1

The Geometry of Euclidean Space

Quaternions came from Hamilton . . . and have been an unmixed evil to those who have touched them in any way. Vector is a useless survival . . . and has never been of the slightest use to any creature.

Lord Kelvin

In this chapter we consider the basic operations on vectors in two- and three-dimensional space: vector addition, scalar multiplication, and the dot and cross products. In Section 1.5 we generalize some of these notions to n-space and review properties of matrices that will be needed in Chapters 2 and 3.

1.1 Vectors in Two- and Three-Dimensional Space

Points P in the plane are represented by ordered pairs of real numbers (a_1, a_2); the numbers a_1 and a_2 are called the ***Cartesian coordinates of*** P. We draw two perpendicular lines, label them as the x and y axes, and then drop perpendiculars from P to these axes, as in Figure 1.1.1. After designating the intersection of the x and y axes as the origin and choosing units on these axes, we produce two signed distances a_1 and a_2 as shown in the figure; a_1 is called the x ***component*** of P, and a_2 is called the y ***component***.

Points in space may be similarly represented as ordered triples of real numbers. To construct such a representation, we choose three mutually perpendicular lines that meet at a point in space. These lines are called x ***axis***, y ***axis***, and z ***axis***, and the point at which they meet is called the ***origin*** (this is our reference point). We choose a scale on these axes, as shown in Figure 1.1.2.

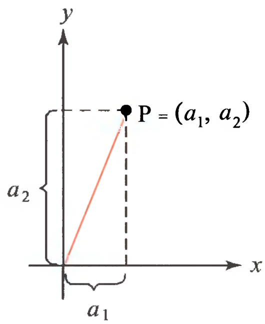


Figure 1.1.1 Cartesian coordinates in the plane.

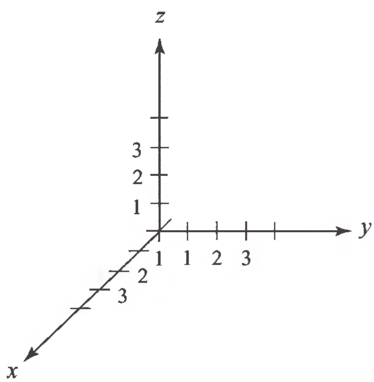


Figure 1.1.2 Cartesian coordinates in space.

The triple (0, 0, 0) corresponds to the origin of the coordinate system, and the arrows on the axes indicate the positive directions. For example, the triple (2, 4, 4) represents a point 2 units from the origin in the positive direction along the x axis, 4 units in the positive direction along the y axis, and 4 units in the positive direction along the z axis (Figure 1.1.3).

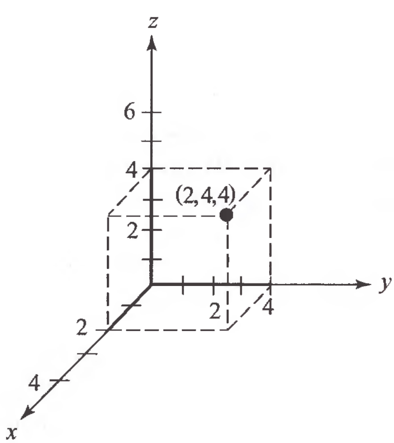


Figure 1.1.3 Geometric representation of the point (2, 4, 4) in Cartesian coordinates.

Because we can associate points in space with ordered triples in this way, we often use the expression "the point (a_1, a_2, a_3)" instead of the longer phrase "the point P that corresponds to the triple (a_1, a_2, a_3)." We say that a_1 is the x ***coordinate*** (or first coordinate), a_2 is the y ***coordinate*** (or second coordinate), and a_3 is the z ***coordinate*** (or third coordinate) of P. It is also common to denote points in space with the letters x, y, and z in place of a_1, a_2, and a_3. Thus, the triple (x, y, z) represents a point whose first coordinate is x, second coordinate is y, and third coordinate is z.

We employ the following notation for the line, the plane, and three-dimensional space:

(i) The real number line is denoted $\mathbb{R}^1$ or simply $\mathbb{R}$.

(ii) The set of all ordered pairs (x, y) of real numbers is denoted $\mathbb{R}^2$.

(iii) The set of all ordered triples (x, y, z) of real numbers is denoted $\mathbb{R}^3$.

When speaking of $\mathbb{R}^1, \mathbb{R}^2$, and $\mathbb{R}^3$ simultaneously, we write $\mathbb{R}^n$, where $n = 1, 2$, or 3; or $\mathbb{R}^m$, where $m = 1, 2, 3$. Starting in Section 1.5 we will also study $\mathbb{R}^n$ for $n = 4, 5, 6, \ldots$, but the cases $n = 1, 2, 3$ are closest to our geometric intuition and will be stressed throughout the book.

Vector Addition and Scalar Multiplication

The operation of addition can be extended from $\mathbb{R}$ to $\mathbb{R}^2$ and $\mathbb{R}^3$. For $\mathbb{R}^3$, this is done as follows. Given the two triples (a_1, a_2, a_3) and (b_1, b_2, b_3), we define their ***sum*** to be

$$(a_1, a_2, a_3) + (b_1, b_2, b_3) = (a_1 + b_1, a_2 + b_2, a_3 + b_3).$$

EXAMPLE 1

$$(1, 1, 1) + (2, -3, 4) = (3, -2, 5),$$
$$(x, y, z) + (0, 0, 0) = (x, y, z),$$
$$(1, 7, 3) + (a, b, c) = (1 + a, 7 + b, 3 + c). \quad \blacktriangle$$

The element $(0, 0, 0)$ is called the ***zero element*** (or just ***zero***) of $\mathbb{R}^3$. The element $(-a_1, -a_2, -a_3)$ is the ***additive inverse*** (or ***negative***) of (a_1, a_2, a_3), and we will write $(a_1, a_2, a_3) - (b_1, b_2, b_3)$ for $(a_1, a_2, a_3) + (-b_1, -b_2, -b_3)$.

The additive inverse, when added to the vector itself, of course produces zero:

$$(a_1, a_2, a_3) + (-a_1, -a_2, -a_3) = (0, 0, 0).$$

There are several important product operations that we will define on $\mathbb{R}^3$. One of these, called the *inner product*, assigns a real number to each pair of elements of $\mathbb{R}^3$. We shall discuss it in detail in Section 1.2. Another product operation for $\mathbb{R}^3$ is called *scalar multiplication* (the word "scalar" is a synonym for "real number"). This product combines scalars (real numbers) and elements of $\mathbb{R}^3$ (ordered triples) to yield elements of $\mathbb{R}^3$ as follows: Given a scalar α and a triple (a_1, a_2, a_3), we define the ***scalar multiple*** by

$$\alpha(a_1, a_2, a_3) = (\alpha a_1, \alpha a_2, \alpha a_3).$$

EXAMPLE 2

$$2(4, e, 1) = (2 \cdot 4, 2 \cdot e, 2 \cdot 1) = (8, 2e, 2),$$
$$6(1, 1, 1) = (6, 6, 6),$$
$$1(u, v, w) = (u, v, w),$$
$$0(p, q, r) = (0, 0, 0). \quad \blacktriangle$$

Addition and scalar multiplication of triples satisfy the following properties:

(i) $(\alpha\beta)(a_1, a_2, a_3) = \alpha[\beta(a_1, a_2, a_3)]$ (associativity)

(ii) $(\alpha + \beta)(a_1, a_2, a_3) = \alpha(a_1, a_2, a_3) + \beta(a_1, a_2, a_3)$ (distributivity)

(iii) $\alpha[(a_1, a_2, a_3) + (b_1, b_2, b_3)] = \alpha(a_1, a_2, a_3) + \alpha(b_1, b_2, b_3)$ (distributivity)

(iv) $\alpha(0, 0, 0) = (0, 0, 0)$ (property of zero)

(v) $0(a_1, a_2, a_3) = (0, 0, 0)$ (property of zero)

(vi) $1(a_1, a_2, a_3) = (a_1, a_2, a_3)$ (property of the unit element)

The identities are proven directly from the definitions of addition and scalar multiplication. For instance,

$$\begin{aligned}(\alpha + \beta)(a_1, a_2, a_3) &= ((\alpha + \beta)a_1, (\alpha + \beta)a_2, (\alpha + \beta)a_3)\\ &= (\alpha a_1 + \beta a_1, \alpha a_2 + \beta a_2, \alpha a_3 + \beta a_3)\\ &= \alpha(a_1, a_2, a_3) + \beta(a_1, a_2, a_3).\end{aligned}$$

For $\mathbb{R}^2$, addition and scalar multiplication are defined just as in $\mathbb{R}^3$, with the third component of each vector dropped off. All the properties (i) to (vi) still hold.

EXAMPLE 3 Interpret the chemical equation $2NH_2 + H_2 = 2NH_3$ as a relation in the algebra of ordered pairs.

SOLUTION We think of the molecule N_xH_y (x atoms of nitrogen, y atoms of hydrogen) as represented by the ordered pair (x, y). Then the chemical equation given is equivalent to $2(1, 2) + (0, 2) = 2(1, 3)$. Indeed, both sides are equal to $(2, 6)$. ▲

Geometry of Vector Operations

Let us turn to the geometry of these operations in $\mathbb{R}^2$ and $\mathbb{R}^3$. For the moment, we define a ***vector*** to be a directed line segment beginning at the origin, that is, a line segment with specified magnitude and direction, and initial point at the origin. Figure 1.1.4 shows several vectors, drawn as arrows beginning at the origin. In print, vectors are usually denoted by boldface letters such as **a**. By hand, we usually write them as $\vec{a}$ or simply as a, possibly with a line or wavy line under it.

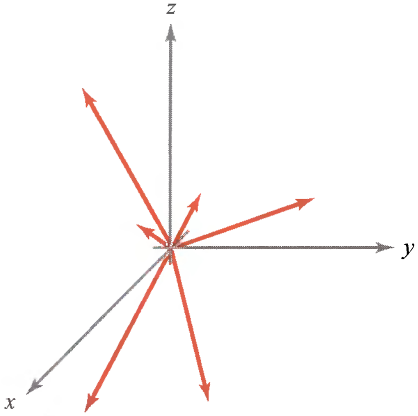


Figure 1.1.4 Geometrically, vectors are thought of as arrows emanating from the origin.

Using this definition of a vector, we associate with each vector **a** the point (a_1, a_2, a_3) where **a** terminates, and conversely, we can associate a vector **a** with each point (a_1, a_2, a_3) in space. Thus, we shall identify **a** with (a_1, a_2, a_3) and write $\mathbf{a} = (a_1, a_2, a_3)$. For this reason, the elements of $\mathbb{R}^3$ not only are ordered triples of real numbers, but are also regarded as vectors. The triple (0, 0, 0) is denoted **0**. We call a_1, a_2, and a_3 the ***components*** of **a**, or when we think of **a** as a point, its ***coordinates***.

Two vectors $\mathbf{a} = (a_1, a_2, a_3)$ and $\mathbf{b} = (b_1, b_2, b_3)$ are equal if and only if $a_1 = b_1$, $a_2 = b_2$, and $a_3 = b_3$. Geometrically this means that **a** and **b** have the same direction and the same length (or "magnitude").

Geometrically, we define vector addition as follows. In the plane containing the vectors $\mathbf{a} = (a_1, a_2, a_3)$ and $\mathbf{b} = (b_1, b_2, b_3)$ (see Figure 1.1.5), form the parallelogram having **a** as one side and **b** as its adjacent side. The sum $\mathbf{a} + \mathbf{b}$ is the directed line segment along the diagonal of the parallelogram.

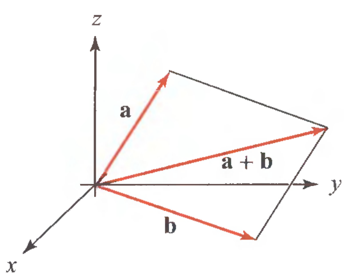


Figure 1.1.5 The geometry of vector addition.

This geometric view of vector addition is useful in many physical situations, as we shall see in the next section. For an easily visualized example, consider a bird or an airplane flying through the air with velocity $\mathbf{v}_1$, but in the presence of a wind with velocity $\mathbf{v}_2$. The resultant velocity, $\mathbf{v}_1 + \mathbf{v}_2$, is what one sees; see Figure 1.1.6.

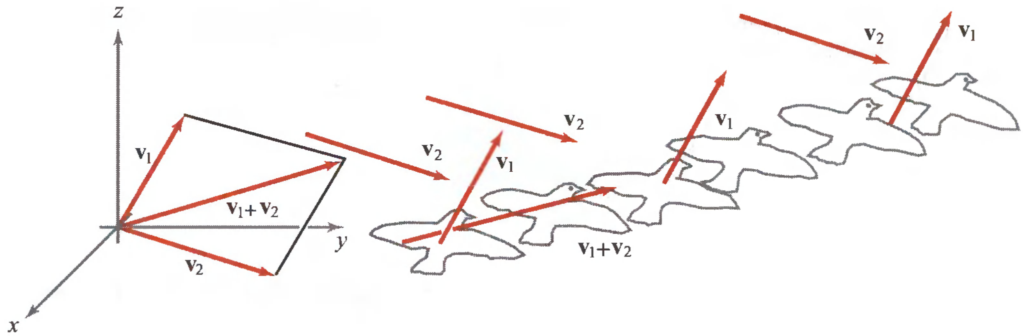


Figure 1.1.6 A physical interpretation of vector addition.

To show that our geometric definition of addition is consistent with our algebraic definition, we demonstrate that $\mathbf{a}+\mathbf{b}=(a_1+b_1, a_2+b_2, a_3+b_3)$. We shall prove this result in the plane and leave the proof in three-dimensional space to the reader. Thus, we wish to show that if $\mathbf{a}=(a_1, a_2)$ and $\mathbf{b}=(b_1, b_2)$, then $\mathbf{a}+\mathbf{b}=(a_1+b_1, a_2+b_2)$.

In Figure 1.1.7 let $\mathbf{a}=(a_1, a_2)$ be the vector ending at the point A, and let $\mathbf{b}=(b_1, b_2)$ be the vector ending at point B. By definition, the vector $\mathbf{a}+\mathbf{b}$ ends at the vertex C of parallelogram OBCA. To verify that $\mathbf{a}+\mathbf{b}=(a_1+b_1, a_2+b_2)$, it suffices to show that the coordinates of C are (a_1+b_1, a_2+b_2). The sides of the triangles OAD and BCG are parallel, and the sides OA and BC have equal lengths, which we write as OA = BC. These triangles are congruent, so BG = OD; since BGFE is a rectangle, EF = BG. Furthermore, OD $= a_1$ and OE $= b_1$. Hence, EF = BG = OD $= a_1$. Since OF = EF + OE, it follows that OF $= a_1 + b_1$. This shows that the x coordinate of $\mathbf{a}+\mathbf{b}$ is a_1+b_1. The proof that the y coordinate is a_2+b_2 is analogous. This argument assumes A and B to be in the first quadrant, but similar arguments hold for the other quadrants.

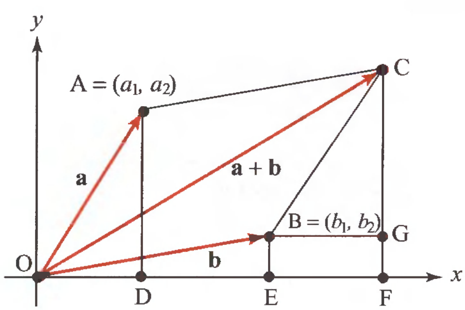


Figure 1.1.7 The construction used to prove that $(a_1, a_2)+(b_1, b_2)=(a_1+b_1, a_2+b_2)$.

Figure 1.1.8(a) illustrates another way of looking at vector addition: in terms of triangles rather than parallelograms. That is, we translate (without rotation) the directed line segment representing the vector $\mathbf{b}$ so that it begins at the end of the vector $\mathbf{a}$. The endpoint of the resulting directed segment is the endpoint of the vector

a + **b**. We note that when **a** and **b** are collinear, the triangle collapses to a line segment, as in Figure 1.1.8(b).

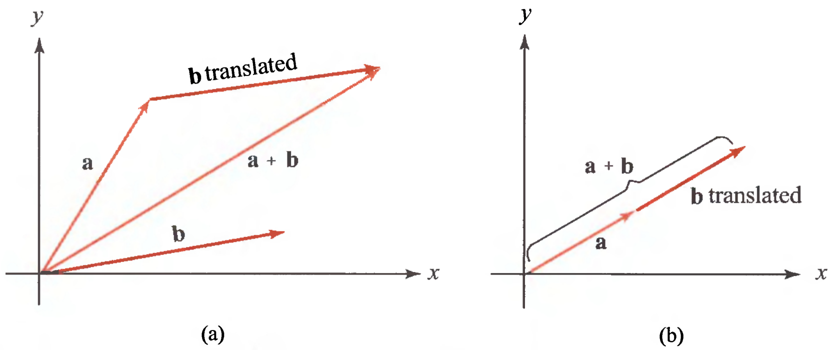


Figure 1.1.8 (a) Vector addition may be visualized in terms of triangles as well as parallelograms. (b) The triangle collapses to a line segment when **a** and **b** are collinear.

In Figure 1.1.8 we have placed **a** and **b** *head to tail*. That is, the tail of **b** is placed at the head of **a**, and the vector **a** + **b** goes from the tail of **a** to the head of **b**. If we do it in the other order, **b** + **a**, we get the same vector by going around the parallelogram the other way. Consistent with this figure, it is useful to let vectors "glide" or "slide," keeping the same magnitude and direction. We want, in fact, to regard two vectors as the *same* if they have the same magnitude and direction. When we insist on vectors beginning at the origin, we will say that we have ***bound vectors***. If we allow vectors to begin at other points, we will speak of ***free vectors*** or just ***vectors***.

Vectors Vectors (also called *free vectors*) are directed line segments in [the plane or] space represented by directed line segments with a beginning (tail) and an end (head). Directed line segments obtained from each other by parallel translation (but not rotation) represent the same vector.

The components (a_1, a_2, a_3) of **a** are the (signed) lengths of the projections of **a** along the three coordinate axes; equivalently, they are defined by placing the tail of **a** at the origin and letting the head be the point (a_1, a_2, a_3). We write $\mathbf{a} = (a_1, a_2, a_3)$.

Two vectors are added by placing them head to tail and drawing the vectors from the tail of the first to the head of the second, as in Figure 1.1.8.

Scalar multiplication of vectors also has a geometric interpretation. If α is a scalar and **a** a vector, we define $\alpha\mathbf{a}$ to be the vector that is $|\alpha|$ times as long as **a**, with the same direction as **a** if $\alpha > 0$, but with the opposite direction if $\alpha < 0$. Figure 1.1.9 illustrates several examples.

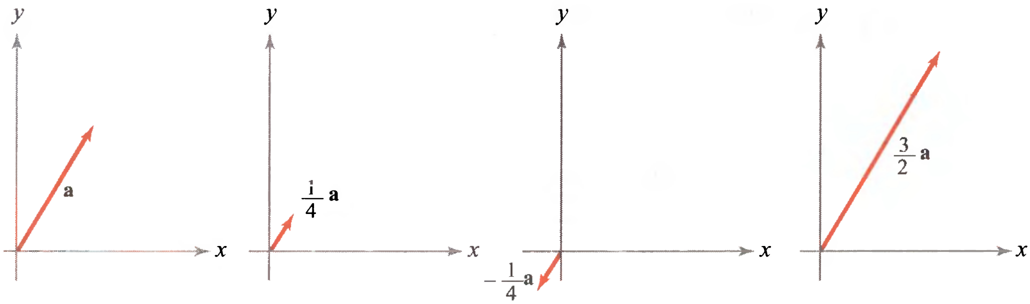


Figure 1.1.9 Some scalar multiples of a vector **a**.

Using an argument based on similar triangles, one finds that if $\mathbf{a} = (a_1, a_2, a_3)$, and α is a scalar, then

$$\alpha\mathbf{a} = (\alpha a_1, \alpha a_2, \alpha a_3).$$

That is, the geometric definition coincides with the algebraic one.

Given two vectors **a** and **b**, how do we represent the vector $\mathbf{b} - \mathbf{a}$ geometrically, that is, what is the geometry of vector subtraction? Because $\mathbf{a} + (\mathbf{b} - \mathbf{a}) = \mathbf{b}$, we see that $\mathbf{b} - \mathbf{a}$ is the vector that one adds to **a** to get **b**. In view of this, we may conclude that $\mathbf{b} - \mathbf{a}$ is the vector parallel to, and with the same magnitude as, the directed line segment beginning at the endpoint of **a** and terminating at the endpoint of **b** when **a** and **b** begin at the same point (see Figure 1.1.10).

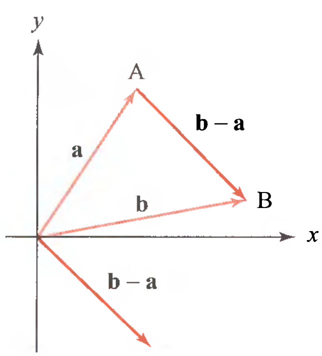


Figure 1.1.10 The geometry of vector subtraction.

EXAMPLE 4 Let **u** and **v** be the vectors shown in Figure 1.1.11. Draw the two vectors $\mathbf{u} + \mathbf{v}$ and $-2\mathbf{u}$. What are their components?

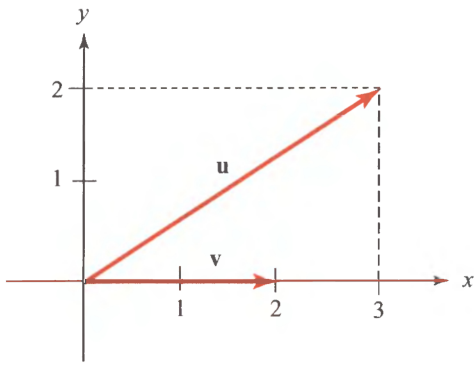


Figure 1.1.11 Find $\mathbf{u}+\mathbf{v}$ and $-2\mathbf{u}$.

SOLUTION Place the tail of $\mathbf{v}$ at the tip of $\mathbf{u}$ to obtain the vector shown in Figure 1.1.12.

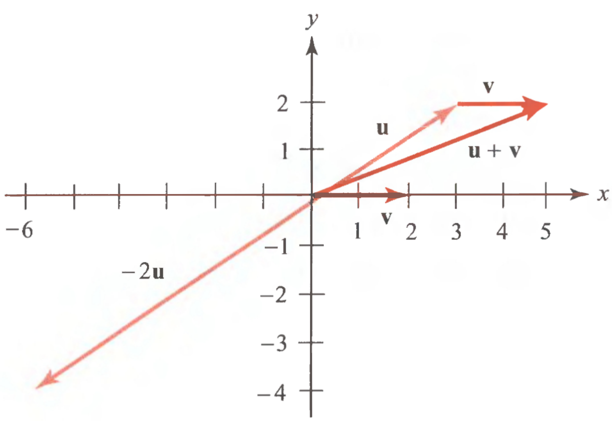


Figure 1.1.12 Computing $\mathbf{u}+\mathbf{v}$ and $-2\mathbf{u}$.

The vector $-2\mathbf{u}$, also shown, has length twice that of $\mathbf{u}$ and points in the opposite direction. From the figure, we see that the vector $\mathbf{u}+\mathbf{v}$ has components (5, 2) and $-2\mathbf{u}$ has components $(-6, -4)$. ▲

EXAMPLE 5

(a) Sketch $-2\mathbf{v}$, where $\mathbf{v}$ has components $(-1, 1, 2)$.

