^2$ and $K = \cos \phi/(R + \cos \phi)$.]

9. Let $\mathbf{\Phi}(u, v) = (u, h(u) \cos v, h(u) \sin v)$, $h > 0$, be a surface of revolution. Show that $K = -h''/h\{1 + (h')^2\}^2$.

**10.** A parametrization $\mathbf{\Phi}$ of a surface $S$ is said to be ***conformal*** (see Section 7.4), provided that $E = G$, $F = 0$. Assume that $\mathbf{\Phi}$ conformally parametrizes $S$.[20] Show that if $H$ and $K$ vanish identically, then $S$ must be part of a plane in $\mathbb{R}^3$.

## REVIEW EXERCISES FOR CHAPTER 7

**1.** Integrate $f(x, y, z) = xyz$ along the following paths:

(a) $\mathbf{c}(t) = (e^t \cos t, e^t \sin t, 3), 0 \le t \le 2\pi$
(b) $\mathbf{c}(t) = (\cos t, \sin t, t), 0 \le t \le 2\pi$
(c) $\mathbf{c}(t) = \frac{3}{2}t^2\mathbf{i} + 2t^2\mathbf{j} + t\mathbf{k}, 0 \le t \le 1$
(d) $\mathbf{c}(t) = t\mathbf{i} + (1/\sqrt{2})t^2\mathbf{j} + \frac{1}{3}t^3\mathbf{k}, 0 \le t \le 1$

**2.** Compute the integral of $f$ along the path $\mathbf{c}$ in each of the following cases:

(a) $f(x, y, z) = x + y + yz$; $\mathbf{c}(t) = (\sin t, \cos t, t), 0 \le t \le 2\pi$
(b) $f(x, y, z) = x + \cos^2 z$; $\mathbf{c}(t) = (\sin t, \cos t, t), 0 \le t \le 2\pi$
(c) $f(x, y, z) = x + y + z$; $\mathbf{c}(t) = (t, t^2, \frac{2}{3}t^3), 0 \le t \le 1$

**3.** Compute each of the following line integrals:

(a) $\int_C (\sin \pi x)\, dy - (\cos \pi y)\, dz$, where $C$ is the triangle whose vertices are $(1, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 1)$, in that order

(b) $\int_C (\sin z)\, dx + (\cos z)\, dy - (xy)^{1/3}\, dz$, where $C$ is the path $\mathbf{c}(\theta) = (\cos^3 \theta, \sin^3 \theta, \theta)$, $0 \le \theta \le 7\pi/2$

**4.** If $\mathbf{F}(\mathbf{x})$ is orthogonal to $\mathbf{c}'(t)$ at each point on the curve $\mathbf{x} = \mathbf{c}(t)$, what can you say about $\int_{\mathbf{c}} \mathbf{F} \cdot d\mathbf{s}$?

**5.** Find the work done by the force $\mathbf{F}(x, y) = (x^2 - y^2)\mathbf{i} + 2xy\mathbf{j}$ in moving a particle counterclockwise around the square with corners $(0, 0)$, $(a, 0)$, $(a, a)$, $(0, a)$, $a > 0$.

**6.** A ring in the shape of the curve $x^2 + y^2 = a^2$ is formed of thin wire weighing $|x| + |y|$ grams per unit length at $(x, y)$. Find the mass of the ring.

**7.** Find a parametrization for each of the following surfaces;

(a) $x^2 + y^2 + z^2 - 4x - 6y = 12$
(b) $2x^2 + y^2 + z^2 - 8x = 1$
(c) $4x^2 + 9y^2 - 2z^2 = 8$

**8.** Find the area of the surface defined by $\mathbf{\Phi}$: $(u, v) \mapsto (x, y, z)$, where

$$x = h(u, v) = u + v, \qquad y = g(u, v) = u, \qquad z = f(u, v) = v;$$

$0 \le u \le 1, 0 \le v \le 1$. Sketch.

---

[20] Gauss proved that conformal parametrization of a surface always exists. The result of this exercise remains valid even if $\mathbf{\Phi}$ is not conformal, but the proof is more difficult.

**9.** Write a formula for the surface area of $\Phi: (r, \theta) \mapsto (x, y, z)$, where

$$x = r\cos\theta, \qquad y = 2r\sin\theta, \qquad z = r;$$

$0 \le r \le 1, 0 \le \theta \le 2\pi$. Describe the surface.

**10.** Suppose $z = f(x, y)$ and $(\partial f/\partial x)^2 + (\partial f/\partial y)^2 = c, c > 0$. Show that the area of the graph of $f$ lying over a region $D$ in the $xy$ plane is $\sqrt{1+c}$ times the area of $D$.

**11.** Compute the integral of $f(x, y, z) = x^2 + y^2 + z^2$ over the surface in Review Exercise 8.

**12.** Find $\iint_S f\, dS$ in each of the following cases:

(a) $f(x, y, z) = x$; $S$ is the part of the plane $x + y + z = 1$ in the positive octant defined by $x \ge 0, y \ge 0, z \ge 0$
(b) $f(x, y, z) = x^2$; $S$ is the part of the plane $x = z$ inside the cylinder $x^2 + y^2 = 1$
(c) $f(x, y, z) = x$; $S$ is the part of the cylinder $x^2 + y^2 = 2x$ with $0 \le z \le \sqrt{x^2 + y^2}$

**13.** Compute the integral of $f(x, y, z) = xyz$ over the rectangle with vertices $(1, 0, 1)$, $(2, 0, 0)$, $(1, 1, 1)$, and $(2, 1, 0)$.