(b) If $\mathbf{v}$ and $\mathbf{w}$ are any two vectors, show that $\mathbf{v}-\frac{1}{3}\mathbf{w}$ and $3\mathbf{v}-\mathbf{w}$ are parallel.

SOLUTION

(a) The vector $-2\mathbf{v}$ is twice as long as $\mathbf{v}$, but points in the opposite direction (see Figure 1.1.13).

(b) $\mathbf{v}-\frac{1}{3}\mathbf{w}=\frac{1}{3}(3\mathbf{v}-\mathbf{w})$; vectors that are multiples of one another are parallel. ▲

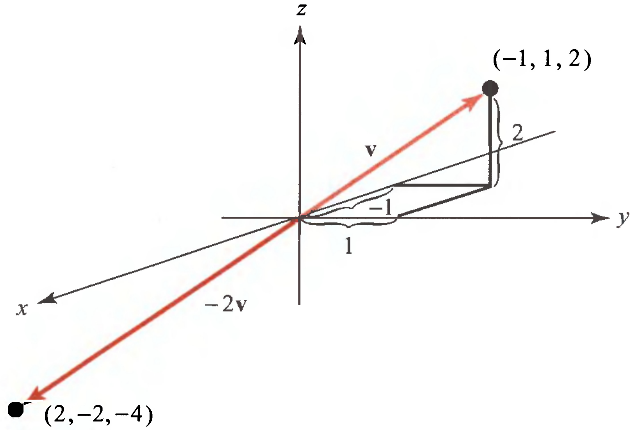


Figure 1.1.13 Multiplying $(-1, 1, 2)$ by -2.

The Standard Basis Vectors

To describe vectors in space, it is convenient to introduce three special vectors along the x, y, and z axes:

i: the vector with components $(1, 0, 0)$

j: the vector with components $(0, 1, 0)$

k: the vector with components $(0, 0, 1)$.

These ***standard basis vectors*** are illustrated in Figure 1.1.14. In the plane one has the standard basis **i** and **j** with components $(1, 0)$ and $(0, 1)$.

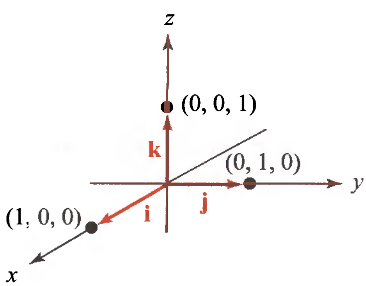


Figure 1.1.14 The standard basis vectors.

Let **a** be any vector, and let (a_1, a_2, a_3) be its components. Then

$$\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k},$$

because the right-hand side is given in components by

$$\begin{aligned} a_1(1, 0, 0) + a_2(0, 1, 0) + a_3(0, 0, 1) &= (a_1, 0, 0) + (0, a_2, 0) + (0, 0, a_3) \\ &= (a_1, a_2, a_3). \end{aligned}$$

Thus, we can express every vector as a sum of scalar multiples of $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$.

The Standard Basis Vectors

1. The vectors $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ are unit vectors along the three coordinate axes, as shown in Figure 1.1.14.
2. If $\mathbf{a}$ has components (a_1, a_2, a_3), then

$$\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}.$$

EXAMPLE 6 Express the vector whose components are $(e, \pi, -\sqrt{3})$ in the standard basis.

SOLUTION Substituting $a_1 = e$, $a_2 = \pi$, and $a_3 = -\sqrt{3}$ into $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ gives

$$\mathbf{v} = e\mathbf{i} + \pi\mathbf{j} - \sqrt{3}\mathbf{k}. \quad \blacktriangle$$

EXAMPLE 7 The vector $(2, 3, 2)$ equals $2\mathbf{i} + 3\mathbf{j} + 2\mathbf{k}$, and the vector $(0, -1, 4)$ is $-\mathbf{j} + 4\mathbf{k}$. Figure 1.1.15 shows $2\mathbf{i} + 3\mathbf{j} + 2\mathbf{k}$; the student should draw in the vector $-\mathbf{j} + 4\mathbf{k}$. $\blacktriangle$

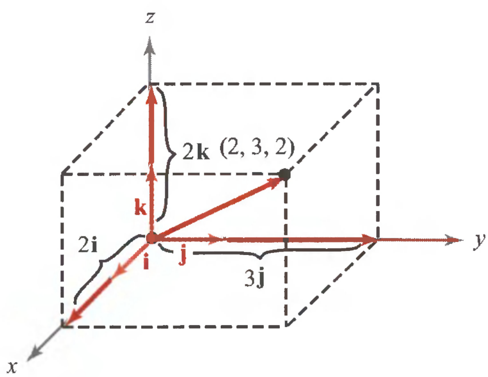


Figure 1.1.15 Representation of (2, 3, 2) in terms of the standard basis vectors **i**, **j**, and **k**.

Addition and scalar multiplication may be written in terms of the standard basis vectors as follows:

$$(a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}) + (b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}) = (a_1 + b_1)\mathbf{i} + (a_2 + b_2)\mathbf{j} + (a_3 + b_3)\mathbf{k}$$

and

$$\alpha(a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}) = (\alpha a_1)\mathbf{i} + (\alpha a_2)\mathbf{j} + \alpha(a_3)\mathbf{k}.$$

The Vector Joining Two Points

To apply vectors to geometric problems, it is useful to assign a vector to a *pair* of points in the plane or in space, as follows. Given two points P and P′, we can draw the vector **v** with tail P and head P′, as in Figure 1.1.16, where we write $\overrightarrow{PP'}$ for **v**.

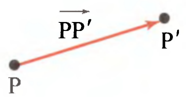


Figure 1.1.16 The vector from P to P′ is denoted $\overrightarrow{PP'}$.

If $P = (x, y, z)$ and $P' = (x', y', z')$, then the vectors from the origin to P and P′ are $\mathbf{a} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ and $\mathbf{a}' = x'\mathbf{i} + y'\mathbf{j} + z'\mathbf{k}$, respectively, so the vector $\overrightarrow{PP'}$ is the difference $\mathbf{a}' - \mathbf{a} = (x' - x)\mathbf{i} + (y' - y)\mathbf{j} + (z' - z)\mathbf{k}$. (See Figure 1.1.17.)

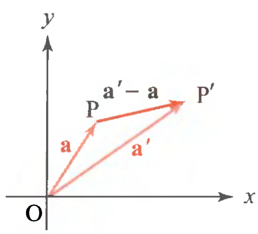


Figure 1.1.17 $\overrightarrow{PP'} = \overrightarrow{OP'} - \overrightarrow{OP}$.

The Vector Joining Two Points If the point P has coordinates (x, y, z) and P′ has coordinates (x', y', z'), then the vector $\overrightarrow{PP'}$ from the tip of P to the tip of P′ has components $(x' - x, y' - y, z' - z)$.

EXAMPLE 8

(a) Find the components of the vector from (3, 5) to (4, 7).

(b) Add the vector **v** from $(-1, 0)$ to $(2, -3)$ and the vector **w** from (2, 0) to (1, 1).

(c) Multiply the vector **v** in (b) by 8. If the resulting vector is represented by the directed line segment from (5, 6) to Q, what is Q?

SOLUTION

(a) As in the preceding box, we subtract the ordered pairs: $(4, 7) - (3, 5) = (1, 2)$. Thus the required components are (1, 2).

(b) The vector **v** has components $(2, -3) - (-1, 0) = (3, -3)$, and **w** has components $(1, 1) - (2, 0) = (-1, 1)$. Therefore, the vector $\mathbf{v} + \mathbf{w}$ has components $(3, -3) + (-1, 1) = (2, -2)$.

(c) The vector $8\mathbf{v}$ has components $8(3, -3) = (24, -24)$. If this vector is represented by the directed line segment from $(5, 6)$ to Q, and Q has coordinates (x, y), then $(x, y) - (5, 6) = (24, -24)$, so $(x, y) = (5, 6) + (24, -24) = (29, -18)$. ▲

EXAMPLE 9 Let $P = (-2, -1)$, $Q = (-3, -3)$, and $R = (-1, -4)$ in the xy plane.

(a) Draw these vectors: $\mathbf{v}$ joining P to Q; $\mathbf{w}$ joining Q to R; $\mathbf{u}$ joining R to P.

(b) What are the components of $\mathbf{v}$, $\mathbf{w}$, and $\mathbf{u}$?

(c) What is $\mathbf{v} + \mathbf{w} + \mathbf{u}$?

SOLUTION

(a) See Figure 1.1.18.

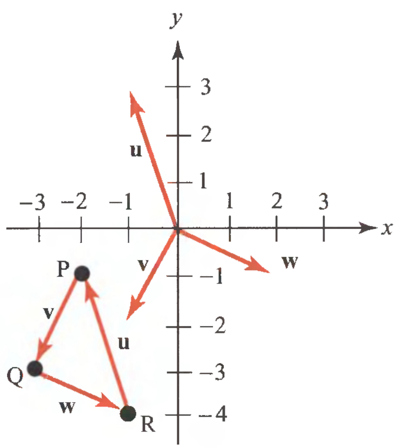


Figure 1.1.18 The vector $\mathbf{v}$ joins P to Q; $\mathbf{w}$ joins Q to R; and $\mathbf{u}$ joins R to P.

(b) Because $\mathbf{v} = \overrightarrow{PQ}$, $\mathbf{w} = \overrightarrow{QR}$, and $\mathbf{u} = \overrightarrow{RP}$, we get

$$\mathbf{v} = (-3, -3) - (-2, -1) = (-1, -2),$$
$$\mathbf{w} = (-1, -4) - (-3, -3) = (2, -1),$$
$$\mathbf{u} = -(-1, -4) + (-2, -1) = (-1, 3).$$

(c) $\mathbf{v} + \mathbf{w} + \mathbf{u} = (-1, -2) + (2, -1) + (-1, 3) = (0, 0)$. ▲

Geometry Theorems by Vector Methods

Many of the theorems of plane geometry can be proved by vector methods. Here is one example.

EXAMPLE 10 Use vectors to prove that the diagonals of a parallelogram bisect each other.

SOLUTION Let OPRQ be the parallelogram, with two adjacent sides represented by the vectors $\mathbf{a} = \overrightarrow{OP}$ and $\mathbf{b} = \overrightarrow{OQ}$. Let M be the midpoint of the diagonal OR, N the midpoint of the other diagonal, PQ. (See Figure 1.1.19.)

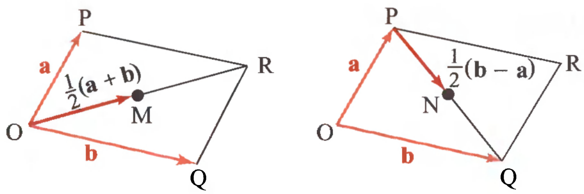


Figure 1.1.19 If the midpoints M and N coincide, then the diagonals OR and PQ bisect each other.

Observe that $\overrightarrow{OR} = \overrightarrow{OP} + \overrightarrow{OQ} = \mathbf{a} + \mathbf{b}$ by the parallelogram rule for vector addition, so $\overrightarrow{OM} = \frac{1}{2}\overrightarrow{OR} = \frac{1}{2}(\mathbf{a} + \mathbf{b})$. On the other hand,

$$\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP} = \mathbf{b} - \mathbf{a}, \qquad \text{so} \qquad \overrightarrow{PN} = \tfrac{1}{2}\overrightarrow{PQ} = \tfrac{1}{2}(\mathbf{b} - \mathbf{a}),$$

and hence

$$\overrightarrow{ON} = \overrightarrow{OP} + \overrightarrow{PN} = \mathbf{a} + \tfrac{1}{2}(\mathbf{b} - \mathbf{a}) = \tfrac{1}{2}(\mathbf{a} + \mathbf{b}).$$

Because $\overrightarrow{OM}$ and $\overrightarrow{ON}$ are equal vectors, the points M and N coincide, so the diagonals bisect each other. ▲

Equations of Lines

Planes and lines are geometric objects that can be represented by equations. We shall defer until Section 1.3 a study of equations representing planes. However, using the geometric interpretation of vector addition and scalar multiplication, we will now find the *equation of a line l that passes through the endpoint of the vector* $\mathbf{a}$, *with the direction of a vector* $\mathbf{v}$ (see Figure 1.1.20).

As t varies through all real values, the points of the form $t\mathbf{v}$ are all scalar multiples of the vector $\mathbf{v}$, and therefore exhaust the points of the line *passing through the origin* in the direction of $\mathbf{v}$. Because every point on l is the endpoint of the diagonal of a parallelogram with sides $\mathbf{a}$ and $t\mathbf{v}$ for some real value of t, we see that all the points on l are of the form $\mathbf{a} + t\mathbf{v}$. Thus, the line l may be expressed by the equation $\mathbf{l}(t) = \mathbf{a} + t\mathbf{v}$. We say that l is expressed ***parametrically***, with t the parameter. At $t = 0$, $\mathbf{l}(t) = \mathbf{a}$. As t increases, the point $\mathbf{l}(t)$ moves away from $\mathbf{a}$ in the direction of $\mathbf{v}$.

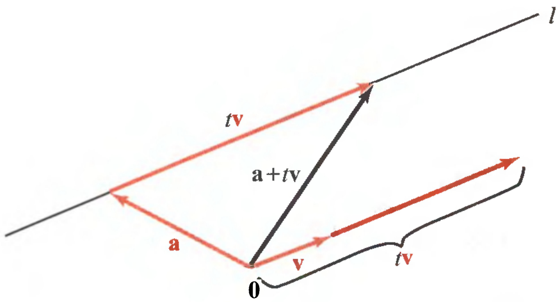


Figure 1.1.20 The line l, parametrically given by $\mathbf{l}(t) = \mathbf{a} + t\mathbf{v}$, lies in the direction $\mathbf{v}$ and passes through the tip of $\mathbf{a}$.

As t decreases from $t = 0$ through negative values, $\mathbf{l}(t)$ moves away from $\mathbf{a}$ in the direction of $-\mathbf{v}$.

> **Point-Direction Form of a Line** The equation of the line l through the tip of $\mathbf{a}$ and pointing in the direction of the vector $\mathbf{v}$ is $\mathbf{l}(t) = \mathbf{a} + t\mathbf{v}$, where the parameter t takes on all real values. In coordinate form, the equations are
>
> $$x = x_1 + at,$$
> $$y = y_1 + bt,$$
> $$z = z_1 + ct,$$
>
> where $\mathbf{a} = (x_1, y_1, z_1)$ and $\mathbf{v} = (a, b, c)$. For lines in the xy plane, one simply drops the z component.

EXAMPLE 11 Determine the equation of the line l passing through $(1, 0, 0)$ in the direction $\mathbf{j}$. See Figure 1.1.21.

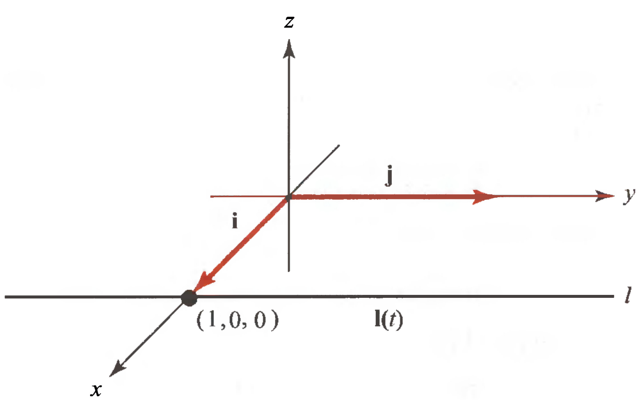


Figure 1.1.21 The line l passes through the tip of $\mathbf{i}$ in the direction $\mathbf{j}$.

SOLUTION The desired line can be expressed parametrically as $\mathbf{l}(t) = \mathbf{i} + t\mathbf{j}$. In terms of coordinates,

$$\mathbf{l}(t) = (1, 0, 0) + t(0, 1, 0) = (1, t, 0). \quad \blacktriangle$$

EXAMPLE 12

(a) Find the equations of the line in space through the point $(3, -1, 2)$ in the direction $2\mathbf{i} - 3\mathbf{j} + 4\mathbf{k}$.

(b) Find the equation of the line in the plane through the point $(1, -6)$ in the direction of $5\mathbf{i} - \pi\mathbf{j}$.

(c) In what direction does the line $x = -3t + 2$, $y = -2(t - 1)$, $z = 8t + 2$ point?

SOLUTION

(a) Here $\mathbf{a} = (3, -1, 2) = (x_1, y_1, z_1)$ and $\mathbf{v} = 2\mathbf{i} - 3\mathbf{j} + 4\mathbf{k}$, so $a = 2$, $b = -3$, and $c = 4$. From the box above, the equations are

$$x = 3 + 2t, \qquad y = -1 - 3t, \qquad z = 2 + 4t.$$

(b) Here $\mathbf{a} = (1, -6)$ and $\mathbf{v} = 5\mathbf{i} - \pi\mathbf{j}$, so the required line is

$$\mathbf{l}(t) = (1, -6) + (5t, -\pi t) = (1 + 5t, -6 - \pi t);$$

that is,

$$x = 1 + 5t, \qquad y = -6 - \pi t.$$

(c) Using the preceding box, we construct the direction $\mathbf{v} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$ from the coefficients of t: $a = -3$, $b = -2$, $c = 8$. Thus, the line points in the direction of $\mathbf{v} = -3\mathbf{i} - 2\mathbf{j} + 8\mathbf{k}$. $\blacktriangle$

EXAMPLE 13 Do the two lines $(x, y, z) = (t, -6t + 1, 2t - 8)$ and $(x, y, z) = (3t + 1, 2t, 0)$ intersect?

SOLUTION If the lines intersect, there must be numbers t_1 and t_2 such that the corresponding points are equal:

$$(t_1, -6t_1 + 1, 2t_1 - 8) = (3t_2 + 1, 2t_2, 0);$$

that is, all three of the following equations hold:

$$\begin{aligned} t_1 &= 3t_2 + 1, \\ -6t_1 + 1 &= 2t_2, \\ 2t_1 - 8 &= 0. \end{aligned}$$

From the third equation, $t_1 = 4$. The first equation then becomes $4 = 3t_2 + 1$; that is, $t_2 = 1$. We must check whether these values satisfy the middle equation:

$$-6t_1 + 1 \stackrel{?}{=} 2t_2.$$

Since $t_1 = 4$ and $t_2 = 1$, this reads

$$-24 + 1 \stackrel{?}{=} 2,$$

which is false, so the lines do not intersect. ▲

Notice that there can be many equations of the same line. Some may be obtained by choosing instead of $\mathbf{a}$, a different point on the given line, and forming the parametric equation of the line beginning at that point and in the direction of $\mathbf{v}$. For example, the endpoint of $\mathbf{a} + \mathbf{v}$ is on the line $\mathbf{l}(t) = \mathbf{a} + t\mathbf{v}$, and thus, $\mathbf{l}_1(t) = (\mathbf{a} + \mathbf{v}) + t\mathbf{v}$ represents the same line. Still other equations may be obtained by observing that if $\alpha \neq 0$, the vector $\alpha\mathbf{v}$ has the same (or opposite) direction as $\mathbf{v}$. Thus, $\mathbf{l}_2(t) = \mathbf{a} + t\alpha\mathbf{v}$ is another equation of $\mathbf{l}(t) = \mathbf{a} + t\mathbf{v}$.

For example, both $\mathbf{l}(t) = (1, 0, 0) + (t, t, 0)$ and $\mathbf{l}_1(s) = (0, -1, 0) + (s, s, 0)$ represent the same line since both are in the direction $\mathbf{i} + \mathbf{j}$ and both pass through the point $(1, 0, 0)$; $\mathbf{l}$ passes through $(1, 0, 0)$ at $t = 0$ and $\mathbf{l}_1$ passes through $(1, 0, 0)$ at $s = 1$.

Therefore, the equation of a line is not uniquely determined. Nevertheless, it is customary to use the term "the" equation of a line. Keeping this in mind, let us derive *the equation of a line passing through the endpoints of two given vectors* $\mathbf{a}$ *and* $\mathbf{b}$. Because the vector $\mathbf{b} - \mathbf{a}$ is parallel to the directed line segment from $\mathbf{a}$ to $\mathbf{b}$, we calculate the parametric equation of the line passing through $\mathbf{a}$ in the direction of $\mathbf{b} - \mathbf{a}$ (Figure 1.1.22). Thus,

$$\mathbf{l}(t) = \mathbf{a} + t(\mathbf{b} - \mathbf{a}); \qquad \text{that is,} \qquad \mathbf{l}(t) = (1 - t)\mathbf{a} + t\mathbf{b}.$$

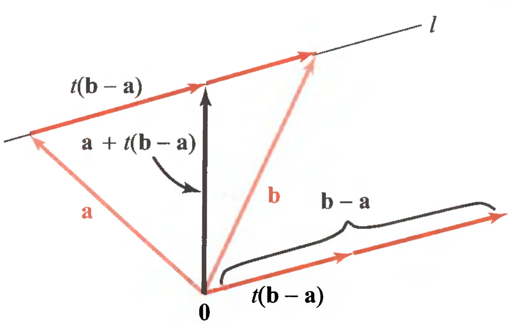


Figure 1.1.22 The line l, parametrically given by $\mathbf{l}(t) = \mathbf{a} + t(\mathbf{b} - \mathbf{a}) = (1 - t)\mathbf{a} + t\mathbf{b}$, passes through the tips of $\mathbf{a}$ and $\mathbf{b}$.

As t increases from 0 to 1, $t(\mathbf{b} - \mathbf{a})$ starts as the zero vector and increases in length (remaining in the direction of $\mathbf{b} - \mathbf{a}$) until at $t = 1$ it is the vector $\mathbf{b} - \mathbf{a}$. Thus, for

$\mathbf{l}(t) = \mathbf{a} + t(\mathbf{b} - \mathbf{a})$, as t increases from 0 to 1, the vector $\mathbf{l}(t)$ moves from the endpoint of $\mathbf{a}$ to the endpoint of $\mathbf{b}$ along the directed line segment from $\mathbf{a}$ to $\mathbf{b}$.

If $P = (x_1, y_1, z_1)$ is the tip of $\mathbf{a}$ and $Q = (x_2, y_2, z_2)$ is the tip of $\mathbf{b}$, then $\mathbf{v} = (x_2 - x_1)\mathbf{i} + (y_2 - y_1)\mathbf{j} + (z_2 - z_1)\mathbf{k}$, and so the equations of the line are

$$x = x_1 + (x_2 - x_1)t,$$
$$y = y_1 + (y_2 - y_1)t,$$
$$z = z_1 + (z_2 - z_1)t.$$

By eliminating t, these can be written as

$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1}.$$

Parametric Equation of a Line: Point–Point Form The parametric equations of the line l through the points $P = (x_1, y_1, z_1)$ and $Q = (x_2, y_2, z_2)$ are

$$x = x_1 + (x_2 - x_1)t,$$
$$y = y_1 + (y_2 - y_1)t,$$
$$z = z_1 + (z_2 - z_1)t,$$

where (x, y, z) is the general point of l, and the parameter t takes on all real values.

EXAMPLE 14 Find the equation of the line through $(2, 1, -3)$ and $(6, -1, -5)$.

SOLUTION Using the preceding box, we choose $(x_1, y_1, z_1) = (2, 1, -3)$ and $(x_2, y_2, z_2) = (6, -1, -5)$, so the equations are

$$x = 2 + (6 - 2)t = 2 + 4t,$$
$$y = 1 + (-1 - 1)t = 1 - 2t,$$
$$z = -3 + (-5 - (-3))t = -3 - 2t. \quad \blacktriangle$$

EXAMPLE 15 Find the equation of the line passing through $(-1, 1, 0)$ and $(0, 0, 1)$ (see Figure 1.1.23).

SOLUTION Letting $\mathbf{a} = -\mathbf{i} + \mathbf{j}$ and $\mathbf{b} = \mathbf{k}$ represent the given points, we have

$$\mathbf{l}(t) = (1 - t)(-\mathbf{i} + \mathbf{j}) + t\mathbf{k} = -(1 - t)\mathbf{i} + (1 - t)\mathbf{j} + t\mathbf{k}.$$

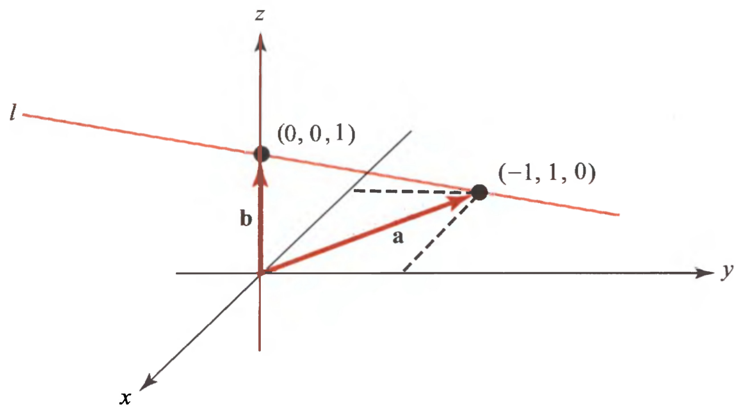


Figure 1.1.23 Finding the equation of the line through two points.

The equation of this line may thus be written as

$$\mathbf{l}(t) = (t-1)\mathbf{i} + (1-t)\mathbf{j} + t\mathbf{k},$$

or, equivalently, if $\mathbf{l}(t) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$,

$$x = t - 1, \qquad y = 1 - t, \qquad z = t. \quad \blacktriangle$$

The description of a line *segment* requires that the domain of the parameter t be restricted, as in the following example.

EXAMPLE 16 Find the equation of the line segment between (1, 1, 1) and (2, 1, 2).

SOLUTION The *line* through (1, 1, 1) and (2, 1, 2) is described in parametric form by $(x, y, z) = (1 + t, 1, 1 + t)$, as t takes on all real values. When $t = 0$, the point (x, y, z) is (1, 1, 1), and when $t = 1$, the point (x, y, z) is (2, 1, 2). Thus, the point (x, y, z) lies between (1, 1, 1) and (2, 1, 2) when $0 \le t \le 1$, so the line *segment* is described by the equations

$$x = 1 + t,$$
$$y = 1,$$
$$z = 1 + t,$$

together with the inequalities $0 \le t \le 1$. ▲

We can also give parametric descriptions of geometric objects other than lines.

EXAMPLE 17 Describe the points that lie within the parallelogram whose adjacent sides are the vectors **a** and **b** based at the origin (“within” includes points on the edges of the parallelogram).

SOLUTION Consider Figure 1.1.24. If P is any point within the given parallelogram and we construct lines l_1 and l_2 through P parallel to the vectors **a** and **b**, respectively, we see that l_1 intersects the side of the parallelogram determined by the vector **b** at some point $t\mathbf{b}$, where $0 \leq t \leq 1$. Likewise, l_2 intersects the side determined by the vector **a** at some point $s\mathbf{a}$, where $0 \leq s \leq 1$.

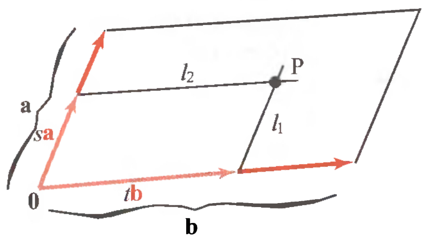


Figure 1.1.24 Describing points within the parallelogram formed by vectors **a** and **b**, with vertex **0**.

Note that P is the endpoint of the diagonal of a parallelogram having adjacent sides $s\mathbf{a}$ and $t\mathbf{b}$; hence, if **v** denotes the vector $\overrightarrow{\mathrm{OP}}$, we see that $\mathbf{v} = s\mathbf{a} + t\mathbf{b}$. We conclude that all the points in the given parallelogram are endpoints of vectors of the form $s\mathbf{a} + t\mathbf{b}$ for $0 \leq s \leq 1$ and $0 \leq t \leq 1$. Reversing our steps, we see that all vectors of this form end within the parallelogram. ▲

As two different lines through the origin determine a plane through the origin, so do two nonparallel vectors. If we apply the same reasoning as in Example 17, we see that the entire plane formed by two nonparallel vectors **v** and **w** consists of all points of the form $s\mathbf{v} + t\mathbf{w}$ where s and t can be any real numbers, as in Figure 1.1.25.

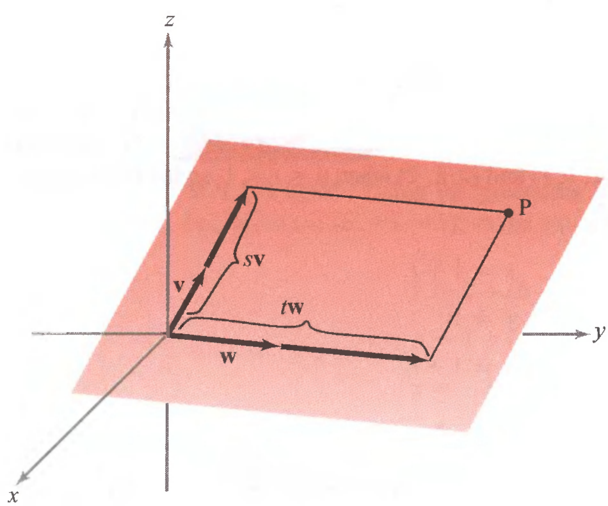


Figure 1.1.25 Describing points P in the plane formed from vectors **v** and **w**.

We have thus described the points in the plane by two parameters. For this reason, we say the plane is ***two-dimensional***. Similarly, a line is called ***one-dimensional*** whether it lies in the plane or in space or is the real number line itself.

The plane determined by **v** and **w** is called the plane ***spanned by*** **v** and **w**. When **v** is a scalar multiple of **w** and $\mathbf{w} \neq \mathbf{0}$, then **v** and **w** are parallel and the plane degenerates to a straight line. When $\mathbf{v} = \mathbf{w} = \mathbf{0}$ (that is, both are zero vectors), we obtain a single point.

There are three particular planes that arise naturally in a coordinate system and that will be useful to us later. We call the plane spanned by vectors **i** and **j** the xy plane, the plane spanned by **j** and **k** the yz plane, and the plane spanned by **i** and **k** the xz plane. These planes are illustrated in Figure 1.1.26.

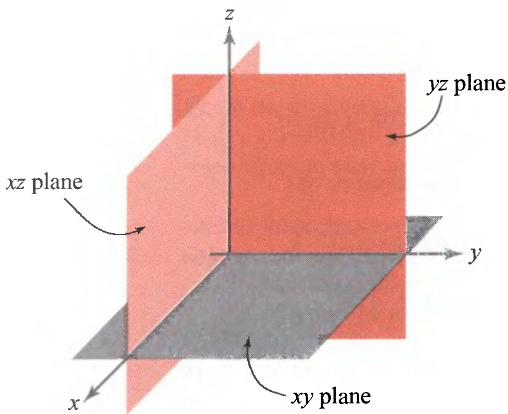


Figure 1.1.26 The three coordinate planes.

EXERCISES

(*Exercises with colored numbers are solved in the Study Guide.*)

Complete the computations in Exercises 1 to 4.

1. $(-21, 23) - (?, 6) = (-25, ?)$
2. $3(133, -0.33, 0) + (-399, 0.99, 0) = (?, ?, ?)$
3. $(8a, -2b, 13c) = (52, 12, 11) + \frac{1}{2}(?, ?, ?)$
4. $(2, 3, 5) - 4\mathbf{i} + 3\mathbf{j} = (?, ?, ?)$

In Exercises 5 to 8, sketch the given vectors **v** *and* **w**. *On your sketch, draw in* $-\mathbf{v}$, $\mathbf{v} + \mathbf{w}$, *and* $\mathbf{v} - \mathbf{w}$.

5. $\mathbf{v} = (2, 1)$ and $\mathbf{w} = (1, 2)$
6. $\mathbf{v} = (0, 4)$ and $\mathbf{w} = (2, -1)$
7. $\mathbf{v} = (2, 3, -6)$ and $\mathbf{w} = (-1, 1, 1)$
8. $\mathbf{v} = (2, 1, 3)$ and $\mathbf{w} = (-2, 0, -1)$

9. What restrictions must be made on x, y, and z so that the triple (x, y, z) will represent a point on the y axis? On the z axis? In the xz plane? In the yz plane?

10. (a) Generalize the geometric construction in Figure 1.1.7 to show that if $\mathbf{v}_1 = (x, y, z)$ and $\mathbf{v}_2 = (x', y', z')$, then $\mathbf{v}_1 + \mathbf{v}_2 = (x + x', y + y', z + z')$.

(b) Using an argument based on similar triangles, prove that $\alpha\mathbf{v} = (\alpha x, \alpha y, \alpha z)$ when $\mathbf{v} = (x, y, z)$.

In Exercises 11 to 17, use set theoretic or vector notation or both to describe the points that lie in the given configurations.

11. The plane spanned by $\mathbf{v}_1 = (2, 7, 0)$ and $\mathbf{v}_2 = (0, 2, 7)$

12. The plane spanned by $\mathbf{v}_1 = (3, -1, 1)$ and $\mathbf{v}_2 = (0, 3, 4)$

13. The line passing through $(-1, -1, -1)$ in the direction of $\mathbf{j}$

14. The line passing through $(0, 2, 1)$ in the direction of $2\mathbf{i} - \mathbf{k}$

15. The line passing through $(-1, -1, -1)$ and $(1, -1, 2)$

16. The line passing through $(-5, 0, 4)$ and $(6, -3, 2)$

17. The parallelogram whose adjacent sides are the vectors $\mathbf{i} + 3\mathbf{k}$ and $-2\mathbf{j}$

18. Find the points of intersection of the line $x = 3 + 2t$, $y = 7 + 8t$, $z = -2 + t$, that is, $\mathbf{l}(t) = (3 + 2t, 7 + 8t, -2 + t)$, with the coordinate planes.

19. Show that there are no points (x, y, z) satisfying $2x - 3y + z - 2 = 0$ and lying on the line $\mathbf{v} = (2, -2, -1) + t(1, 1, 1)$.

20. Show that every point on the line $\mathbf{v} = (1, -1, 2) + t(2, 3, 1)$ satisfies the equation $5x - 3y - z - 6 = 0$.

21. Determine whether the lines $x = 3t + 2$, $y = t - 1$, $z = 6t + 1$, and $x = 3s - 1$, $y = s - 2$, $z = s$ intersect.

22. Do the lines $(x, y, z) = (t + 4, 4t + 5, t - 2)$ and $(x, y, z) = (2s + 3, s + 1, 2s - 3)$ intersect?

In Exercises 23 to 25, use vector methods to describe the given configurations.

23. The parallelepiped with edges the vectors $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ emanating from the origin.

24. The points within the parallelogram with one corner at (x_0, y_0, z_0) whose sides extending from that corner are equal in magnitude and direction to vectors $\mathbf{a}$ and $\mathbf{b}$.

25. The plane determined by the three points (x_0, y_0, z_0), (x_1, y_1, z_1), and (x_2, y_2, z_2).

Prove the statements in Exercises 26 to 28.

26. The line segment joining the midpoints of two sides of a triangle is parallel to and has half the length of the third side.

27. If PQR is a triangle in space and $b > 0$ is a number, then there is a triangle with sides parallel to those of PQR and side lengths b times those of PQR.

28. The medians of a triangle intersect at a point, and this point divides each median in a ratio of 2 : 1.

Problems 29 and 30 require some knowledge of chemical notation.

29. Write the chemical equation $CO + H_2O = H_2 + CO_2$ as an equation in ordered triples (x_1, x_2, x_3) where x_1, x_2, x_3 are the number of carbon, hydrogen, and oxygen atoms, respectively, in each molecule.

30. **(a)** Write the chemical equation $pC_3H_4O_3 + qO_2 = rCO_2 + sH_2O$ as an equation in ordered triples with unknown coefficients p, q, r, and s.
(b) Find the smallest positive integer solution for p, q, r, and s.
(c) Illustrate the solution by a vector diagram in space.

31. Find a line that lies entirely in the set defined by the equation $x^2 + y^2 - z^2 = 1$.

1.2 The Inner Product, Length, and Distance

In this section and the next we shall discuss two products of vectors: the inner product and the cross product. These are very useful in physical applications and have interesting geometric interpretations. The first product we shall consider is called the *inner product*. The name *dot product* is often used instead.

The Inner Product

Suppose we have two vectors **a** and **b** in $\mathbb{R}^3$ (Figure 1.2.1) and we wish to determine the angle between them, that is, the smaller angle subtended by **a** and **b** in the plane

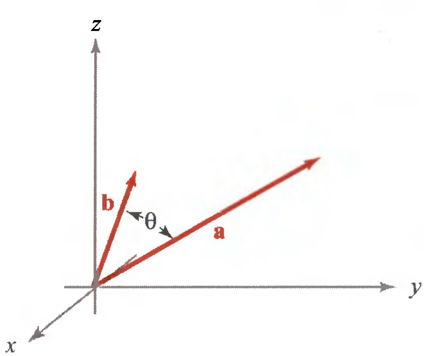


Figure 1.2.1 θ is the angle between the vectors **a** and **b**.

that they span. The inner product enables us to do this. Let us first develop the concept formally and then prove that this product does what we claim. Let $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ and $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$. We define the ***inner product*** of $\mathbf{a}$ and $\mathbf{b}$, written $\mathbf{a}\cdot\mathbf{b}$, to be the real number

$$\mathbf{a}\cdot\mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3.$$

Note that the inner product of two vectors is a scalar quantity. Sometimes the inner product is denoted $\langle \mathbf{a}, \mathbf{b}\rangle$; *thus,* $\langle \mathbf{a}, \mathbf{b}\rangle$ *and* $\mathbf{a}\cdot\mathbf{b}$ *mean exactly the same thing.*

EXAMPLE 1

(a) If $\mathbf{a} = 3\mathbf{i} + \mathbf{j} - 2\mathbf{k}$ and $\mathbf{b} = \mathbf{i} - \mathbf{j} + \mathbf{k}$, calculate $\mathbf{a}\cdot\mathbf{b}$.

(b) Calculate $(2\mathbf{i} + \mathbf{j} - \mathbf{k})\cdot(3\mathbf{k} - 2\mathbf{j})$.

SOLUTION

(a) $\mathbf{a}\cdot\mathbf{b} = 3\cdot 1 + 1\cdot(-1) + (-2)\cdot 1 = 3 - 1 - 2 = 0.$

(b) $$\begin{aligned}(2\mathbf{i} + \mathbf{j} - \mathbf{k})\cdot(3\mathbf{k} - 2\mathbf{j}) &= (2\mathbf{i} + \mathbf{j} - \mathbf{k})\cdot(0\mathbf{i} - 2\mathbf{j} + 3\mathbf{k})\\ &= 2\cdot 0 - 1\cdot 2 - 1\cdot 3 = -5.\end{aligned}$$ ▲

Certain properties of the inner product follow from the definition. If $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ are vectors in $\mathbb{R}^3$ and α and β are real numbers, then

(i) $\mathbf{a}\cdot\mathbf{a} \geq 0$;

$\mathbf{a}\cdot\mathbf{a} = 0$ if and only if $\mathbf{a} = \mathbf{0}$.

(ii) $\alpha\mathbf{a}\cdot\mathbf{b} = \alpha(\mathbf{a}\cdot\mathbf{b})$ and $\mathbf{a}\cdot\beta\mathbf{b} = \beta(\mathbf{a}\cdot\mathbf{b})$.

(iii) $\mathbf{a}\cdot(\mathbf{b} + \mathbf{c}) = \mathbf{a}\cdot\mathbf{b} + \mathbf{a}\cdot\mathbf{c}$ and $(\mathbf{a} + \mathbf{b})\cdot\mathbf{c} = \mathbf{a}\cdot\mathbf{c} + \mathbf{b}\cdot\mathbf{c}$.

(iv) $\mathbf{a}\cdot\mathbf{b} = \mathbf{b}\cdot\mathbf{a}$.

To prove the first of these properties, observe that if $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$, then $\mathbf{a}\cdot\mathbf{a} = a_1^2 + a_2^2 + a_3^2$. Because a_1, a_2, and a_3 are real numbers, we know $a_1^2 \geq 0$, $a_2^2 \geq 0$, $a_3^2 \geq 0$. Thus, $\mathbf{a}\cdot\mathbf{a} \geq 0$. Moreover, if $a_1^2 + a_2^2 + a_3^2 = 0$, then $a_1 = a_2 = a_3 = 0$; therefore, $\mathbf{a} = \mathbf{0}$ (zero vector). The proofs of the other properties of the inner product are also easily obtained.

It follows from the Pythagorean theorem that the ***length*** of the vector $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ is $\sqrt{a_1^2 + a_2^2 + a_3^2}$ (see Figure 1.2.2). The length of the vector $\mathbf{a}$ is denoted by $\|\mathbf{a}\|$. This quantity is often called the ***norm*** of $\mathbf{a}$. Because $\mathbf{a}\cdot\mathbf{a} = a_1^2 + a_2^2 + a_3^2$, it follows that

$$\|\mathbf{a}\| = (\mathbf{a}\cdot\mathbf{a})^{1/2}.$$

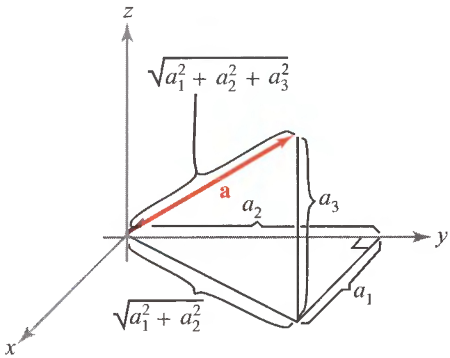


Figure 1.2.2 The length of the vector $\mathbf{a} = (a_1, a_2, a_3)$ is given by the Pythagorean formula: $\sqrt{a_1^2 + a_2^2 + a_3^2}$.

Unit Vectors

Vectors with norm 1 are called ***unit vectors***. For example, the vectors $\mathbf{i}, \mathbf{j}, \mathbf{k}$ are unit vectors. Observe that for any nonzero vector $\mathbf{a}$, $\mathbf{a}/\|\mathbf{a}\|$ is a unit vector; when we divide $\mathbf{a}$ by $\|\mathbf{a}\|$, we say that we have ***normalized*** $\mathbf{a}$.

EXAMPLE 2

(a) Normalize $\mathbf{v} = 2\mathbf{i} + 3\mathbf{j} - \frac{1}{2}\mathbf{k}$.

(b) Find unit vectors $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ in the plane such that $\mathbf{b} + \mathbf{c} = \mathbf{a}$.

SOLUTION

(a) We have $\|\mathbf{v}\| = \sqrt{2^2 + 3^2 + (1/2)^2} = (1/2)\sqrt{53}$, so the normalization of $\mathbf{v}$ is

$$\mathbf{u} = \frac{1}{\|\mathbf{v}\|}\mathbf{v} = \frac{4}{\sqrt{53}}\mathbf{i} + \frac{6}{\sqrt{53}}\mathbf{j} - \frac{1}{\sqrt{53}}\mathbf{k}.$$

(b) Because all three vectors are to have length 1, a triangle with sides $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ must be equilateral, as in Figure 1.2.3. Orienting the triangle as in the figure, we take $\mathbf{a} = \mathbf{i}$, then necessarily

$$\mathbf{b} = \frac{1}{2}\mathbf{i} + \frac{\sqrt{3}}{2}\mathbf{j}, \qquad \text{and} \qquad \mathbf{c} = \frac{1}{2}\mathbf{i} - \frac{\sqrt{3}}{2}\mathbf{j}.$$

Note that indeed $\|\mathbf{a}\| = \|\mathbf{b}\| = \|\mathbf{c}\| = 1$ and that $\mathbf{b} + \mathbf{c} = \mathbf{a}$. ▲

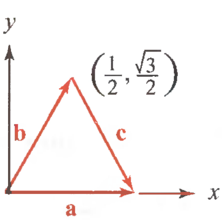


Figure 1.2.3 The vectors $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ are represented by the sides of an equilateral triangle.

In the plane, define the vector $\mathbf{i}_\theta = (\cos\theta)\mathbf{i} + (\sin\theta)\mathbf{j}$, which is the unit vector making an angle θ with the x axis (see Figure 1.2.4).

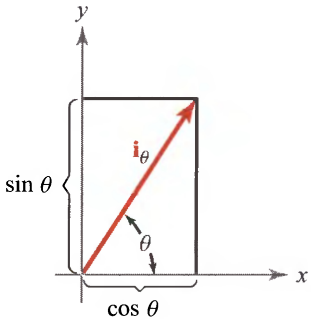


Figure 1.2.4 The coordinates of $\mathbf{i}_\theta$ are $\cos\theta$ and $\sin\theta$; it is a unit vector because $\cos^2\theta + \sin^2\theta = 1$.

Distance

If $\mathbf{a}$ and $\mathbf{b}$ are vectors, we have seen that the vector $\mathbf{b} - \mathbf{a}$ is parallel to and has the same magnitude as the directed line segment from the endpoint of $\mathbf{a}$ to the endpoint of $\mathbf{b}$. It follows that the distance from the endpoint of $\mathbf{a}$ to the endpoint of $\mathbf{b}$ is $\|\mathbf{b} - \mathbf{a}\|$ (see Figure 1.2.5).

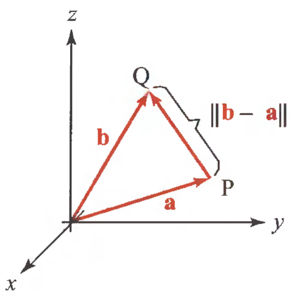


Figure 1.2.5 The distance between the tips of $\mathbf{a}$ and $\mathbf{b}$ is $\|\mathbf{b} - \mathbf{a}\|$.

Inner Product, Length, and Distance Letting $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ and $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$, their ***inner product*** is

$$\mathbf{a}\cdot\mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3,$$

while the ***length*** of $\mathbf{a}$ is

$$\|\mathbf{a}\| = \sqrt{\mathbf{a}\cdot\mathbf{a}} = \sqrt{a_1^2 + a_2^2 + a_3^2}.$$

To ***normalize*** a vector $\mathbf{a}$, form the vector

$$\frac{\mathbf{a}}{\|\mathbf{a}\|}.$$

The ***distance between*** the endpoints of $\mathbf{a}$ and $\mathbf{b}$ is $\|\mathbf{a} - \mathbf{b}\|$, and the ***distance between*** P and Q is $\|\overrightarrow{\mathrm{PQ}}\|$.

EXAMPLE 3 Find the distance from the endpoint of the vector **i**, that is, the point (1, 0, 0), to the endpoint of the vector **j**, that is, the point (0, 1, 0).

SOLUTION $\|\mathbf{j}-\mathbf{i}\| = \sqrt{(0-1)^2+(1-0)^2+(0-0)^2} = \sqrt{2}$. ▲

The Angle Between Two Vectors

Let us now show that the inner product does indeed measure the angle between two vectors.