**14.** Compute the integral of $x + y$ over the surface of the unit sphere.

**15.** Compute the surface integral of $x$ over the triangle with vertices $(1, 1, 1)$, $(2, 1, 1)$, and $(2, 0, 3)$.

**16.** A paraboloid of revolution $S$ is parametrized by $\Phi(u, v) = (u\cos v, u\sin v, u^2)$, $0 \le u \le 2, 0 \le v \le 2\pi$.

(a) Find an equation in $x, y$, and $z$ describing the surface.
(b) What are the geometric meanings of the parameters $u$ and $v$?
(c) Find a unit vector orthogonal to the surface at $\Phi(u, v)$.
(d) Find the equation for the tangent plane at $\Phi(u_0, v_0) = (1, 1, 2)$ and express your answer in the following two ways:

(i) parametrized by $u$ and $v$; and
(ii) in terms of $x, y$, and $z$.

(e) Find the area of $S$.

**17.** Let $f(x, y, z) = xe^y \cos \pi z$.

(a) Compute $\mathbf{F} = \nabla f$.
(b) Evaluate $\int_C \mathbf{F} \cdot d\mathbf{s}$ where $\mathbf{c}(t) = (3\cos^4 t, 5\sin^7 t, 0), 0 \le t \le \pi$.

**18.** Let $\mathbf{F}(x, y, z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$. Evaluate $\iint_S \mathbf{F} \cdot d\mathbf{S}$ where $S$ is the upper hemisphere of the unit sphere $x^2 + y^2 + z^2 = 1$.

**19.** Let $\mathbf{F}(x, y, z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$. Evaluate $\int_{\mathbf{c}} \mathbf{F} \cdot d\mathbf{s}$ where $\mathbf{c}(t) = (e^t, t, t^2), 0 \le t \le 1$.

**20.** Let $\mathbf{F} = \nabla f$ for a given scalar function. Let $\mathbf{c}(t)$ be a closed curve, that is, $\mathbf{c}(b) = \mathbf{c}(a)$. Show that $\int_{\mathbf{c}} \mathbf{F} \cdot d\mathbf{s} = 0$.

**21.** Consider the surface $\Phi(u, v) = (u^2 \cos v, u^2 \sin v, u)$. Compute the unit normal at $u = 1, v = 0$. Compute the equation of the tangent plane at this point.

**22.** Let $S$ be the part of the cone $z^2 = x^2 + y^2$ with $z$ between 1 and 2 oriented by the normal pointing out of the cone. Compute $\iint_S \mathbf{F} \cdot d\mathbf{S}$ where $\mathbf{F}(x, y, z) = (x^2, y^2, z^2)$.

**23.** Let $\mathbf{F} = x\mathbf{i} + x^2\mathbf{j} + yz\mathbf{k}$ represent the velocity field of a fluid (velocity measured in meters per second). Compute how many cubic meters of fluid per second are crossing the $xy$ plane through the square $0 \leq x \leq 1, 0 \leq y \leq 1$.

**24.** Show that the surface area of the part of the sphere $x^2 + y^2 + z^2 = 1$ lying above the rectangle $[-a, a] \times [-a, a]$, where $2a^2 < 1$, in the $xy$ plane is

$$A = 2\int_{-a}^{a} \sin^{-1}\left(\frac{a}{\sqrt{1-x^2}}\right) dx.$$

**25.** Let $S$ be a surface and $C$ a closed curve bounding $S$. Verify the equality

$$\iint_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S} = \int_C \mathbf{F} \cdot d\mathbf{s}$$

if $\mathbf{F}$ is a gradient field (use Review Exercise 20).

**26.** Calculate $\iint_S \mathbf{F} \cdot d\mathbf{S}$ where $\mathbf{F}(x, y, z) = (x, y, -y)$ and $S$ is the cylindrical surface defined by $x^2 + y^2 = 1, 0 \leq z \leq 1$, with normal pointing out of the cylinder.

**27.** Let $S$ be the portion of the cylinder $x^2 + y^2 = 4$ between the planes $z = 0$ and $z = x + 3$. Compute the following:

(a) $\iint_S x^2\, dS$ (b) $\iint_S y^2\, dS$ (c) $\iint_S z^2\, dS$

**28.** Let $\Gamma$ be the curve of intersection of the plane $z = ax + by$ with the cylinder $x^2 + y^2 = 1$. Find all values of the real numbers $a$ and $b$ such that $a^2 + b^2 = 1$ and

$$\int_\Gamma y\, dx + (z - x)\, dy - y\, dz = 0.$$

**29.** A circular helix that lies on the cylinder $x^2 + y^2 = R^2$ with pitch $p$ may be described parametrically by

$$x = R\cos\theta, \qquad y = R\sin\theta, \qquad z = p\theta, \qquad \theta \geq 0.$$

A particle slides under the action of gravity (which acts parallel to the $z$ axis) without friction along the helix. If the particle starts out at the height $z_0 > 0$, then when it reaches the height $z$, $0 < z < z_0$, along the helix, its speed is given by

$$\frac{ds}{dt} = \sqrt{(z_0 - z)2g},$$

where $s$ is arc length along the helix, $g$ is the constant of gravity, and $t$ is time.

(a) Find the length of the part of the helix between the planes $z = z_0$ and $z = z_1$, $0 < z_1 < z_0$.

(b) Compute the time $T_0$ it takes the particle to reach the plane $z = 0$.