THEOREM 1 Let **a** and **b** be two vectors in $\mathbb{R}^3$ and let θ, where $0 \le \theta \le \pi$, be the angle between them (Figure 1.2.6). Then

$$\mathbf{a}\cdot\mathbf{b} = \|\mathbf{a}\|\,\|\mathbf{b}\|\cos\theta.$$

It follows from the equation $\mathbf{a}\cdot\mathbf{b} = \|\mathbf{a}\|\,\|\mathbf{b}\|\cos\theta$ that if **a** and **b** are nonzero, we may express the angle between them as

$$\theta = \cos^{-1}\left(\frac{\mathbf{a}\cdot\mathbf{b}}{\|\mathbf{a}\|\,\|\mathbf{b}\|}\right).$$

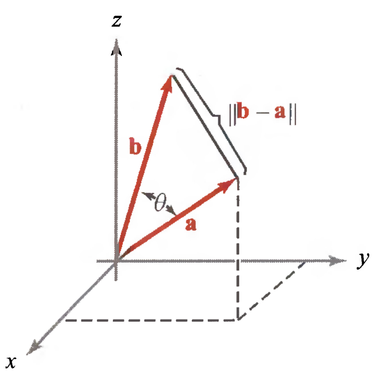


Figure 1.2.6 The vectors **a**, **b**, and the angle θ between them; the geometry for Theorem 1 and its proof.

PROOF If we apply the law of cosines from trigonometry to the triangle with one vertex at the origin and adjacent sides determined by the vectors **a** and **b** (as in the figure), it follows that

$$\|\mathbf{b}-\mathbf{a}\|^2 = \|\mathbf{a}\|^2 + \|\mathbf{b}\|^2 - 2\|\mathbf{a}\|\,\|\mathbf{b}\|\cos\theta.$$

Because $\|\mathbf{b}-\mathbf{a}\|^2 = (\mathbf{b}-\mathbf{a})\cdot(\mathbf{b}-\mathbf{a})$, $\|\mathbf{a}\|^2 = \mathbf{a}\cdot\mathbf{a}$, and $\|\mathbf{b}\|^2 = \mathbf{b}\cdot\mathbf{b}$, we can rewrite the above equation as

$$(\mathbf{b}-\mathbf{a})\cdot(\mathbf{b}-\mathbf{a}) = \mathbf{a}\cdot\mathbf{a}+\mathbf{b}\cdot\mathbf{b}-2\|\mathbf{a}\|\|\mathbf{b}\|\cos\theta.$$

We can also expand $(\mathbf{b}-\mathbf{a})\cdot(\mathbf{b}-\mathbf{a})$ as follows:

$$\begin{aligned}(\mathbf{b}-\mathbf{a})\cdot(\mathbf{b}-\mathbf{a}) &= \mathbf{b}\cdot(\mathbf{b}-\mathbf{a})-\mathbf{a}\cdot(\mathbf{b}-\mathbf{a})\\ &= \mathbf{b}\cdot\mathbf{b}-\mathbf{b}\cdot\mathbf{a}-\mathbf{a}\cdot\mathbf{b}+\mathbf{a}\cdot\mathbf{a}\\ &= \mathbf{a}\cdot\mathbf{a}+\mathbf{b}\cdot\mathbf{b}-2\mathbf{a}\cdot\mathbf{b}.\end{aligned}$$

Thus,

$$\mathbf{a}\cdot\mathbf{a}+\mathbf{b}\cdot\mathbf{b}-2\mathbf{a}\cdot\mathbf{b} = \mathbf{a}\cdot\mathbf{a}+\mathbf{b}\cdot\mathbf{b}-2\|\mathbf{a}\|\|\mathbf{b}\|\cos\theta.$$

That is,

$$\mathbf{a}\cdot\mathbf{b} = \|\mathbf{a}\|\|\mathbf{b}\|\cos\theta. \quad \blacksquare$$

EXAMPLE 4 Find the angle between the vectors $\mathbf{i}+\mathbf{j}+\mathbf{k}$ and $\mathbf{i}+\mathbf{j}-\mathbf{k}$ (see Figure 1.2.7).

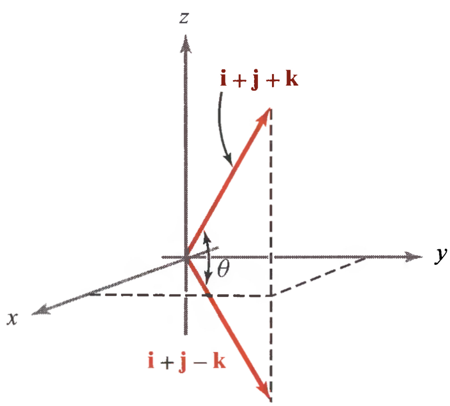


Figure 1.2.7 Finding the angle between $\mathbf{a}=\mathbf{i}+\mathbf{j}+\mathbf{k}$ and $\mathbf{b}=\mathbf{i}+\mathbf{j}-\mathbf{k}$.

SOLUTION Using Theorem 1, we have

$$(\mathbf{i}+\mathbf{j}+\mathbf{k})\cdot(\mathbf{i}+\mathbf{j}-\mathbf{k}) = \|\mathbf{i}+\mathbf{j}+\mathbf{k}\|\|\mathbf{i}+\mathbf{j}-\mathbf{k}\|\cos\theta,$$

and so

$$1+1-1 = (\sqrt{3})(\sqrt{3})\cos\theta.$$

Hence,

$$\cos\theta = \tfrac{1}{3}.$$

That is,

$$\theta = \cos^{-1}(\tfrac{1}{3}) \approx 1.23 \text{ radians } (71^\circ). \quad \blacktriangle$$

The Cauchy–Schwarz Inequality

Theorem 1 shows that the inner product of two vectors is the product of their lengths times the cosine of the angle between them. This relationship is often of value in problems of a geometric nature. An important consequence of Theorem 1 is:

COROLLARY: Cauchy–Schwarz Inequality For any two vectors $\mathbf{a}$ and $\mathbf{b}$, we have

$$|\mathbf{a}\cdot\mathbf{b}| \leq \|\mathbf{a}\|\,\|\mathbf{b}\|$$

with equality if and only if $\mathbf{a}$ is a scalar multiple of $\mathbf{b}$, or one of them is $\mathbf{0}$.

PROOF If $\mathbf{a}$ is not a scalar multiple of $\mathbf{b}$, then θ, the angle between them, is not zero or π, and so $|\cos\theta| < 1$, and thus the inequality holds; in fact, if $\mathbf{a}$ and $\mathbf{b}$ are both nonzero, *strict* inequality holds in this case. When $\mathbf{a}$ is a scalar multiple of $\mathbf{b}$, then $\theta = 0$ or π and $|\cos\theta| = 1$, so equality holds in this case. ■

EXAMPLE 5 Verify the Cauchy–Schwarz inequality for $\mathbf{a} = -\mathbf{i}+\mathbf{j}+\mathbf{k}$ and $\mathbf{b} = 3\mathbf{i}+\mathbf{k}$.

SOLUTION The dot product is $\mathbf{a}\cdot\mathbf{b} = -3+0+1 = -2$, so $|\mathbf{a}\cdot\mathbf{b}| = 2$. Also, $\|\mathbf{a}\| = \sqrt{1+1+1} = \sqrt{3}$ and $\|\mathbf{b}\| = \sqrt{9+1} = \sqrt{10}$, and it is true that $2 \leq \sqrt{3}\cdot\sqrt{10}$ because $\sqrt{3}\cdot\sqrt{10} > \sqrt{3}\cdot\sqrt{3} = 3 \geq 2$. ▲

If $\mathbf{a}$ and $\mathbf{b}$ are nonzero vectors in $\mathbb{R}^3$ and θ is the angle between them, we see that $\mathbf{a}\cdot\mathbf{b} = 0$ if and only if $\cos\theta = 0$. Thus, *the inner product of two nonzero vectors is zero if and only if the vectors are perpendicular*. Hence, the inner product provides us with a convenient method for determining whether two vectors are perpendicular. Often we say that perpendicular vectors are ***orthogonal***. The standard basis vectors $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ are mutually orthogonal and of length 1; any such system is called ***orthonormal***. We shall adopt the convention that the zero vector is orthogonal to all vectors.

EXAMPLE 6 The vectors $\mathbf{i}_\theta = (\cos\theta)\mathbf{i} + (\sin\theta)\mathbf{j}$ and $\mathbf{j}_\theta = -(\sin\theta)\mathbf{i} + (\cos\theta)\mathbf{j}$ are orthogonal, because

$$\mathbf{i}_\theta \cdot \mathbf{j}_\theta = -\cos\theta\sin\theta + \sin\theta\cos\theta = 0$$

(see Figure 1.2.8). ▲

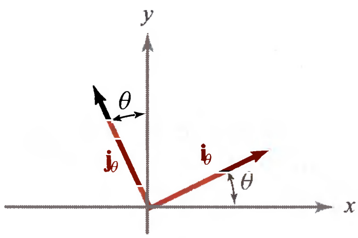


Figure 1.2.8 The vectors $\mathbf{i}_\theta$ and $\mathbf{j}_\theta$ are orthogonal and of unit length, that is, they are orthonormal.

EXAMPLE 7 Let $\mathbf{a}$ and $\mathbf{b}$ be two nonzero orthogonal vectors. If $\mathbf{c}$ is a vector in the plane spanned by $\mathbf{a}$ and $\mathbf{b}$, then there are scalars α and β such that $\mathbf{c} = \alpha\mathbf{a} + \beta\mathbf{b}$. Use the inner product to determine α and β (see Figure 1.2.9).

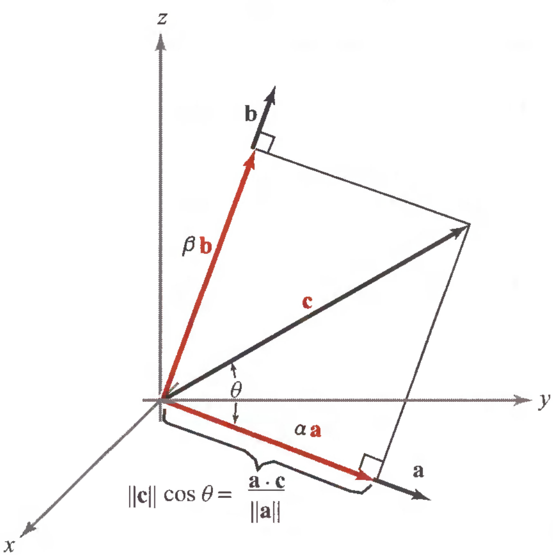


Figure 1.2.9 The geometry for finding α and β, where $\mathbf{c} = \alpha\mathbf{a} + \beta\mathbf{b}$.

SOLUTION Taking the inner product of $\mathbf{a}$ and $\mathbf{c}$, we have

$$\mathbf{a}\cdot\mathbf{c} = \mathbf{a}\cdot(\alpha\mathbf{a} + \beta\mathbf{b}) = \alpha\mathbf{a}\cdot\mathbf{a} + \beta\mathbf{a}\cdot\mathbf{b}.$$

Because $\mathbf{a}$ and $\mathbf{b}$ are orthogonal, $\mathbf{a} \cdot \mathbf{b} = 0$, and so

$$\alpha = \frac{\mathbf{a} \cdot \mathbf{c}}{\mathbf{a} \cdot \mathbf{a}} = \frac{\mathbf{a} \cdot \mathbf{c}}{\|\mathbf{a}\|^2}.$$

Similarly,

$$\beta = \frac{\mathbf{b} \cdot \mathbf{c}}{\mathbf{b} \cdot \mathbf{b}} = \frac{\mathbf{b} \cdot \mathbf{c}}{\|\mathbf{b}\|^2}. \quad \blacktriangle$$

Orthogonal Projection

In the preceding example, the vector $\alpha\mathbf{a}$ is called the ***projection of*** $\mathbf{c}$ ***along*** $\mathbf{a}$, and $\beta\mathbf{b}$ is its ***projection along*** $\mathbf{b}$. Let us formulate this idea more generally. If $\mathbf{v}$ is a vector, and l is the line through the origin in the direction of a vector $\mathbf{a}$, then the ***orthogonal projection of*** $\mathbf{v}$ ***on*** $\mathbf{a}$ is the vector $\mathbf{p}$ whose tip is obtained by dropping a perpendicular line to l from the tip of $\mathbf{v}$, as in Figure 1.2.10.

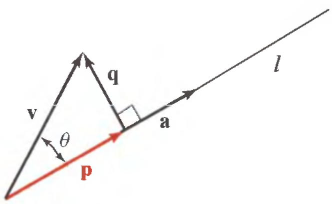


Figure 1.2.10 $\mathbf{p}$ is the orthogonal projection of $\mathbf{v}$ on $\mathbf{a}$.

Referring to the figure, we see that $\mathbf{p}$ is a multiple of $\mathbf{a}$ and that $\mathbf{v}$ is the sum of $\mathbf{p}$ and a vector $\mathbf{q}$ perpendicular to $\mathbf{a}$. Thus,

$$\mathbf{v} = c\mathbf{a} + \mathbf{q},$$

where $\mathbf{p} = c\mathbf{a}$ and $\mathbf{a} \cdot \mathbf{q} = 0$. Taking the dot product of $\mathbf{a}$ with both sides of $\mathbf{v} = c\mathbf{a} + \mathbf{q}$, we find $\mathbf{a} \cdot \mathbf{v} = c\mathbf{a} \cdot \mathbf{a}$, so $c = (\mathbf{a} \cdot \mathbf{v})/(\mathbf{a} \cdot \mathbf{a})$, and hence

$$\mathbf{p} = \frac{\mathbf{a} \cdot \mathbf{v}}{\|\mathbf{a}\|^2}\mathbf{a}.$$

The length of $\mathbf{p}$ is

$$\|\mathbf{p}\| = \frac{|\mathbf{a} \cdot \mathbf{v}|}{\|\mathbf{a}\|^2}\|\mathbf{a}\| = \frac{|\mathbf{a} \cdot \mathbf{v}|}{\|\mathbf{a}\|} = \|\mathbf{v}\| \cos\theta.$$

Orthogonal Projection The ***orthogonal projection*** of $\mathbf{v}$ on $\mathbf{a}$ is the vector

$$\mathbf{p} = \frac{\mathbf{a} \cdot \mathbf{v}}{\|\mathbf{a}\|^2}\mathbf{a}.$$

EXAMPLE 8 Find the orthogonal projection of $\mathbf{i}+\mathbf{j}$ on $\mathbf{i}-2\mathbf{j}$.

SOLUTION With $\mathbf{a}=\mathbf{i}-2\mathbf{j}$ and $\mathbf{v}=\mathbf{i}+\mathbf{j}$, the orthogonal projection of $\mathbf{v}$ on $\mathbf{a}$ is

$$\frac{\mathbf{a}\cdot\mathbf{v}}{\mathbf{a}\cdot\mathbf{a}}\mathbf{a}=\frac{1-2}{1+4}(\mathbf{i}-2\mathbf{j})=-\frac{1}{5}(\mathbf{i}-2\mathbf{j})$$

(see Figure 1.2.11). ▲

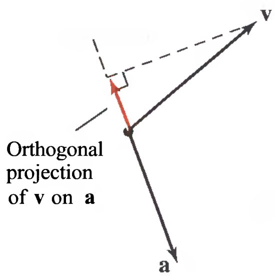


Figure 1.2.11 The orthogonal projection of $\mathbf{v}$ on $\mathbf{a}$ equals $-\frac{1}{5}\mathbf{a}$.

The Triangle Inequality

A useful consequence of the Cauchy–Schwarz inequality, which is called the ***triangle inequality***, relates the lengths of vectors $\mathbf{a}$ and $\mathbf{b}$ and of their sum $\mathbf{a}+\mathbf{b}$. Geometrically, the triangle inequality says that the length of any side of a triangle is no greater than the sum of the lengths of the other two sides (see Figure 1.2.12).

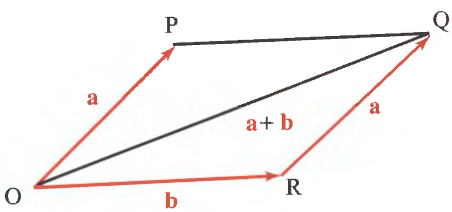


Figure 1.2.12 This geometry shows that $\|OQ\| \le \|OR\| + \|RQ\|$ or, in vector notation, that $\|\mathbf{a}+\mathbf{b}\| \le \|\mathbf{a}\| + \|\mathbf{b}\|$, which is the triangle inequality.

THEOREM 2: Triangle Inequality For vectors $\mathbf{a}$ and $\mathbf{b}$ in space,

$$\|\mathbf{a}+\mathbf{b}\| \le \|\mathbf{a}\| + \|\mathbf{b}\|.$$

PROOF While this result may be clear geometrically, it is useful to give a proof using the Cauchy–Schwarz inequality, as it will generalize to n-dimensional vectors. We consider the square of the left-hand side:

$$\|\mathbf{a}+\mathbf{b}\|^2 = (\mathbf{a}+\mathbf{b})\cdot(\mathbf{a}+\mathbf{b}) = \|\mathbf{a}\|^2 + 2\mathbf{a}\cdot\mathbf{b} + \|\mathbf{b}\|^2.$$

By the Cauchy–Schwarz inequality, we have

$$\|\mathbf{a}\|^2 + 2\mathbf{a}\cdot\mathbf{b} + \|\mathbf{b}\|^2 \le \|\mathbf{a}\|^2 + 2\|\mathbf{a}\|\,\|\mathbf{b}\| + \|\mathbf{b}\|^2 = (\|\mathbf{a}\| + \|\mathbf{b}\|)^2.$$

Thus,

$$\|\mathbf{a}+\mathbf{b}\|^2 \le (\|\mathbf{a}\| + \|\mathbf{b}\|)^2;$$

taking square roots proves the result. ■

EXAMPLE 9

(a) Verify the triangle inequality for $\mathbf{a} = \mathbf{i} + \mathbf{j}$ and $\mathbf{b} = 2\mathbf{i} + \mathbf{j} + \mathbf{k}$.

(b) Prove that $\|\mathbf{u}-\mathbf{v}\| \le \|\mathbf{u}-\mathbf{w}\| + \|\mathbf{w}-\mathbf{v}\|$ for any vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$. Illustrate with a figure in which $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ have the same base point.

SOLUTION

(a) We have $\mathbf{a}+\mathbf{b} = 3\mathbf{i} + 2\mathbf{j} + \mathbf{k}$, so $\|\mathbf{a}+\mathbf{b}\| = \sqrt{9+4+1} = \sqrt{14}$. On the other hand, $\|\mathbf{a}\| = \sqrt{2}$ and $\|\mathbf{b}\| = \sqrt{6}$, so the triangle inequality asserts that $\sqrt{14} \le \sqrt{2} + \sqrt{6}$. The numbers bear this out: $\sqrt{14} \approx 3.74$, while $\sqrt{2} + \sqrt{6} \approx 1.41 + 2.45 = 3.86$.

(b) We find that $\mathbf{u}-\mathbf{v} = (\mathbf{u}-\mathbf{w}) + (\mathbf{w}-\mathbf{v})$, so the result follows from the triangle inequality with $\mathbf{a}$ replaced by $\mathbf{u}-\mathbf{w}$ and $\mathbf{b}$ replaced by $\mathbf{w}-\mathbf{v}$. Geometrically, we are considering the shaded triangle in Figure 1.2.13. ▲

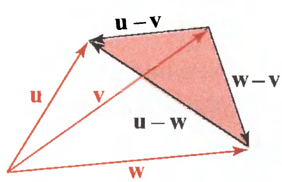


Figure 1.2.13 Illustrating the inequality $\|\mathbf{u}-\mathbf{v}\| \le \|\mathbf{u}-\mathbf{w}\| + \|\mathbf{w}-\mathbf{v}\|$.

Physical Applications of Vectors

A simple example of a physical quantity represented by a vector is a displacement. Suppose that, on a part of the earth's surface small enough to be considered flat, we introduce coordinates so that the x axis points east, the y axis points north, and the unit of length is the kilometer. If we are at a point P and wish to get to a point Q, the ***displacement vector*** $\mathbf{d} = \overrightarrow{PQ}$ joining P to Q tells us the direction and distance we have to travel. If x and y are the components of this vector, the displacement of P to Q is "x kilometers east, y kilometers north."

EXAMPLE 10 Suppose that two navigators who cannot see one another but can communicate by radio wish to determine the relative position of their ships. Explain how they can do this if they can each determine their displacement vector to the same lighthouse.

SOLUTION Let P_1 and P_2 be the positions of the ships, and let Q be the position of the lighthouse. The displacement of the lighthouse from the ith ship is the vector $\mathbf{d}_i$ joining P_i to Q. The displacement of the second ship from the first is the vector $\mathbf{d}$ joining P_1 to P_2. We have $\mathbf{d} + \mathbf{d}_2 = \mathbf{d}_1$ (Figure 1.2.14), and so $\mathbf{d} = \mathbf{d}_1 - \mathbf{d}_2$. That is, the displacement from one ship to the other is the difference between the displacements from the ships to the lighthouse. ▲

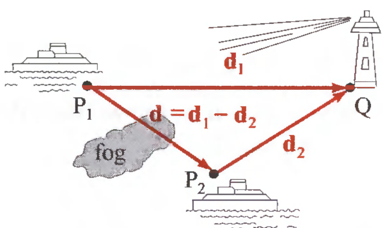


Figure 1.2.14 Vector methods can be used to locate objects.

We can also represent the velocity of a moving object as a vector. For the moment, we will consider only objects moving at uniform speed along straight lines. Suppose, for example, that a boat is steaming across a lake at 10 kilometers per hour (km/h) in the northeast direction. After 1 hour of travel, the displacement is $(10/\sqrt{2}, 10/\sqrt{2}) \approx (7.07, 7.07)$; see Figure 1.2.15.

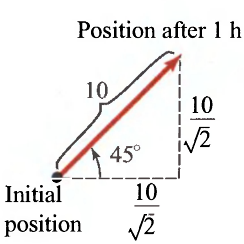


Figure 1.2.15 If an object moves northeast at 10 km/h, its velocity vector has components $(10/\sqrt{2}, 10/\sqrt{2}) = 10(1/\sqrt{2}, 1/\sqrt{2})$, where $(1/\sqrt{2}, 1/\sqrt{2})$ is the unit vector of the northeast direction.

The vector whose components are $(10/\sqrt{2}, 10/\sqrt{2})$ is called the *velocity vector* of the boat. In general, if an object is moving uniformly along a straight line, *its* ***velocity vector*** *is the displacement vector from the position at any moment to the position* 1 *unit of time later*. If a current appears on the lake, moving due eastward at 2 km/h, and the boat continues to point in the same direction with its engine running at the same rate, its displacement after 1 hour will have components given by $(10/\sqrt{2}+2, 10/\sqrt{2})$; see Figure 1.2.16. The new velocity vector, therefore, has components $(10/\sqrt{2}+2, 10/\sqrt{2})$. We note that this is the sum of the original velocity vector $(10/\sqrt{2}, 10/\sqrt{2})$ of the boat and the velocity vector $(2, 0)$ of the current.

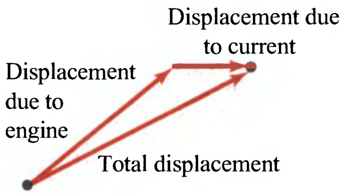


Figure 1.2.16 The total displacement is the sum of the displacements due to the engine and the current.

> **Displacement and Velocity** If an object has a (constant) velocity vector $\mathbf{v}$, then in t units of time the resulting displacement vector of the object is $\mathbf{d} = t\mathbf{v}$; thus, after time $t = 1$, the displacement vector equals the velocity vector. See Figure 1.2.17.

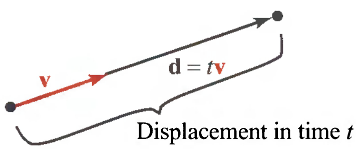


Figure 1.2.17 Displacement = time × velocity.

EXAMPLE 11 A bird is flying in a straight line with velocity vector $10\mathbf{i}+6\mathbf{j}+\mathbf{k}$ (in kilometers per hour). Suppose that (x, y) are its coordinates on the ground and z is its height above the ground.

(a) If the bird is at position $(1, 2, 3)$ at a certain moment, what is its location 1 hour later? 1 minute later?

(b) How many seconds does it take the bird to climb 10 meters?

SOLUTION (a) The displacement vector from $(1, 2, 3)$ after 1 hour is given by $10\mathbf{i}+6\mathbf{j}+\mathbf{k}$, so the new position is $(1, 2, 3)+(10, 6, 1)=(11, 8, 4)$. After 1 minute, the displacement vector from $(1, 2, 3)$ is $\frac{1}{60}(10\mathbf{i}+6\mathbf{j}+\mathbf{k}) = \frac{1}{6}\mathbf{i}+\frac{1}{10}\mathbf{j}+\frac{1}{60}\mathbf{k}$, and so the new position is $(1, 2, 3)+(\frac{1}{6}, \frac{1}{10}, \frac{1}{60}) = (\frac{7}{6}, \frac{21}{10}, \frac{181}{60})$.

(b) After t seconds ($= t/3600$ hours), the displacement vector from $(1, 2, 3)$ is $(t/3600)(10\mathbf{i}+6\mathbf{j}+\mathbf{k}) = (t/360)\mathbf{i}+(t/600)\mathbf{j}+(t/3600)\mathbf{k}$. The increase in altitude is the z component, namely, $t/3600$. This will equal 10 m ($= \frac{1}{100}$ km) when $t/3600 = \frac{1}{100}$, that is, when $t = 36$ s. ▲

EXAMPLE 12 Physical forces have magnitude and direction and may thus be represented by vectors. If several forces act at once on an object, the resultant force is represented by the sum of the individual force vectors. Suppose that forces $\mathbf{i}+\mathbf{k}$ and $\mathbf{j}+\mathbf{k}$ are acting on a body. What third force $\mathbf{F}$ must we impose to counteract the two—that is, to make the total force equal to zero?

SOLUTION The force $\mathbf{F}$ should be chosen so that $(\mathbf{i}+\mathbf{k})+(\mathbf{j}+\mathbf{k})+\mathbf{F}=\mathbf{0}$; that is, $\mathbf{F}=-(\mathbf{i}+\mathbf{k})-(\mathbf{j}+\mathbf{k})=-\mathbf{i}-\mathbf{j}-2\mathbf{k}$. (Recall that $\mathbf{0}$ is the *zero vector*, the vector whose components are all zero.) ▲

EXERCISES

1. Calculate $(3\mathbf{i}+2\mathbf{j}+\mathbf{k})\cdot(\mathbf{i}+2\mathbf{j}-\mathbf{k})$.

2. Calculate $\mathbf{a}\cdot\mathbf{b}$ where $\mathbf{a}=2\mathbf{i}+10\mathbf{j}-12\mathbf{k}$ and $\mathbf{b}=-3\mathbf{i}+4\mathbf{k}$.

3. Find the angle between $7\mathbf{j}+19\mathbf{k}$ and $-2\mathbf{i}-\mathbf{j}$ (to the nearest degree).

4. Compute $\mathbf{u}\cdot\mathbf{v}$, where $\mathbf{u}=\sqrt{3}\mathbf{i}-315\mathbf{j}+22\mathbf{k}$ and $\mathbf{v}=\mathbf{u}/\|\mathbf{u}\|$.

5. Is $\|8\mathbf{i}-12\mathbf{k}\|\cdot\|6\mathbf{j}+\mathbf{k}\|-|(8\mathbf{i}-12\mathbf{k})\cdot(6\mathbf{j}+\mathbf{k})|$ equal to zero? Explain.

In Exercises 6 to 11, compute $\|\mathbf{u}\|$, $\|\mathbf{v}\|$, *and* $\mathbf{u}\cdot\mathbf{v}$ *for the given vectors in* $\mathbb{R}^3$.

6. $\mathbf{u}=15\mathbf{i}-2\mathbf{j}+4\mathbf{k}$, $\mathbf{v}=\pi\mathbf{i}+3\mathbf{j}-\mathbf{k}$

7. $\mathbf{u}=2\mathbf{j}-\mathbf{i}$, $\mathbf{v}=-\mathbf{j}+\mathbf{i}$

8. $\mathbf{u}=5\mathbf{i}-\mathbf{j}+2\mathbf{k}$, $\mathbf{v}=\mathbf{i}+\mathbf{j}-\mathbf{k}$

9. $\mathbf{u}=-\mathbf{i}+3\mathbf{j}+\mathbf{k}$, $\mathbf{v}=-2\mathbf{i}-3\mathbf{j}-7\mathbf{k}$

10. $\mathbf{u}=-\mathbf{i}+3\mathbf{k}$, $\mathbf{v}=4\mathbf{j}$

11. $\mathbf{u}=-\mathbf{i}+2\mathbf{j}-3\mathbf{k}$, $\mathbf{v}=-\mathbf{i}-3\mathbf{j}+4\mathbf{k}$

12. Normalize the vectors in Exercises 6 to 8. (Only the solution corresponding to Exercise 7 is in the Student Guide.)

13. Find the angle between the vectors in Exercises 9 to 11. If necessary, express your answer in terms of $\cos^{-1}$.

14. Find the projection of $\mathbf{u}=-\mathbf{i}+\mathbf{j}+\mathbf{k}$ onto $\mathbf{v}=2\mathbf{i}+\mathbf{j}-3\mathbf{k}$.

15. Find the projection of $\mathbf{v}=2\mathbf{i}+\mathbf{j}-3\mathbf{k}$ onto $\mathbf{u}=-\mathbf{i}+\mathbf{j}+\mathbf{k}$.

16. What restrictions must be made on the scalar b so that the vector $2\mathbf{i}+b\mathbf{j}$ is orthogonal to (a) $-3\mathbf{i}+2\mathbf{j}+\mathbf{k}$ and (b) $\mathbf{k}$?

17. Find two nonparallel vectors both orthogonal to $(1, 1, 1)$.

18. Find the line through $(3, 1, -2)$ that intersects and is perpendicular to the line $x = -1 + t, y = -2 + t, z = -1 + t$. [HINT: If (x_0, y_0, z_0) is the point of intersection, find its coordinates.]

19. A ship at position $(1, 0)$ on a nautical chart (with north in the positive y direction) sights a rock at position $(2, 4)$. What is the vector joining the ship to the rock? What angle θ does this vector make with due north? (This is called the ***bearing*** of the rock from the ship.)

20. Suppose that the ship in Exercise 19 is pointing due north and traveling at a speed of 4 knots relative to the water. There is a current flowing due east at 1 knot. The units on the chart are nautical miles; 1 knot = 1 nautical mile per hour.

(a) If there were no current, what vector $\mathbf{u}$ would represent the velocity of the ship relative to the sea bottom?
(b) If the ship were just drifting with the current, what vector $\mathbf{v}$ would represent its velocity relative to the sea bottom?
(c) What vector $\mathbf{w}$ represents the total velocity of the ship?
(d) Where would the ship be after 1 hour?
(e) Should the captain change course?
(f) What if the rock were an iceberg?

21. An airplane is located at position $(3, 4, 5)$ at noon and traveling with velocity $400\mathbf{i} + 500\mathbf{j} - \mathbf{k}$ kilometers per hour. The pilot spots an airport at position $(23, 29, 0)$.

(a) At what time will the plane pass directly over the airport? (Assume that the plane is flying over flat ground and that the vector $\mathbf{k}$ points straight up.)
(b) How high above the airport will the plane be when it passes?

22. The wind velocity $\mathbf{v}_1$ is 40 miles per hour (mi/h) from east to west while an airplane travels with air speed $\mathbf{v}_2$ of 100 mi/h due north. The speed of the airplane relative to the ground is the vector sum $\mathbf{v}_1 + \mathbf{v}_2$.

(a) Find $\mathbf{v}_1 + \mathbf{v}_2$.
(b) Draw a figure to scale.

23. A force of 50 lb is directed 50° above horizontal, pointing to the right. Determine its horizontal and vertical components. Display all results in a figure.

24. Two persons pull horizontally on ropes attached to a post, the angle between the ropes being 60°. Person A pulls with a force of 150 lb, while B pulls with a force of 110 lb.

(a) The resultant force is the vector sum of the two forces. Draw a figure to scale that graphically represents the three forces.
(b) Using trigonometry, determine formulas for the vector components of the two forces in a conveniently chosen coordinate system. Perform the algebraic addition, and find the angle the resultant force makes with A.

25. A 1-kilogram (1-kg) mass located at the origin is suspended by ropes attached to the two points $(1, 1, 1)$ and $(-1, -1, 1)$. If the force of gravity is pointing in the direction of the vector $-\mathbf{k}$, what is the vector describing the force along each rope? [HINT: Use the symmetry of the problem. A 1-kg mass weighs 9.8 newtons (N).]

26. Suppose that an object moving in direction $\mathbf{i} + \mathbf{j}$ is acted on by a force given by the vector $2\mathbf{i} + \mathbf{j}$. Express this force as a sum of a force in the direction of motion and a force perpendicular to the direction of motion.

27. A force of 6 N (newtons) makes an angle of $\pi/4$ radian with the y axis, pointing to the right. The force acts against the movement of an object along the straight line connecting $(1, 2)$ to $(5, 4)$.

(a) Find a formula for the force vector $\mathbf{F}$.

(b) Find the angle θ between the displacement direction $\mathbf{D} = (5 - 1)\mathbf{i} + (4 - 2)\mathbf{j}$ and the force direction $\mathbf{F}$.

(c) The ***work done*** is $\mathbf{F} \cdot \mathbf{D}$, or equivalently, $\|\mathbf{F}\|\,\|\mathbf{D}\| \cos\theta$. Compute the work from both formulas and compare.

1.3 Matrices, Determinants, and the Cross Product

In Section 1.2 we defined a product of vectors that was a scalar. In this section we shall define a product of vectors that is a vector; that is, we shall show how, given two vectors $\mathbf{a}$ and $\mathbf{b}$, we can produce a third vector $\mathbf{a} \times \mathbf{b}$, called the *cross product* of $\mathbf{a}$ and $\mathbf{b}$. This new vector will have the pleasing geometric property that it is perpendicular to the plane spanned (determined) by $\mathbf{a}$ and $\mathbf{b}$. The definition of the cross product is based on the notions of the matrix and the determinant, and so these are developed first. Once this has been accomplished, we can study the geometric implications of the mathematical structure we have built.

2 × 2 Matrices

We define a 2×2 ***matrix*** to be an array

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

where a_{11}, a_{12}, a_{21}, and a_{22} are four scalars. For example,

$$\begin{bmatrix} 2 & 1 \\ 0 & 4 \end{bmatrix}, \quad \begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix}, \quad \text{and} \quad \begin{bmatrix} 13 & 7 \\ 6 & 11 \end{bmatrix}$$

are 2×2 matrices. The ***determinant***

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

of such a matrix is the real number defined by the equation

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} =: a_{11}a_{22} - a_{12}a_{21}. \tag{1}$$

EXAMPLE 1

$$\begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} = 1 - 1 = 0; \qquad \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = 4 - 6 = -2; \qquad \begin{vmatrix} 5 & 6 \\ 7 & 8 \end{vmatrix} = 40 - 42 = -2. \quad \blacktriangle$$

3 × 3 Matrices

A 3×3 ***matrix*** is an array

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

where, again, each a_{ij} is a scalar; a_{ij} denotes the entry in the array that is in the ith row and the jth column. We define the ***determinant*** of a 3×3 matrix by the rule

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}. \tag{2}$$

Without some mnemonic device, formula (2) would be difficult to memorize. The rule to learn is that you move along the first row, multiplying a_{1j} by the determinant of the 2×2 matrix obtained by canceling out the first row and the jth column, and then you add these up, remembering to put a minus in front of the a_{12} term. For example, the determinant multiplied by the middle term of formula (2), namely,

$$\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix},$$

is obtained by crossing out the first row and the second column of the given 3×3 matrix:

$$\begin{bmatrix} \cancel{a_{11}} & \cancel{a_{12}} & \cancel{a_{13}} \\ a_{21} & \cancel{a_{22}} & a_{23} \\ a_{31} & \cancel{a_{32}} & a_{33} \end{bmatrix}.$$

EXAMPLE 2

$$\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1 \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} - 0 \begin{vmatrix} 0 & 0 \\ 0 & 1 \end{vmatrix} + 0 \begin{vmatrix} 0 & 1 \\ 0 & 0 \end{vmatrix} = 1.$$

$$\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} = 1 \begin{vmatrix} 5 & 6 \\ 8 & 9 \end{vmatrix} - 2 \begin{vmatrix} 4 & 6 \\ 7 & 9 \end{vmatrix} + 3 \begin{vmatrix} 4 & 5 \\ 7 & 8 \end{vmatrix} = -3 + 12 - 9 = 0. \quad \blacktriangle$$

Properties of Determinants

An important property of determinants is that interchanging two rows or two columns results in a change of sign. For 2×2 determinants, this is a consequence of the definition as follows: For rows, we have

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{21}a_{12} = -(a_{21}a_{12} - a_{11}a_{22}) = -\begin{vmatrix} a_{21} & a_{22} \\ a_{11} & a_{12} \end{vmatrix}$$

and for columns,

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = -(a_{12}a_{21} - a_{11}a_{22}) = -\begin{vmatrix} a_{12} & a_{11} \\ a_{22} & a_{21} \end{vmatrix}.$$

We leave it to the reader to verify this property for the 3×3 case.

A second fundamental property of determinants is that *we can factor scalars out of any row or column.* For 2×2 determinants, this means

$$\begin{vmatrix} \alpha a_{11} & a_{12} \\ \alpha a_{21} & a_{22} \end{vmatrix} = \begin{vmatrix} a_{11} & \alpha a_{12} \\ a_{21} & \alpha a_{22} \end{vmatrix} = \alpha \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = \begin{vmatrix} \alpha a_{11} & \alpha a_{12} \\ a_{21} & a_{22} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} \\ \alpha a_{21} & \alpha a_{22} \end{vmatrix}.$$

Similarly, for 3×3 determinants we have

$$\begin{vmatrix} \alpha a_{11} & \alpha a_{12} & \alpha a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \alpha \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11} & \alpha a_{12} & a_{13} \\ a_{21} & \alpha a_{22} & a_{23} \\ a_{31} & \alpha a_{32} & a_{33} \end{vmatrix},$$

and so on. These results follow from the definitions. In particular, if any row or column consists of zeros, then the value of the determinant is zero.

A third fundamental fact about determinants is the following: *If we change a row (or column) by adding another row (or, respectively, column) to it, the value of the determinant remains the same.* For the 2×2 case, this means that

$$\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = \begin{vmatrix} a_1 + b_1 & a_2 + b_2 \\ b_1 & b_2 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 \\ b_1 + a_1 & b_2 + a_2 \end{vmatrix}$$

$$= \begin{vmatrix} a_1 + a_2 & a_2 \\ b_1 + b_2 & b_2 \end{vmatrix} = \begin{vmatrix} a_1 & a_1 + a_2 \\ b_1 & b_1 + b_2 \end{vmatrix}.$$

For the 3×3 case, this means

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \begin{vmatrix} a_1+b_1 & a_2+b_2 & a_3+b_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \begin{vmatrix} a_1+a_2 & a_2 & a_3 \\ b_1+b_2 & b_2 & b_3 \\ c_1+c_2 & c_2 & c_3 \end{vmatrix},$$

and so on. Again, this property can be proved using the definition of the determinant.

EXAMPLE 3 Suppose

$$\mathbf{a} = \alpha\mathbf{b} + \beta\mathbf{c}; \qquad \text{that is, } \mathbf{a} = (a_1, a_2, a_3) = \alpha(b_1, b_2, b_3) + \beta(c_1, c_2, c_3).$$

Show that

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = 0.$$

SOLUTION We shall prove the case $\alpha \neq 0$, $\beta \neq 0$. The case $\alpha = 0 = \beta$ is trivial, and the case where exactly one of α, β is zero is a simple modification of the case we prove. Using the fundamental properties of determinants, the determinant in question is

$$\begin{vmatrix} \alpha b_1+\beta c_1 & \alpha b_2+\beta c_2 & \alpha b_3+\beta c_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$$= -\frac{1}{\alpha}\begin{vmatrix} \alpha b_1+\beta c_1 & \alpha b_2+\beta c_2 & \alpha b_3+\beta c_3 \\ -\alpha b_1 & -\alpha b_2 & -\alpha b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

(factoring $-1/\alpha$ out of the second row)

$$= \left(-\frac{1}{\alpha}\right)\left(-\frac{1}{\beta}\right)\begin{vmatrix} \alpha b_1+\beta c_1 & \alpha b_2+\beta c_2 & \alpha b_3+\beta c_3 \\ -\alpha b_1 & -\alpha b_2 & -\alpha b_3 \\ -\beta c_1 & -\beta c_2 & -\beta c_3 \end{vmatrix}$$

(factoring $-1/\beta$ out of the third row)

$$= \frac{1}{\alpha\beta}\begin{vmatrix} \beta c_1 & \beta c_2 & \beta c_3 \\ -\alpha b_1 & -\alpha b_2 & -\alpha b_3 \\ -\beta c_1 & -\beta c_2 & -\beta c_3 \end{vmatrix} \qquad \text{(adding the second row to the first row)}$$

$$= \frac{1}{\alpha\beta}\begin{vmatrix} 0 & 0 & 0 \\ -\alpha b_1 & -\alpha b_2 & -\alpha b_3 \\ -\beta c_1 & -\beta c_2 & -\beta c_3 \end{vmatrix} \qquad \text{(adding the third row to the first row)}$$

$$= 0. \quad \blacktriangle$$

Closely related to these properties is the fact that *we can expand a* 3×3 *determinant along any row or column* using the signs in the following checkerboard pattern:

$$\begin{vmatrix} + & - & + \\ - & + & - \\ + & - & + \end{vmatrix}$$

For instance, the reader can check that we can expand "by minors" along the middle row:

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = -a_{21}\begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} + a_{22}\begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} - a_{23}\begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix}.$$

Let us redo the second determinant in Example 2 using this formula:

$$\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} = -4\begin{vmatrix} 2 & 3 \\ 8 & 9 \end{vmatrix} + 5\begin{vmatrix} 1 & 3 \\ 7 & 9 \end{vmatrix} - 6\begin{vmatrix} 1 & 2 \\ 7 & 8 \end{vmatrix} = (-4)(-6) + (5)(12) + (-6)(6) = 0.$$

Historical Note

Determinants appear to have been invented and first used by Leibniz in 1693, in connection with solutions of linear equations. Maclaurin and Cramer developed their properties between 1729 and 1750; in particular, they showed that the solution of the system of equations

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 &= b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 &= b_3 \end{aligned}$$

is

$$x_1 = \frac{1}{\Delta}\begin{vmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{vmatrix}, \qquad x_2 = \frac{1}{\Delta}\begin{vmatrix} a_{11} & b_1 & a_{13} \\ a_{21} & b_2 & a_{23} \\ a_{31} & b_3 & a_{33} \end{vmatrix}, \qquad x_3 = \frac{1}{\Delta}\begin{vmatrix} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \\ a_{31} & a_{32} & b_3 \end{vmatrix}$$

where

$$\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix},$$

a fact now known as ***Cramer's rule***. While this method is rather inefficient from a numerical point of view, it is of theoretical importance in matrix theory. Later, Vandermonde (1772) and Cauchy (1812), treating determinants as a separate topic worthy of special attention, developed the field more systematically, with contributions by Laplace, Jacobi, and others. Formulas for volumes of parallelepipeds in terms of determinants are due to Lagrange (1775). We shall study these later in this section. Although during the nineteenth century mathematicians studied matrices and determinants, the subjects were considered separate. For the full history up to 1900, see T. Muir, *The Theory of Determinants in the Historical Order of Development* (reprinted by Dover, New York, 1960).

Cross Products

Now that we have established the necessary properties of determinants and discussed their history, we are ready to proceed with the cross product of vectors.

DEFINITION: The Cross Product Suppose that $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ and $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$ are vectors in $\mathbb{R}^3$. The ***cross product*** or ***vector product*** of $\mathbf{a}$ and $\mathbf{b}$, denoted $\mathbf{a} \times \mathbf{b}$, is defined to be the vector

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \mathbf{k},$$

or, symbolically,

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}.$$

Even though we only defined determinants for arrays of *real* numbers, this formal expression involving *vectors* is a useful memory aid for the cross product.

EXAMPLE 4 Find $(3\mathbf{i} - \mathbf{j} + \mathbf{k}) \times (\mathbf{i} + 2\mathbf{j} - \mathbf{k})$.

SOLUTION

$$(3\mathbf{i} - \mathbf{j} + \mathbf{k}) \times (\mathbf{i} + 2\mathbf{j} - \mathbf{k}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & -1 & 1 \\ 1 & 2 & -1 \end{vmatrix} = -\mathbf{i} + 4\mathbf{j} + 7\mathbf{k}. \quad \blacktriangle$$

Certain algebraic properties of the cross product follow from the definition. If $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ are vectors and α, β, and γ are scalars, then

(i) $\mathbf{a} \times \mathbf{b} = -(\mathbf{b} \times \mathbf{a})$

(ii) $\mathbf{a} \times (\beta\mathbf{b} + \gamma\mathbf{c}) = \beta(\mathbf{a} \times \mathbf{b}) + \gamma(\mathbf{a} \times \mathbf{c})$ and $(\alpha\mathbf{a} + \beta\mathbf{b}) \times \mathbf{c} = \alpha(\mathbf{a} \times \mathbf{c}) + \beta(\mathbf{b} \times \mathbf{c})$.

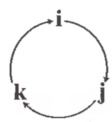
Note that $\mathbf{a} \times \mathbf{a} = -(\mathbf{a} \times \mathbf{a})$, by property (i). Thus, $\mathbf{a} \times \mathbf{a} = \mathbf{0}$. In particular,

$$\mathbf{i} \times \mathbf{i} = \mathbf{0}, \qquad \mathbf{j} \times \mathbf{j} = \mathbf{0}, \qquad \mathbf{k} \times \mathbf{k} = \mathbf{0}.$$

Also,

$$\mathbf{i} \times \mathbf{j} = \mathbf{k}, \qquad \mathbf{j} \times \mathbf{k} = \mathbf{i}, \qquad \mathbf{k} \times \mathbf{i} = \mathbf{j},$$

which can be remembered by cyclicly permuting $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ like this:



To give a geometric interpretation of the cross product, we first introduce the triple product. Given three vectors $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$, the real number

$$(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}$$

is called the ***triple product*** of $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ (in that order). To obtain a formula for it, let $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$, $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$, and $\mathbf{c} = c_1\mathbf{i} + c_2\mathbf{j} + c_3\mathbf{k}$. Then

$$\begin{aligned}(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} &= \left(\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}\mathbf{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix}\mathbf{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}\mathbf{k}\right) \cdot (c_1\mathbf{i} + c_2\mathbf{j} + c_3\mathbf{k}) \\ &= \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} c_1 - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} c_2 + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} c_3.\end{aligned}$$

This is the expansion by minors of the third row of the determinant, so

$$(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}.$$

If $\mathbf{c}$ is a vector in the plane spanned by the vectors $\mathbf{a}$ and $\mathbf{b}$, then the third row in the determinant expression for $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}$ is a linear combination of the first and second rows, and therefore $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} = 0$. In other words, *the vector* $\mathbf{a} \times \mathbf{b}$ *is orthogonal to any vector in the plane spanned by* $\mathbf{a}$ *and* $\mathbf{b}$, *in particular to both* $\mathbf{a}$ *and* $\mathbf{b}$.

Next, we calculate the length of $\mathbf{a} \times \mathbf{b}$. Note that

$$\begin{aligned}\|\mathbf{a} \times \mathbf{b}\|^2 &= \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}^2 + \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix}^2 + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}^2 \\ &= (a_2b_3 - a_3b_2)^2 + (a_1b_3 - b_1a_3)^2 + (a_1b_2 - b_1a_2)^2.\end{aligned}$$

If we expand the terms in the last expression, we can recollect them to give

$$(a_1^2 + a_2^2 + a_3^2)(b_1^2 + b_2^2 + b_3^2) - (a_1b_1 + a_2b_2 + a_3b_3)^2,$$

which equals

$$\|\mathbf{a}\|^2\|\mathbf{b}\|^2 - (\mathbf{a}\cdot\mathbf{b})^2 = \|\mathbf{a}\|^2\|\mathbf{b}\|^2 - \|\mathbf{a}\|^2\|\mathbf{b}\|^2\cos^2\theta = \|\mathbf{a}\|^2\|\mathbf{b}\|^2\sin^2\theta$$

where θ is the angle between **a** and **b**, $0 \le \theta \le \pi$. Taking square roots and using $\sqrt{k^2} = |k|$, we find that $\|\mathbf{a}\times\mathbf{b}\| = \|\mathbf{a}\|\|\mathbf{b}\||\sin\theta|$.

Combining our results, we conclude that $\mathbf{a}\times\mathbf{b}$ is a *vector perpendicular to the plane $\mathcal{P}$ spanned by* **a** *and* **b** *with length* $\|\mathbf{a}\|\|\mathbf{b}\||\sin\theta|$. We see from Figure 1.3.1 that this length is also the area of the parallelogram (with base $\|\mathbf{a}\|$ and height $\|\mathbf{b}\sin\theta\|$) spanned by **a** and **b**. There are still two possible vectors that satisfy these conditions, because there are two choices of direction that are perpendicular (or normal) to $\mathcal{P}$. This is clear from Figure 1.3.1, which shows the two choices $\mathbf{n}_1$ and $-\mathbf{n}_1$ perpendicular to $\mathcal{P}$, with $\|\mathbf{n}_1\| = \|-\mathbf{n}_1\| = \|\mathbf{a}\|\|\mathbf{b}\||\sin\theta|$.

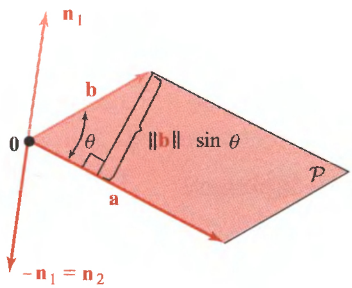


Figure 1.3.1 $\mathbf{n}_1$ and $\mathbf{n}_2$ are the two possible vectors orthogonal to both **a** and **b**, and with norm $\|\mathbf{a}\|\ \|\mathbf{b}\|\ |\sin\theta|$.

Which vector represents $\mathbf{a}\times\mathbf{b}$, $\mathbf{n}_1$ or $-\mathbf{n}_1$? The answer is $\mathbf{n}_1$. Try a few cases such as $\mathbf{k} = \mathbf{i}\times\mathbf{j}$ to verify this. The following "right-hand rule" determines the direction of $\mathbf{a}\times\mathbf{b}$ in general. Take your *right hand* and place it so your fingers curl from **a** toward **b** through the *acute* angle θ, as in Figure 1.3.2. Then your thumb points in the direction of $\mathbf{a}\times\mathbf{b}$.

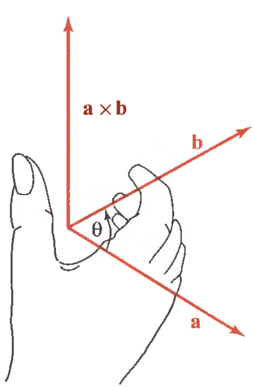


Figure 1.3.2 The right-hand rule for determining in which of the two possible directions $\mathbf{a}\times\mathbf{b}$ points.

The Cross Product

Geometric definition: $\mathbf{a} \times \mathbf{b}$ is the vector such that:

(1) $\|\mathbf{a} \times \mathbf{b}\| = \|\mathbf{a}\|\,\|\mathbf{b}\| \sin\theta$, the area of the parallelogram spanned by $\mathbf{a}$ and $\mathbf{b}$ (θ is the angle between $\mathbf{a}$ and $\mathbf{b}$; $0 \le \theta \le \pi$); see Figure 1.3.3.

(2) $\mathbf{a} \times \mathbf{b}$ is perpendicular to $\mathbf{a}$ and $\mathbf{b}$, and the triple $(\mathbf{a}, \mathbf{b}, \mathbf{a} \times \mathbf{b})$ obeys the right-hand rule.

Component formula:

$$(a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}) \times (b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

$$= (a_2b_3 - a_3b_2)\mathbf{i} - (a_1b_3 - a_3b_1)\mathbf{j} + (a_1b_2 - a_2b_1)\mathbf{k}.$$

Algebraic rules:

1. $\mathbf{a} \times \mathbf{b} = \mathbf{0}$ if and only if $\mathbf{a}$ and $\mathbf{b}$ are parallel or $\mathbf{a}$ or $\mathbf{b}$ is zero.
2. $\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$.
3. $\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c}$.
4. $(\mathbf{a} + \mathbf{b}) \times \mathbf{c} = \mathbf{a} \times \mathbf{c} + \mathbf{b} \times \mathbf{c}$.
5. $(\alpha\mathbf{a}) \times \mathbf{b} = \alpha(\mathbf{a} \times \mathbf{b})$.

Multiplication table:

		Second factor		
	$\times$	$\mathbf{i}$	$\mathbf{j}$	$\mathbf{k}$
	$\mathbf{i}$	$\mathbf{0}$	$\mathbf{k}$	$-\mathbf{j}$
First	$\mathbf{j}$	$-\mathbf{k}$	$\mathbf{0}$	$\mathbf{i}$
factor	$\mathbf{k}$	$\mathbf{j}$	$-\mathbf{i}$	$\mathbf{0}$

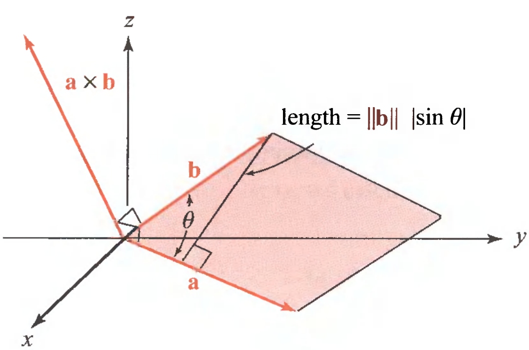


Figure 1.3.3 The length of $\mathbf{a} \times \mathbf{b}$ is the area of the parallelogram formed by $\mathbf{a}$ and $\mathbf{b}$.

EXAMPLE 5 Find the area of the parallelogram spanned by the two vectors $\mathbf{a} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$ and $\mathbf{b} = -\mathbf{i} - \mathbf{k}$.

SOLUTION We calculate the cross product of **a** and **b** by applying the component or determinant formula, with $a_1 = 1$, $a_2 = 2$, $a_3 = 3$, $b_1 = -1$, $b_2 = 0$, $b_3 = -1$:

$$\begin{aligned}\mathbf{a} \times \mathbf{b} &= [(2)(-1) - (3)(0)]\mathbf{i} + [(3)(-1) - (1)(-1)]\mathbf{j} + [(1)(0) - (2)(-1)]\mathbf{k} \\ &= -2\mathbf{i} - 2\mathbf{j} + 2\mathbf{k}.\end{aligned}$$

Thus, the area is

$$\|\mathbf{a} \times \mathbf{b}\| = \sqrt{(-2)^2 + (-2)^2 + (2)^2} = 2\sqrt{3}. \quad \blacktriangle$$

EXAMPLE 6 Find a unit vector orthogonal to the vectors $\mathbf{i} + \mathbf{j}$ and $\mathbf{j} + \mathbf{k}$.

SOLUTION A vector perpendicular to both $\mathbf{i} + \mathbf{j}$ and $\mathbf{j} + \mathbf{k}$ is their cross product, namely, the vector

$$(\mathbf{i} + \mathbf{j}) \times (\mathbf{j} + \mathbf{k}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{vmatrix} = \mathbf{i} - \mathbf{j} + \mathbf{k}.$$

Because $\|\mathbf{i} - \mathbf{j} + \mathbf{k}\| = \sqrt{3}$, the vector

$$\frac{1}{\sqrt{3}}(\mathbf{i} - \mathbf{j} + \mathbf{k})$$

is a unit vector perpendicular to $\mathbf{i} + \mathbf{j}$ and $\mathbf{j} + \mathbf{k}$. $\blacktriangle$

EXAMPLE 7 Derive an identity relating the dot and cross products from the formulas

$$\|\mathbf{u} \times \mathbf{v}\| = \|\mathbf{u}\|\,\|\mathbf{v}\| \sin\theta \qquad \text{and} \qquad \mathbf{u} \cdot \mathbf{v} = \|\mathbf{u}\|\,\|\mathbf{v}\| \cos\theta$$

by eliminating θ.

SOLUTION Seeing $\sin\theta$ and $\cos\theta$ multiplied by the same expression suggests squaring the two formulas and adding the results. We get

$$\|\mathbf{u} \times \mathbf{v}\|^2 + (\mathbf{u} \cdot \mathbf{v})^2 = \|\mathbf{u}\|^2\|\mathbf{v}\|^2(\sin^2\theta + \cos^2\theta) = \|\mathbf{u}\|^2\|\mathbf{v}\|^2,$$

so

$$\|\mathbf{u} \times \mathbf{v}\|^2 = \|\mathbf{u}\|^2\|\mathbf{v}\|^2 - (\mathbf{u} \cdot \mathbf{v})^2.$$

This identity is interesting because it establishes a link between the dot and cross products. $\blacktriangle$

Geometry of Determinants

Using the cross product, we may obtain a basic geometric interpretation of 2×2 and 3×3 determinants. Let $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j}$ and $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j}$ be two vectors in the plane. If θ is the angle between $\mathbf{a}$ and $\mathbf{b}$, we have seen that $\|\mathbf{a} \times \mathbf{b}\| = \|\mathbf{a}\|\|\mathbf{b}\||\sin\theta|$ is the area of the parallelogram with adjacent sides $\mathbf{a}$ and $\mathbf{b}$. The cross product as a determinant is

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & 0 \\ b_1 & b_2 & 0 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \mathbf{k}.$$

Thus, the area $\|\mathbf{a} \times \mathbf{b}\|$ is the absolute value of the determinant

$$\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = a_1 b_2 - a_2 b_1.$$

Geometry of 2×2 Determinants The absolute value of the determinant $\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$ is the area of the parallelogram whose adjacent sides are the vectors $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j}$ and $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j}$. The sign of the determinant is + when, rotating in the counterclockwise direction, the angle from $\mathbf{a}$ to $\mathbf{b}$ is less than π.

EXAMPLE 8 Find the area of the triangle with vertices at the points (1, 1), (0, 2), and (3, 2) (Figure 1.3.4).

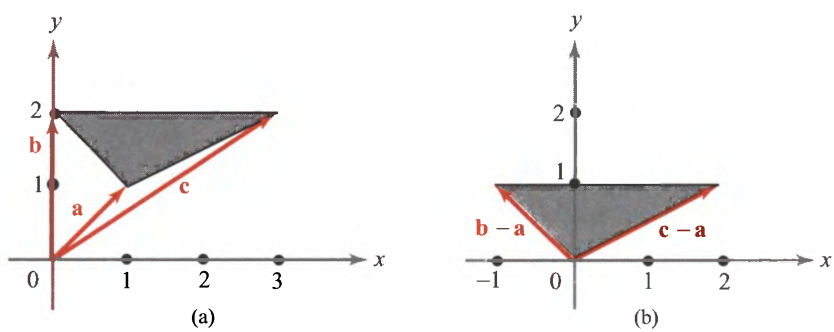


Figure 1.3.4 (a) Find the area A of the shaded triangle by expressing the sides as vector differences (b) to get $A = \|(\mathbf{b} - \mathbf{a}) \times (\mathbf{c} - \mathbf{a})\|/2$.

SOLUTION Let $\mathbf{a} = \mathbf{i} + \mathbf{j}$, $\mathbf{b} = 2\mathbf{j}$, and $\mathbf{c} = 3\mathbf{i} + 2\mathbf{j}$. It is clear that the triangle whose vertices are the endpoints of the vectors $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ has the same area as the triangle with vertices at $\mathbf{0}$, $\mathbf{b} - \mathbf{a}$, and $\mathbf{c} - \mathbf{a}$ (Figure 1.3.4). Indeed, the latter is merely a translation of the former triangle. Because the area of this translated triangle is one-half the area of the parallelogram with adjacent sides $\mathbf{b} - \mathbf{a} = -\mathbf{i} + \mathbf{j}$, and $\mathbf{c} - \mathbf{a} = 2\mathbf{i} + \mathbf{j}$, we find that the area of the triangle with vertices (1, 1), (0, 2), and (3, 2) is the absolute value of

$$\frac{1}{2}\begin{vmatrix} -1 & 1 \\ 2 & 1 \end{vmatrix} = -\frac{3}{2},$$

that is, 3/2. ▲

There is an interpretation of determinants of 3×3 matrices as volumes that is analogous to the interpretation of determinants of 2×2 matrices as areas.

Geometry of 3×3 Determinants The absolute value of the determinant

$$D = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

is the volume of the parallelepiped whose adjacent sides are the vectors

$$\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}, \qquad \mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}, \qquad \text{and} \qquad \mathbf{c} = c_1\mathbf{i} + c_2\mathbf{j} + c_3\mathbf{k}.$$

To prove the statement in the box above, we refer to Figure 1.3.5 and note that the length of the cross product, namely, $\|\mathbf{a} \times \mathbf{b}\|$, is the area of the parallelogram with adjacent sides $\mathbf{a}$ and $\mathbf{b}$. Moreover, $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} = \|\mathbf{a} \times \mathbf{b}\| \|\mathbf{c}\| \cos \psi$, where ψ is the angle that $\mathbf{c}$ makes with the normal to the plane spanned by $\mathbf{a}$ and $\mathbf{b}$. Because the volume of the parallelepiped with adjacent sides $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ is the product of the area of the base $\|\mathbf{a} \times \mathbf{b}\|$ and the altitude $\|\mathbf{c}\| |\cos \psi|$, it follows that the volume is $|(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c})|$. We saw earlier that $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} = D$, so the volume equals the absolute value of D.

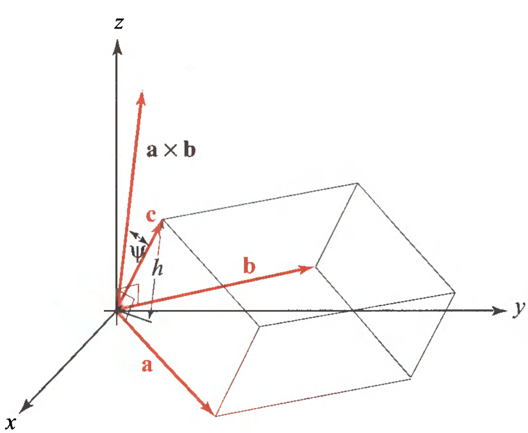


Figure 1.3.5 The volume of the parallelepiped spanned by **a**, **b**, **c** is the absolute value of the determinant of the 3×3 matrix having **a**, **b**, **c** as its rows.

EXAMPLE 9 Find the volume of the parallelepiped spanned by the three vectors $\mathbf{i} + 3\mathbf{k}$, $2\mathbf{i} + \mathbf{j} - 2\mathbf{k}$, and $5\mathbf{i} + 4\mathbf{k}$.

SOLUTION The volume is the absolute value of

$$\begin{vmatrix} 1 & 0 & 3 \\ 2 & 1 & -2 \\ 5 & 0 & 4 \end{vmatrix}.$$

If we expand this determinant by minors by going down the second column, the only nonzero term is

$$\begin{vmatrix} 1 & 3 \\ 5 & 4 \end{vmatrix}(1) = -11,$$

so the volume equals 11. ▲

Equations of Planes

Let $\mathcal{P}$ be a plane in space, $\mathrm{P}_0 = (x_0, y_0, z_0)$ a point on that plane, and suppose that $\mathbf{n} = A\mathbf{i} + B\mathbf{j} + C\mathbf{k}$ is a vector normal to that plane (see Figure 1.3.6). Let $\mathrm{P} = (x, y, z)$ be a point in $\mathbb{R}^3$. Then P lies on the plane $\mathcal{P}$ if and only if the vector $\overrightarrow{\mathrm{P}_0\mathrm{P}} = (x - x_0)\mathbf{i} + (y - y_0)\mathbf{j} + (z - z_0)\mathbf{k}$ is perpendicular to $\mathbf{n}$, that is, $\overrightarrow{\mathrm{P}_0\mathrm{P}} \cdot \mathbf{n} = 0$, or, equivalently,

$$(A\mathbf{i} + B\mathbf{j} + C\mathbf{k}) \cdot [(x - x_0)\mathbf{i} + (y - y_0)\mathbf{j} + (z - z_0)\mathbf{k}] = 0.$$

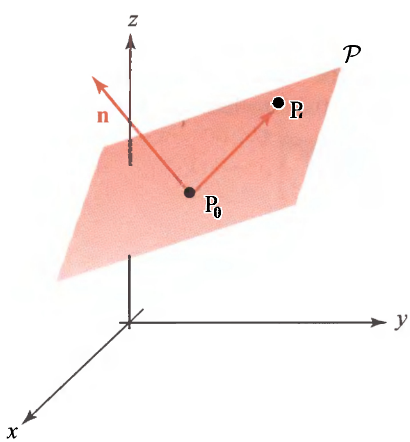


Figure 1.3.6 The points P of the plane through P_0 and perpendicular to $\mathbf{n}$ satisfy the equation $\overrightarrow{\mathrm{P}_0\mathrm{P}} \cdot \mathbf{n} = 0$.

Thus,

$$A(x - x_0) + B(y - y_0) + C(z - z_0) = 0.$$

Equation of a Plane in Space The equation of the plane $\mathcal{P}$ through (x_0, y_0, z_0) that has a normal vector $\mathbf{n} = A\mathbf{i} + B\mathbf{j} + C\mathbf{k}$ is

$$A(x - x_0) + B(y - y_0) + C(z - z_0) = 0;$$

that is, $(x, y, z) \in \mathcal{P}$ if and only if

$$Ax + By + Cz + D = 0$$

where $D = -Ax_0 - By_0 - Cz_0$.

The four numbers A, B, C, and D are not determined uniquely by the plane $\mathcal{P}$. To see this, note that (x, y, z) satisfies the equation $Ax + By + Cz + D = 0$ if and only if it also satisfies the relation

$$(\lambda A)x + (\lambda B)y + (\lambda C)z + (\lambda D) = 0$$

for any constant $\lambda \neq 0$. Furthermore, if A, B, C, D and A', B', C', D' determine the same plane $\mathcal{P}$, then $A = \lambda A'$, $B = \lambda B'$, $C = \lambda C'$, $D = \lambda D'$ for a scalar λ. Consequently, A, B, C, D are ***determined by $\mathcal{P}$ up to a scalar multiple***.

EXAMPLE 10 Determine an equation for the plane that is perpendicular to the vector $\mathbf{i} + \mathbf{j} + \mathbf{k}$ and contains the point $(1, 0, 0)$.

SOLUTION Using the general form $A(x - x_0) + B(y - y_0) + C(z - z_0) = 0$, the plane is $1(x - 1) + 1(y - 0) + 1(z - 0) = 0$; that is, $x + y + z = 1$. ▲

EXAMPLE 11 Find an equation for the plane containing the three points $(1, 1, 1)$, $(2, 0, 0)$, and $(1, 1, 0)$.

SOLUTION *Method 1.* This is a "brute force" method that you can use if you have forgotten the vector methods. The equation for any plane is of the form $Ax + By + Cz + D = 0$. Because the points $(1, 1, 1)$, $(2, 0, 0)$, and $(1, 1, 0)$ lie in the plane, we have

$$\begin{aligned} A + B + C + D &= 0, \\ 2A \qquad\quad + D &= 0, \\ A + B \qquad + D &= 0. \end{aligned}$$

Proceeding by elimination, we reduce this system of equations to the form

$$\begin{aligned} 2A + D &= 0 \quad \text{(second equation)} \\ 2B + D &= 0 \quad (2 \times \text{third} - \text{second}), \\ C &= 0 \quad (\text{first} - \text{third}). \end{aligned}$$

Because the numbers A, B, C, and D are determined only up to a scalar multiple, we can fix the value of one of them, say $A = 1$, and then the others will be determined uniquely. We get $A = 1$, $D = -2$, $B = 1$, $C = 0$. Thus, an equation of the plane that contains the given points is $x + y - 2 = 0$.

Method 2. Let $\mathrm{P} = (1, 1, 1)$, $\mathrm{Q} = (2, 0, 0)$, $\mathrm{R} = (1, 1, 0)$. Any vector normal to the plane must be orthogonal to the vectors $\overrightarrow{\mathrm{QP}}$ and $\overrightarrow{\mathrm{RP}}$, which are parallel to the plane, because their endpoints lie on the plane. Thus, $\mathbf{n} = \overrightarrow{\mathrm{QP}} \times \overrightarrow{\mathrm{RP}}$ is normal to the plane. Computing the cross product, we have

$$\mathbf{n} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 1 & 1 \\ 0 & 0 & 1 \end{vmatrix} = \mathbf{i} + \mathbf{j}.$$

Because the point $(2, 0, 0)$ lies on the plane, we conclude that the equation is given by $(x - 2) + (y - 0) + 0 \cdot (z - 0) = 0$; that is, $x + y - 2 = 0$. ▲

Two planes are called ***parallel*** when their normal vectors are parallel. Thus, the planes $A_1x + B_1y + C_1z + D_1 = 0$ and $A_2x + B_2y + C_2z + D_2 = 0$ are parallel when $\mathbf{n}_1 = A_1\mathbf{i} + B_1\mathbf{j} + C_1\mathbf{k}$ and $\mathbf{n}_2 = A_2\mathbf{i} + B_2\mathbf{j} + C_2\mathbf{k}$ are parallel; that is, $\mathbf{n}_1 = \sigma\mathbf{n}_2$ for a constant σ. For example, the planes

$$x - 2y + z = 0 \qquad \text{and} \qquad -2x + 4y - 2z = 10$$

are parallel, but the planes

$$x - 2y + z = 0 \qquad \text{and} \qquad 2x - 2y + z = 10$$

are not parallel.

Distance: Point to Plane

Let us now determine the distance from a point $\mathrm{E} = (x_1, y_1, z_1)$ to the plane $\mathcal{P}$ described by the equation $A(x - x_0) + B(y - y_0) + C(z - z_0) = Ax + By + Cz + D = 0$. To do so, consider the unit normal vector

$$\mathbf{n} = \frac{A\mathbf{i} + B\mathbf{j} + C\mathbf{k}}{\sqrt{A^2 + B^2 + C^2}},$$

which is a unit vector normal to the plane. Drop a perpendicular from E to the plane and construct the triangle REQ shown in Figure 1.3.7. The distance $d = \|\overrightarrow{\mathrm{EQ}}\|$ is the length of the projection of $\mathbf{v} = \overrightarrow{\mathrm{RE}}$ (the vector from R to E) onto $\mathbf{n}$;

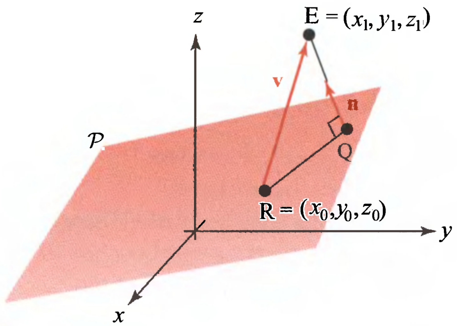


Figure 1.3.7 The geometry for determining the distance from the point E to plane $\mathcal{P}$.

thus,

$$\text{Distance} = |\mathbf{v} \cdot \mathbf{n}| = |[(x_1 - x_0)\mathbf{i} + (y_1 - y_0)\mathbf{j} + (z_1 - z_0)\mathbf{k}] \cdot \mathbf{n}|$$
$$= \frac{|A(x_1 - x_0) + B(y_1 - y_0) + C(z_1 - z_0)|}{\sqrt{A^2 + B^2 + C^2}}.$$

If the plane is given in the form $Ax + By + Cz + D = 0$, then for any point (x_0, y_0, z_0) on it, $D = -(Ax_0 + By_0 + Cz_0)$. Substitution into the previous formula gives the following:

Distance from a Point to a Plane The distance from (x_1, y_1, z_1) to the plane $Ax + By + Cz + D = 0$ is

$$\text{Distance} = \frac{|Ax_1 + By_1 + Cz_1 + D|}{\sqrt{A^2 + B^2 + C^2}}.$$

EXAMPLE 12 Find the distance from Q = $(2, 0, -1)$ to the plane $3x - 2y + 8z + 1 = 0$.

SOLUTION We substitute into the formula in the preceding box the values $x_1 = 2$ $y_1 = 0, z_1 = -1$ (the point) and $A = 3, B = -2, C = 8, D = 1$ (the plane) to give

$$\text{Distance} = \frac{|3 \cdot 2 + (-2) \cdot 0 + 8(-1) + 1|}{\sqrt{3^2 + (-2)^2 + 8^2}} = \frac{|-1|}{\sqrt{77}} = \frac{1}{\sqrt{77}}. \quad \blacktriangle$$

Historical Note

The Origins of the Vector, Scalar, Dot, and Cross Products

Quadratic Equations, Cubic Equations, and Imaginary Numbers. We know from Babylonian clay tablets that this great civilization possessed the quadratic formula, enabling them (in verbal form) to solve quadratic equations. Because the concept of negative numbers had to wait until the sixteenth century to see the light of day, the Babylonians did not consider either negative (or imaginary) solutions.

With the Renaissance and the rediscovery of ancient learning, Italian mathematicians began to wonder about the solutions of cubic equations, $x^3 + ax^2 + bx + c = 0$, where a, b, and c are positive numbers.

Around 1500, Scipione del Ferro, a professor in Bologna (the oldest European university) was able to solve cubics of the form $x^3 + bx = c$, but kept his discovery secret. Before his death, he passed his formula to his successor, Antonio Fior, who for a while also kept the formula to himself. It remained a secret until a brilliant, self-taught mathematician named Nicolo Fontana, also known as Tartaglia (the stammerer), appeared on the scene. Tartaglia claimed he could solve the cubic, and Fior felt he needed to protect the priority of del Ferro, and so in response challenged Tartaglia to a public competition.

We are told that Tartaglia was able to solve all of the thirty cubic equations posed by Fior. Amazingly, some scholars believe that Tartaglia discovered the formula for solutions to $x^3 + cx = d$ only days before the contest was to take place.

The greatest mathematician of the sixteenth century, Gerolamo Cardano (1501–1576)—a Renaissance scholar, mathematician, physician, and fortuneteller—gave the first published solution of the general cubic. Although born of modest means, he (like Tartaglia) rose, through effort and natural brilliance, to great fame. Cardano is the author of the first book on games of chance (marking the beginning of modern probability theory) and also of *Ars Magna* (the Great Art), which marks the beginning of modern algebra. It was in this book that Cardano published the solution to the *general* cubic $x^3 + ax^2 + bx + c = 0$. How did he get it?

While working on his algebra book, and aware that Tartaglia was able to solve forms of cubic, Cardano, in 1539, wrote to Tartaglia asking for a meeting. After some cajoling, Tartaglia agreed. It was at this meeting that, in exchange for a pledge of secrecy (and we know how these generally go), Tartaglia revealed his solution, from which Cardano was able to derive a solution to the general equation, which then appeared in *Ars Magna*. Feeling betrayed, Tartaglia led a scathing attack on Cardano, leading to a small soap opera.

What is important for us, at the moment, is that as a consequence of the method of solution, something very strange occurred. Consider the cubic $x^3 - 15x = 4$. Its only positive root is 4. However, the Tartaglia–Cardano solution formula yields

$$x = \sqrt[3]{2 + \sqrt{-121}} + \sqrt[3]{2 - \sqrt{-121}} \tag{1}$$

as the positive root. Thus, this number must be equal to 4. Yet this must be *nonsense*, because inside the cube root we are taking the square root of a negative number—at the time, an absolute impossibility. This was a real shock. Over 100 years later, in 1702, when Leibniz, codiscoverer of calculus, showed the great Dutch scientist Christian Huygens the formula

$$\sqrt{6} = \sqrt{1 + \sqrt{-3}} + \sqrt{1 - \sqrt{-3}} \tag{2}$$

Huygens was completely flabbergasted, and remarked that this equality "defies all human understanding." [Try, informally, to verify both formulas (1) and (2) for yourself.]

Whether nonsense or not, Tartaglia and Cardano's formula forced mathematicians to confront square roots of negative numbers (or *imaginary numbers*, as they are called today). This historical incident is another example that negates the (widespread) view that mathematics is "made up" by mathematicians. As is often the case, *it is the mathematics itself that speaks to us.*

The Maturing of Complex Numbers. For well over two centuries, numbers like $i = \sqrt{-1}$ were looked at with great suspicion. The square root of any negative number can be written in terms of i; for example, $\sqrt{-a} = \sqrt{(a)(-1)} = \sqrt{a}\sqrt{-1}$. In the middle of the eighteenth century, the Swiss mathematician Leonhard Euler connected the universal cosmic numbers e and π with the imaginary number i. Whatever i was or meant, it necessarily follows that

$$e^{\pi i} = -1,$$

that is, e "raised to the power πi equals -1. Thus, these cosmic numbers, reflecting perhaps some deeper mystery, are in fact connected to each other by a very simple formula.

At the beginning of the nineteenth century, the German mathematician Karl Friedrich Gauss was able to prove the *fundamental theorem of algebra*, which says that any nth-degree polynomial has n roots (some or all of which may be imaginary; that is, the roots have the form $a + bi$, where, as earlier, $i = \sqrt{-1}$ and where a and b are real numbers).

By the middle of the nineteenth century, the French mathematician Augusten-Louis Cauchy and the German mathematician Bernhard Riemann had developed the differential calculus for functions of one complex variable. An example of such a function is $F(z) = z^n$, where

$z = a + bi$. In this case, the usual formula for the derivative, $F'(z) = nz^{n-1}$, still holds. However, by introducing imaginary numbers, Cauchy was able to evaluate "real integrals" that heretofore could not be evaluated. For example, it is possible to show that

$$\int_0^\infty \frac{\sin x}{x}\, dx = \frac{\pi}{2}$$

and that

$$\int_0^\pi \log \sin x \, dx = -\pi \log 2,$$

These were stunning results.

In summary, the solution of the cubic equation, the fundamental theorem of algebra, and the evaluation of real integrals proved how valuable it was to consider imaginary numbers $a + bi$, even though they were not (at least not yet) on *terra firma*. Did they really exist or were they simply phantoms of our *imagination*, and thus truly *imaginary*?

Hamilton's Definition of Complex Numbers. Many mathematicians after Cardano made important contributions to imaginary (or complex) numbers, including Argand, Wessel, and Gauss—all of whom represented them geometrically. However, the modern, intellectually rigorous definition of a complex number is due to the great Irish mathematician William Rowan Hamilton (see Figure 1.3.8). After Newton, who created the vector concept through his invention of the notion of force, Hamilton was, beyond any doubt, the most important and singular figure in

Figure 1.3.8 Sir William Rowan Hamilton (1805–1865).

the development of vector calculus. It was Hamilton who gave us the terms *vector* and *scalar quantity*.

William Rowan Hamilton was born in Dublin, Ireland, at midnight on August 3, 1805. In 1823, he entered Trinity College, Dublin. His university career, by any standard, was phenomenal. By his third year, Trinity offered him a professorship, the Andrew's Chair of Astronomy, and the State named him Royal Astronomer of Ireland. These honors were based on his theoretical prediction (in 1824) of two entirely new and unexpected optical phenomena, namely, internal and external conical refraction.

By 1827 he had become interested in imaginary numbers. He wrote that "the symbol $\sqrt{-1}$ is absurd, and denotes an impossible extraction . . ." He set out to put the idea of a complex number on a firm logical foundation. His solution was to define a complex number $a + bi$ as a point (a, b) in the plane $\mathbb{R}^2$, much as we do today. Thus, the imaginary number bi for Hamilton was simply the point $(0, b)$ on the y axis. The difference between complex numbers and the Cartesian plane was that Hamilton followed the proforma multiplication of complex numbers:

$$(a + bi)(c + di) = (ac - bd) + (ad + bc)i,$$

and defined a new multiplication on the complex plane:

$$(a, b) \cdot (c, d) = (ac - bd, ad + bc).$$

Thus, $i = \sqrt{-1}$ just disappears into the point $(0, 1)$, and the mystery and confusion over complex numbers disappears along with it.

From Complex Numbers to Quaternions. From Hamilton's interpretation, *complex numbers are nothing more than the extension of real numbers into a new dimension, two dimensions*. Hamilton, however, also did fundamental work in mechanics, and he knew well that two dimensions were too limiting for the space analysis necessary for understanding the physics of the three-dimensional world. Therefore, Hamilton set out to find a triplet system; that is, an acceptable[1] multiplication scheme on points (a,b,c) in $\mathbb{R}^3$, or, as it were, on vectors $a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$.

By 1843, Hamilton realized that his quest was hopeless. But then, on October 16, 1843, Hamilton discovered that what he could not achieve for $\mathbb{R}^3$ he could achieve for $\mathbb{R}^4$; he discovered *quaternions*, an entirely new number system.

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[1] For him, "acceptable" meant that the associative law of multiplication would hold.

Let us revisit the important historical moment in Hamilton's own words:

> But on the 16^{th} day of the same month—which happened to be a Monday, and a Council Day of the Royal Irish Academy—I was walking in to attend and preside, and your mother was walking with me, along the Royal Canal, to which she had perhaps driven; and although she talked with me now and then, yet another *under-current* of thought was going on in my mind, which gave at last a *result*, whereof it is not too much to say that I felt at *once* the importance. An *electric* circuit seemed to *close*; and a spark flashed forth, the herald (as I *foresaw, immediately*) of many long years to come of definitely directed thought and work, by *myself* if spared, and at all events on the parts of *others*, if I should even be allowed to live long enough to distinctly communicate the discovery. Nor could I resist the impulse—unphilosophical as it may have been—to cut with a knife on a stone of Brougham Bridge, as we passed it, the fundamental formula with the symbols i, j, k; namely
>
> $$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1,$$
>
> which contains the *Solution of the Problem*, but of course, as an inscription, has long since moldered away.[2]

Hamilton had realized that the multiplication he had been searching for could be introduced on 4-tuples (a, b, c, d), which he had denoted by

$$a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k}.$$

The a was called the *scalar part* and $b\mathbf{i} + c\mathbf{j} + d\mathbf{k}$ was called the *vector part*, which in reality, as with complex numbers, meant the point (a, b, c, d) in $\mathbb{R}^4$. The multiplication table he introduced was

$$\mathbf{ij} = \mathbf{k} = -\mathbf{ji}$$
$$\mathbf{ki} = \mathbf{j} = -\mathbf{ik}$$
$$\mathbf{jk} = \mathbf{i} = -\mathbf{kj}$$
$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1.$$

Hamilton continued to *passionately* believe in his quaternions until the end of his life. Unfortunately, historical development went in another direction.

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[2] *North British Review* **14** (1858): 57.

The first step away from the quaternions was in fact taken by a firm believer in the importance of quaternions, namely, Peter Guthrie Tait, who was born in 1831 near Edinburgh, Scotland. In 1860, Tait was appointed to the Chair of Natural Philosophy at Edinburgh University, where he remained until his death in 1901. In 1867, he wrote his *Elementary Treatises on Quaternions*, a text stressing physical applications. His third chapter was most significant. It was here that Tait looked at the quaternionic product of two vectors:

$$\mathbf{v} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k} \qquad \text{and} \qquad \mathbf{w} = a'\mathbf{i} + b'\mathbf{j} + c'\mathbf{k}.$$

Then the product $\mathbf{vw}$, as defined by Hamilton, yields:

$$\begin{aligned}&(a\mathbf{i} + b\mathbf{j} + c\mathbf{k})(a'\mathbf{i} + b'\mathbf{j} + c'\mathbf{k})\\ &\qquad = -(aa' + bb' + cc') + (bc' - cb')\mathbf{i} + (ac' - ca')\mathbf{j} + (ab' - ba')\mathbf{k}\end{aligned}$$

or, in modern form:

$$\mathbf{vw} = -(\mathbf{v} \cdot \mathbf{w}) + \mathbf{v} \times \mathbf{w},$$

where $\cdot$ is the modern dot or inner product of vectors and $\times$ is the cross product. Tait discovered the formulas

$$\mathbf{v} \cdot \mathbf{w} = \|\mathbf{v}\|\|\mathbf{w}\| \cos\theta \qquad \text{and} \qquad \|\mathbf{v} \times \mathbf{w}\| = \|\mathbf{v}\|\|\mathbf{w}\| \sin\theta,$$

where θ is the angle formed by $\mathbf{v}$ and $\mathbf{w}$. Moreover, he showed that $\mathbf{v} \times \mathbf{w}$ was orthogonal to $\mathbf{v}$ and $\mathbf{w}$, therefore giving a *geometric* interpretation of the quaternionic product of two vectors.

This began the move away from the study of quaternions and back to Newton's vectors, with the quaternionic product eventually being replaced by two separate products, the inner product and the cross product.

By the way, you might wonder why Hamilton did not at first discover the cross product, since it is a product on $\mathbb{R}^3$. The reason is that it did not have a fundamental property that he required—namely, it was not associative:[3]

$$0 = (\mathbf{i} \times \mathbf{i}) \times \mathbf{k} \neq \mathbf{i} \times (\mathbf{i} \times \mathbf{k}) = -\mathbf{k}$$

[3]Interestingly, if one is willing to continue to live with nonassociativity, there is also a vector product with most of the properties of the cross product in $\mathbb{R}^7$; this involves yet another number system called the *octonians*, which exists in $\mathbb{R}^8$. The nonexistence of a cross product in other dimensions is a result that goes beyond the scope of this text. For further information, see the *American Mathematical Monthly*, **74** (1967), pp. 188–194, and **90** (1983), p. 697, as well as J. Baez, "The Octonians,"*Bulletin of the American Mathematical Society*, **39** (2002), pp. 145–206. One can show that systems like the quaternions and octonians occur only in dimension 1 (the reals $\mathbb{R}$), dimension 2 (the complex numbers), dimension 4 (the quaternions), and dimension 8 (the octonians). On the other hand, the "right"way to extend the cross product is to introduce the notion of *differential forms*, which exists in *any* dimension. We discuss their construction in Section 8.6.

The Move Away from Quaternions. The scientists ultimately responsible for the demise of quaternions were James Clerk Maxwell (see Figure 1.3.9), Oliver Heaviside, and Josiah Willard Gibbs, a founder of statistical mechanics. In the 1860s, Maxwell wrote down his monumental equations of electricity and magnetism. No vector notation was used (it did not exist). Instead, Maxwell wrote out his equations in what we would now call "component form." Around 1870, Tait began to correspond with Maxwell, piquing his interest in quaternions.

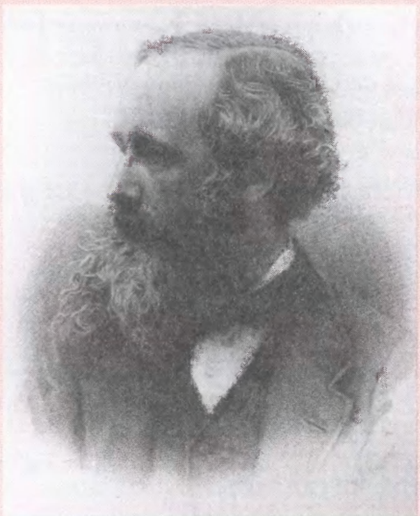


Figure 1.3.9 James Clerk Maxwell (1831–1879).

In 1873, Maxwell published his epic work, *Treatise on Electricity and Magnetism*. Here (as we shall do in Chapter 8), Maxwell wrote down the equations of the electromagnetic field using quaternions, thus motivating physicists and mathematicians alike to take a closer look at them. From this manuscript many have concluded that Maxwell was a supporter of the "quaternionic approach" to physics. The truth, however, is that Maxwell was reluctant to use quaternions. It was Maxwell, in fact, who began the process of separating the *vector* part of a product of two quaternions (the cross product) from its *scalar* part (the dot product).

It is known that Maxwell was troubled by the fact that the scalar part of the "square" of a vector ($\mathbf{vv}$) was always negative ($-\mathbf{v} \cdot \mathbf{v}$), which in the case of a velocity vector could be interpreted as negative kinetic energy—an unacceptable idea!

It was Heaviside and Gibbs who made the final push away from quaternions. Heaviside, an independent researcher interested in electricity and magnetism, and Gibbs, a professor of mathematical physics at Yale, almost simultaneously—and independently—created our modern system of vector analysis, which we have just started to study.

In 1879, Gibbs taught a course at Yale in vector analysis with applications to electricity and magnetism. This treatise was clearly motivated by the advent of Maxwell's equations, which we will be studying in Chapter 8. In 1884, he published his *Elements of Vector Analysis*, a book in which all the properties of the dot and cross products are fully developed. Knowing that much of what Gibbs wrote was in fact due to Tait, Gibbs's contemporaries did not view his book as highly original. However, it is one of the sources from which modern vector analysis has come into existence.

Heaviside was also largely motivated by Maxwell's brilliant work. His great *Electromagnetic Theory* was published in three volumes. Volume I (1893) contained the first extensive treatment of modern vector analysis.

We all owe a great debt to E. B. Wilson's 1901 book *Vector Analysis: A Textbook for the Use of Students of Mathematics and Physics Founded upon the Lectures of J. Willard Gibbs*. Wilson was reluctant to take Gibbs's course, because he had just completed a full-year course in quaternions at Harvard under J. M. Pierce, a champion of quaternionic methods; but he was forced by a dean to add the course to his program, and he did so in 1899. Wilson was later asked by the editor of the Yale Bicentennial Series to write a book based on Gibbs's lectures. For a picture of Gibbs and for additional historical comments on divergence and curl, see the Historical Note in Section 4.4.

EXERCISES

1. Verify that interchanging the first two rows of the 3×3 determinant

$$\begin{vmatrix} 1 & 2 & 1 \\ 3 & 0 & 1 \\ 2 & 0 & 2 \end{vmatrix}$$

changes the sign of the determinant.

2. Evaluate the determinants

(a) $\begin{vmatrix} 2 & -1 & 0 \\ 4 & 3 & 2 \\ 3 & 0 & 1 \end{vmatrix}$

(b) $\begin{vmatrix} 36 & 18 & 17 \\ 45 & 24 & 20 \\ 3 & 5 & --2 \end{vmatrix}$

(c) $\begin{vmatrix} 1 & 4 & 9 \\ 4 & 9 & 16 \\ 9 & 16 & 25 \end{vmatrix}$

(d) $\begin{vmatrix} 2 & 3 & 5 \\ 7 & 11 & 13 \\ 17 & 19 & 23 \end{vmatrix}$

3. Compute $\mathbf{a} \times \mathbf{b}$, where $\mathbf{a} = \mathbf{i} - 2\mathbf{j} + \mathbf{k}$, $\mathbf{b} = 2\mathbf{i} + \mathbf{j} + \mathbf{k}$.

4. Compute $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$, where $\mathbf{a}$ and $\mathbf{b}$ are as in Exercise 3 and $\mathbf{c} = 3\mathbf{i} - \mathbf{j} + 2\mathbf{k}$.

5. Find the area of the parallelogram with sides $\mathbf{a}$ and $\mathbf{b}$ given in Exercise 3.

6. A triangle has vertices $(0, 0, 0)$, $(1, 1, 1)$, and $(0, -2, 3)$. Find its area.

7. What is the volume of the parallelepiped with sides $2\mathbf{i} + \mathbf{j} - \mathbf{k}$, $5\mathbf{i} - 3\mathbf{k}$, and $\mathbf{i} - 2\mathbf{j} + \mathbf{k}$?

8. What is the volume of the parallelepiped with sides $\mathbf{i}$, $3\mathbf{j} - \mathbf{k}$, and $4\mathbf{i} + 2\mathbf{j} - \mathbf{k}$?

In Exercises 9 to 12, describe all unit vectors orthogonal to both of the given vectors.

9. $\mathbf{i}, \mathbf{j}$

10. $-5\mathbf{i} + 9\mathbf{j} - 4\mathbf{k}$, $7\mathbf{i} + 8\mathbf{j} + 9\mathbf{k}$

11. $-5\mathbf{i} + 9\mathbf{j} - 4\mathbf{k}$, $7\mathbf{i} + 8\mathbf{j} + 9\mathbf{k}$, $\mathbf{0}$

12. $2\mathbf{i} - 4\mathbf{j} + 3\mathbf{k}$, $-4\mathbf{i} + 8\mathbf{j} - 6\mathbf{k}$

13. Compute $\mathbf{u} + \mathbf{v}$, $\mathbf{u} \cdot \mathbf{v}$, $\|\mathbf{u}\|$, $\|\mathbf{v}\|$, and $\mathbf{u} \times \mathbf{v}$, where $\mathbf{u} = \mathbf{i} - 2\mathbf{j} + \mathbf{k}$, $\mathbf{v} = 2\mathbf{i} - \mathbf{j} + 2\mathbf{k}$.

14. Repeat Exercise 13 for $\mathbf{u} = 3\mathbf{i} + \mathbf{j} - \mathbf{k}$, $\mathbf{v} = -6\mathbf{i} - 2\mathbf{j} - 2\mathbf{k}$.

15. Find an equation for the plane that

    (a) is perpendicular to $\mathbf{v} = (1, 1, 1)$ and passes through $(1, 0, 0)$.
    (b) is perpendicular to $\mathbf{v} = (1, 2, 3)$ and passes through $(1, 1, 1)$.
    (c) is perpendicular to the line $\mathbf{l}(t) = (5, 0, 2)t + (3, -1, 1)$ and passes through $(5, -1, 0)$.
    (d) is perpendicular to the line $\mathbf{l}(t) = (-1, -2, 3)t + (0, 7, 1)$ and passes through $(2, 4, -1)$.

16. Find an equation for the plane that passes through

    (a) $(0, 0, 0)$, $(2, 0, -1)$, and $(0, 4, -3)$.
    (b) $(1, 2, 0)$, $(0, 1, -2)$, and $(4, 0, 1)$.
    (c) $(2, -1, 3)$, $(0, 0, 5)$, and $(5, 7, -1)$.

17. (a) Show that two parallel planes are either identical or they never intersect.
    (b) How do two nonparallel planes intersect?

18. Find the intersection of the planes $x + 2y + z = 0$ and $x - 3y - z = 0$.

19. Find the intersection of the planes $x + (y - 1) + z = 0$ and $-x + (y + 1) - z = 0$.

20. Find the intersection of the two planes with equations $3(x - 1) + 2y + (z + 1) = 0$ and $(x - 1) + 4y - (z + 1) = 0$.

21. (a) Prove the two triple-vector-product identities

$$(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{b} \cdot \mathbf{c})\mathbf{a} \qquad \text{and} \qquad \mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}.$$

    (b) Prove $(\mathbf{u} \times \mathbf{v}) \times \mathbf{w} = \mathbf{u} \times (\mathbf{v} \times \mathbf{w})$ if and only if $(\mathbf{u} \times \mathbf{w}) \times \mathbf{v} = \mathbf{0}$.

(c) Also prove that $(\mathbf{u} \times \mathbf{v}) \times \mathbf{w} + (\mathbf{v} \times \mathbf{w}) \times \mathbf{u} + (\mathbf{w} \times \mathbf{u}) \times \mathbf{v} = \mathbf{0}$ (called the ***Jacobi identity***).

22. (a) Prove, without recourse to geometry, that

$$\begin{aligned} \mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) &= \mathbf{v} \cdot (\mathbf{w} \times \mathbf{u}) = \mathbf{w} \cdot (\mathbf{u} \times \mathbf{v}) = -\mathbf{u} \cdot (\mathbf{w} \times \mathbf{v}) \\ &= -\mathbf{w} \cdot (\mathbf{v} \times \mathbf{u}) = -\mathbf{v} \cdot (\mathbf{u} \times \mathbf{w}). \end{aligned}$$

(b) Use part (a) and Exercise 21(a) to prove that

$$(\mathbf{u} \times \mathbf{v}) \cdot (\mathbf{u}' \times \mathbf{v}') = (\mathbf{u} \cdot \mathbf{u}')(\mathbf{v} \cdot \mathbf{v}') - (\mathbf{u} \cdot \mathbf{v}')(\mathbf{u}' \cdot \mathbf{v}) = \begin{vmatrix} \mathbf{u} \cdot \mathbf{u}' & \mathbf{u} \cdot \mathbf{v}' \\ \mathbf{u}' \cdot \mathbf{v} & \mathbf{v} \cdot \mathbf{v}' \end{vmatrix}.$$

23. Verify Cramer's rule.

24. Find an equation for the plane that passes through the point $(2, -1, 3)$ and is perpendicular to the line $\mathbf{v} = (1, -2, 2) + t(3, -2, 4)$.

25. Find an equation for the plane that passes through the point $(1, 2, -3)$ and is perpendicular to the line $\mathbf{v} = (0, -2, 1) + t(1, -2, 3)$.

26. Find the equation of the line that passes through the point $(1, -2, -3)$ and is perpendicular to the plane $3x - y - 2z + 4 = 0$.

27. Find an equation for the plane containing the two (parallel) lines

$$\mathbf{v}_1 = (0, 1, -2) + t(2, 3, -1) \qquad \text{and} \qquad \mathbf{v}_2 = (2, -1, 0) + t(2, 3, -1).$$

28. Find the distance from the point $(2, 1, -1)$ to the plane $x - 2y + 2z + 5 = 0$.

29. Find an equation for the plane that contains the line $\mathbf{v} = (-1, 1, 2) + t(3, 2, 4)$ and is perpendicular to the plane $2x + y - 3z + 4 = 0$.

30. Find an equation for the plane that passes through $(3, 2, -1)$ and $(1, -1, 2)$ and that is parallel to the line $\mathbf{v} = (1, -1, 0) + t(3, 2, -2)$.

31. Redo Exercises 19 and 20 of Section 1.1 using the dot product and what you know about normals to planes.

32. Given vectors $\mathbf{a}$ and $\mathbf{b}$, do the equations $\mathbf{x} \times \mathbf{a} = \mathbf{b}$ and $\mathbf{x} \cdot \mathbf{a} = \|\mathbf{a}\|$ determine a unique vector $\mathbf{x}$? Argue both geometrically and analytically.

33. Determine the distance from the plane $12x + 13y + 5z + 2 = 0$ to the point $(1, 1, -5)$.

34. Find the distance to the point $(6, 1, 0)$ from the plane through the origin that is perpendicular to $\mathbf{i} - 2\mathbf{j} + \mathbf{k}$.

35. (a) In mechanics, the ***moment** M **of a force*** $\mathbf{F}$ *about a point* O is defined to be the magnitude of $\mathbf{F}$ times the perpendicular distance d from O to the line of action of $\mathbf{F}$. The ***vector moment*** $\mathbf{M}$ is the vector of magnitude M whose direction is perpendicular to the plane of O and $\mathbf{F}$, determined by the right-hand rule. Show that $\mathbf{M} = \mathbf{R} \times \mathbf{F}$, where $\mathbf{R}$ is any vector from O to the line of action of $\mathbf{F}$. (See Figure 1.3.10.)

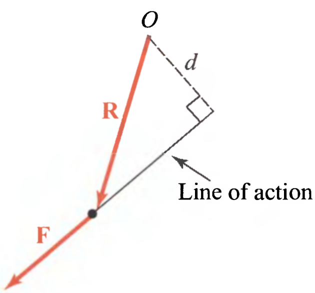


Figure 1.3.10 Moment of a force.

(b) Find the moment of the force vector $\mathbf{F} = \mathbf{i} - \mathbf{j} + 2\mathbf{k}$ newtons about the origin if the line of action is $x = 1 + t$, $y = 1 - t$, $z = 2t$.

36. Show that the plane that passes through the three points $A = (a_1, a_2, a_3)$, $B = (b_1, b_2, b_3)$, and $C = (c_1, c_2, c_3)$ consists of the points $\mathrm{P} = (x, y, z)$ given by

$$\begin{vmatrix} a_1 - x & a_2 - y & a_3 - z \\ b_1 - x & b_2 - y & b_3 - z \\ c_1 - x & c_2 - y & c_3 - z \end{vmatrix} = 0.$$

(HINT: Write the determinant as a triple product.)

37. Two media with indices of refraction n_1 and n_2 are separated by a plane surface perpendicular to the unit vector $\mathbf{N}$. Let $\mathbf{a}$ and $\mathbf{b}$ be unit vectors along the incident and refracted rays, respectively, their directions being those of the light rays. Show that $n_1(\mathbf{N} \times \mathbf{a}) = n_2(\mathbf{N} \times \mathbf{b})$ by using *Snell's law*, $\sin\theta_1 / \sin\theta_2 = n_2/n_1$, where θ_1 and θ_2 are the angles of incidence and refraction, respectively. (See Figure 1.3.11.)

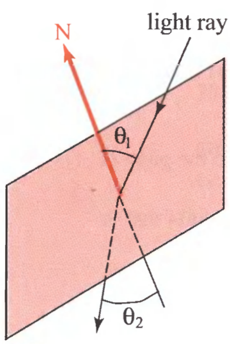


Figure 1.3.11 Snell's law.

38. Justify the steps in the following computation:

$$\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 10 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 7 & 8 & 10 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & -6 & -11 \end{vmatrix} = \begin{vmatrix} -3 & -6 \\ -6 & -11 \end{vmatrix} = 33 - 36 = -3.$$

39. Show that adding a multiple of the first row of a matrix to the second row leaves the determinant unchanged; that is,

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 + \lambda a_1 & b_2 + \lambda b_1 & c_2 + \lambda c_1 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}.$$

[In fact, adding a multiple of any row (column) of a matrix to another row (column) leaves the determinant unchanged.]

1.4 Cylindrical and Spherical Coordinates

A standard way to represent a point in the plane $\mathbb{R}^2$ is by means of rectangular coordinates (x, y). However, as the reader has probably learned in elementary calculus, polar coordinates in the plane can be extremely useful. As portrayed in Figure 1.4.1, the coordinates (r, θ) are related to (x, y) by the formulas

$$x = r\cos\theta \qquad \text{and} \qquad y = r\sin\theta,$$

where we usually take $r \geq 0$ and $0 \leq \theta < 2\pi$.

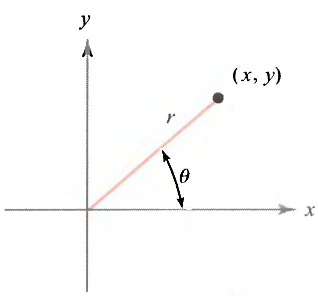


Figure 1.4.1 The polar coordinates of (x, y) are (r, θ).

Readers not familiar with polar coordinates are advised to study the relevant section of their calculus texts. We now set forth two ways of representing points in space other than by using rectangular Cartesian coordinates (x, y, z). These alternative coordinate systems are particularly well suited for certain types of problems, such as the evaluation of integrals using a change of variables.

Historical Note

In 1671, Isaac Newton wrote a manuscript entitled *The Method of Fuxions and Infinite Series*, which contains many uses of coordinate geometry to sketch the solutions of equations. In particular, he introduces the polar coordinate system, among various other coordinate systems.

In 1691, Jacob Bernoulli published a paper also containing polar coordinates. Because Newton's manuscript was not published until after his death in 1727, credit for the discovery of polar coordinates is usually attributed to Bernoulli.

Cylindrical Coordinates

DEFINITION The ***cylindrical coordinates*** (r, θ, z) of a point (x, y, z) are defined by (see Figure 1.4.2)

$$x = r\cos\theta, \qquad y = r\sin\theta, \qquad z = z. \tag{1}$$

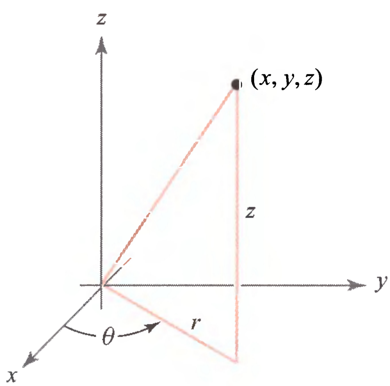


Figure 1.4.2 Representing a point (x, y, z) in terms of its cylindrical coordinates r, θ, and z.

To express r, θ, and z in terms of x, y, and z, and to ensure that θ lies between 0 and 2π, we can write

$$r = \sqrt{x^2 + y^2}, \qquad \theta = \begin{cases} \tan^{-1}(y/x) & \text{if } x > 0 \text{ and } y \geq 0 \\ \pi + \tan^{-1}(y/x) & \text{if } x < 0 \\ 2\pi + \tan^{-1}(y/x) & \text{if } x > 0 \text{ and } y < 0, \end{cases} \qquad z = z,$$

where $\tan^{-1}(y/x)$ is taken to lie between $-\pi/2$ and $\pi/2$. The requirement that $0 \leq \theta < 2\pi$ uniquely determines θ and $r \geq 0$ for a given x and y. If $x = 0$, then $\theta = \pi/2$ for $y > 0$ and $3\pi/2$ for $y < 0$. If $x = y = 0$, θ is undefined.

In other words, for any point (x, y, z), we represent the first and second coordinates in terms of polar coordinates and leave the third coordinate unchanged. Formula (1) shows that, given (r, θ, z), the triple (x, y, z) is completely determined, and vice versa, if we restrict θ to the interval $[0, 2\pi)$ (sometimes the range $(-\pi, \pi]$ is convenient) and require that $r > 0$.

To see why we use the term *cylindrical coordinates*, note that if the conditions $0 \leq \theta < 2\pi, -\infty < z < \infty$ hold and if $r = a$ is some positive constant, then the locus of these points is a cylinder of radius a (see Figure 1.4.3).

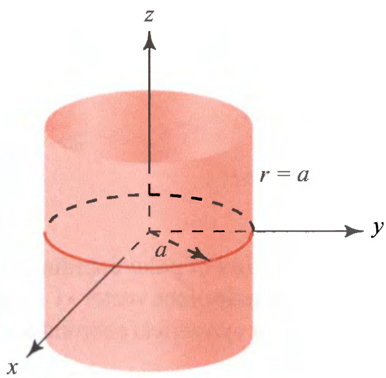


Figure 1.4.3 The graph of the points whose cylindrical coordinates satisfy $r = a$ is a cylinder.

EXAMPLE 1 (a) Find and plot the cylindrical coordinates of (6, 6, 8). (b) If a point has cylindrical coordinates $(8, 2\pi/3, -3)$, what are its Cartesian coordinates? Plot.

SOLUTION For part (a), we have $r = \sqrt{6^2 + 6^2} = 6\sqrt{2}$ and $\theta = \tan^{-1}(6/6) = \tan^{-1}(1) = \pi/4$. Thus, the cylindrical coordinates are $(6\sqrt{2}, \pi/4, 8)$. This is point P in Figure 1.4.4. For part (b), note that $2\pi/3 = \pi/2 + \pi/6$ and compute

$$x = r\cos\theta = 8\cos\frac{2\pi}{3} = -\frac{8}{2} = -4$$

and

$$y = r\sin\theta = 8\sin\frac{2\pi}{3} = 8\frac{\sqrt{3}}{2} = 4\sqrt{3}.$$

Thus, the Cartesian coordinates are $(-4, 4\sqrt{3}, -3)$. This is point Q in the figure. ▲

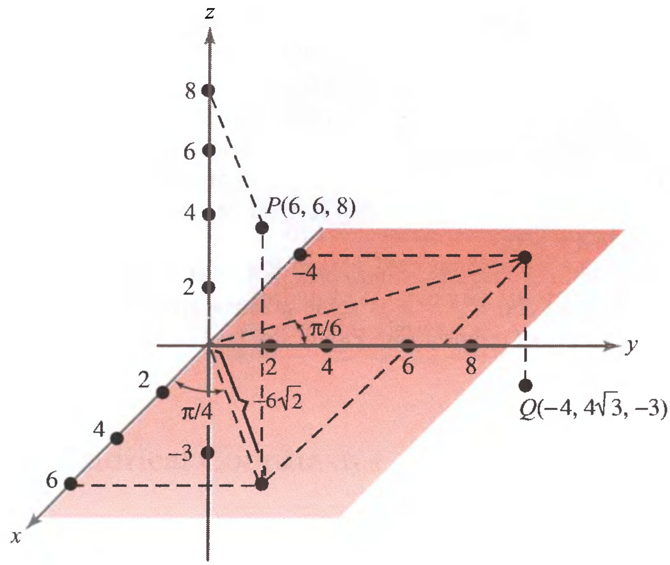


Figure 1.4.4 Some examples of the conversion between Cartesian and cylindrical coordinates.

Spherical Coordinates

Cylindrical coordinates are not the only possible generalization of polar coordinates to three dimensions. Recall that in two dimensions the magnitude of the vector $x\mathbf{i} + y\mathbf{j}$ (that is, $\sqrt{x^2 + y^2}$) is the r in the polar coordinate system. For cylindrical coordinates, the length of the vector $x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, namely,

$$\rho = \sqrt{x^2 + y^2 + z^2},$$

is not one of the coordinates of that system—instead, we used the magnitude $r = \sqrt{x^2 + y^2}$, the angle θ, and the "height" z.

We now modify this by introducing the *spherical coordinate* system, which *does use* ρ as a coordinate. Spherical coordinates are often useful for problems that possess spherical symmetry (symmetry about a point), whereas cylindrical coordinates can be applied when cylindrical symmetry (symmetry about a line) is involved.

Given a point $(x, y, z) \in \mathbb{R}^3$, let

$$\rho = \sqrt{x^2 + y^2 + z^2}$$

and represent x and y by polar coordinates in the xy plane:

$$x = r\cos\theta, \qquad y = r\sin\theta \tag{2}$$

where $r = \sqrt{x^2 + y^2}$ and θ is determined by formula (1) [see the expression for θ following formula (1)]. The coordinate z is given by

$$z = \rho\cos\phi,$$

where ϕ is the angle (chosen to lie between 0 and π, inclusive) that the radius vector $\mathbf{v} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ makes with the positive z axis, in the plane containing the vector $\mathbf{v}$ and the z axis (see Figure 1.4.5). Using the dot product, we can express ϕ as follows:

$$\cos\phi = \frac{\mathbf{v}\cdot\mathbf{k}}{\|\mathbf{v}\|}, \qquad \text{that is,} \qquad \phi = \cos^{-1}\left(\frac{\mathbf{v}\cdot\mathbf{k}}{\|\mathbf{v}\|}\right).$$

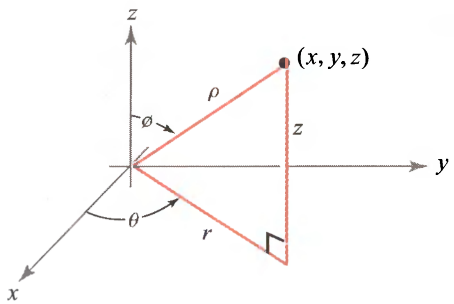


Figure 1.4.5 Spherical coordinates (ρ, θ, ϕ); the graph of points satisfying $\rho = a$ is a sphere.

We take as our coordinates the quantities ρ, θ, ϕ. Because

$$r = \rho \sin\phi,$$

we can use formula (2) to find x, y, z in terms of the spherical coordinates ρ, θ, ϕ.

DEFINITION The ***spherical coordinates*** of (x, y, z) is the triple (ρ, θ, ϕ), defined as follows:

$$x = \rho\sin\phi\cos\theta, \qquad y = \rho\sin\phi\sin\theta, \qquad z = \rho\cos\phi \tag{3}$$

where

$$\rho \ge 0, \qquad 0 \le \theta < 2\pi, \qquad 0 \le \phi \le \pi.$$

Historical Note

In 1773, Joseph Louis Lagrange was working on Newton's gravitational theory as it applied to ellipsoids of revolution. In attempting to calculate the total gravitational attraction of such an ellipsoid, he encountered an integral that was difficult to evaluate. Motivated by this application, he introduced spherical coordinates, which allowed him to calculate the integral. We will be discussing the method of changing coordinates as it applies to multiple integrals in Section 6.2, and applications to gravitation in Section 6.3, where we show how the inverse square law of gravity allowed Newton to consider spherical masses as point masses.

Spherical coordinates are also closely connected to navigation by latitude and longitude. To see the connection, first note that the sphere of radius a centered at the origin is described by a very simple equation in spherical coordinates, namely, $\rho = a$. Fixing the radius a, the spherical coordinates θ and ϕ are similar to the geographic coordinates of longitude and latitude if we take the earth's axis to be the z axis. There are differences, though: The geographical longitude is $|\theta|$ and is called east or west longitude, according to whether θ is a positive or negative measure from the Greenwich meridian; the geographical latitude is $|\pi/2 - \phi|$ and is called north or south latitude, according to whether $\pi/2 - \phi$ is positive or negative.

EXAMPLE 2

(a) Find the spherical coordinates of the Cartesian point $(1, -1, 1)$ and plot.

(b) Find the Cartesian coordinates of the spherical coordinate point $(3, \pi/6, \pi/4)$ and plot.

(c) Let a point have Cartesian coordinates $(2, -3, 6)$. Find its spherical coordinates and plot.

(d) Let a point have spherical coordinates $(1, -\pi/2, \pi/4)$. Find its Cartesian coordinates and plot.

SOLUTION

(a) $\rho = \sqrt{x^2 + y^2 + z^2} = \sqrt{1^2 + (-1)^2 + 1^2} = \sqrt{3}$,

$$\theta = 2\pi + \tan^{-1}\left(\frac{y}{x}\right) = 2\pi + \tan^{-1}\left(\frac{-1}{1}\right) = 2\pi - \frac{\pi}{4} = \frac{7\pi}{4}$$

$$\phi = \cos^{-1}\left(\frac{z}{\rho}\right) = \cos^{-1}\left(\frac{1}{\sqrt{3}}\right) \approx 0.955 \approx 54.74^\circ.$$

See Figure 1.4.6(a) and the formula for θ following formula (1).

(b) $$x = \rho \sin\phi \cos\theta = 3\sin\left(\frac{\pi}{4}\right)\cos\left(\frac{\pi}{6}\right) = 3\left(\frac{1}{\sqrt{2}}\right)\frac{\sqrt{3}}{2} = \frac{3\sqrt{3}}{2\sqrt{2}},$$

$$y = \rho \sin\phi \sin\theta = 3\sin\left(\frac{\pi}{4}\right)\sin\left(\frac{\pi}{6}\right) = 3\left(\frac{1}{\sqrt{2}}\right)\left(\frac{1}{2}\right) = \frac{3}{2\sqrt{2}},$$

$$z = \rho\cos\phi = 3\cos\left(\frac{\pi}{4}\right) = \frac{3}{\sqrt{2}} = \frac{3\sqrt{2}}{2}.$$

See Figure 1.4.6(b).

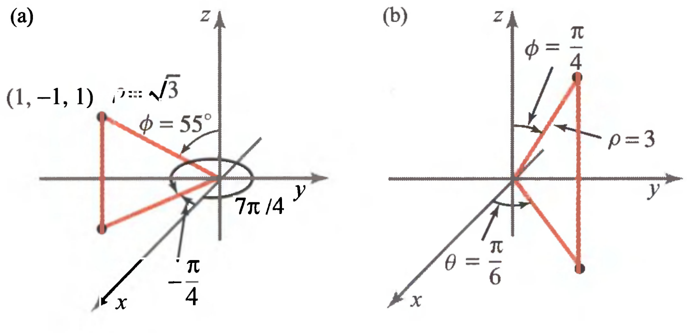


Figure 1.4.6 Finding (a) the spherical coordinates of the point (1, –1, 1), and (b) the Cartesian coordinates of $(3, \pi/6, \pi/4)$.

(c) $$\rho = \sqrt{x^2 + y^2 + z^2} = \sqrt{2^2 + (-3)^2 + 6^2} = \sqrt{49} = 7,$$

$$\theta = 2\pi + \tan^{-1}\left(\frac{y}{x}\right) = 2\pi + \tan^{-1}\left(\frac{-3}{2}\right) \approx 5.3004 \text{ radians} \approx 303.69°,$$

$$\phi = \cos^{-1}\left(\frac{z}{\rho}\right) = \cos^{-1}\left(\frac{6}{7}\right) \approx 0.541 \approx 31.0°.$$

See Figure 1.4.7(a).

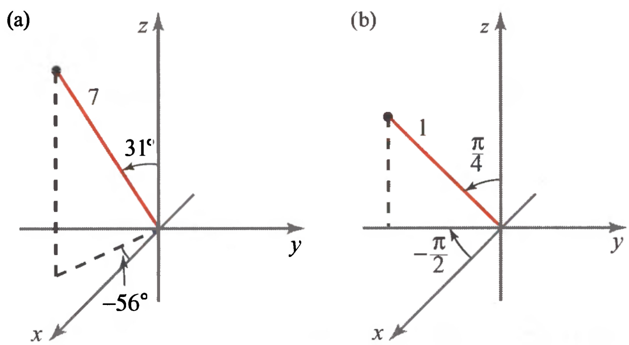


Figure 1.4.7 Finding (a) the spherical coordinates of the point (2, –3, 6), and (b) the Cartesian coordinates of $(1, -\pi/2, \pi/4)$.

(d) $$x = \rho \sin\phi \cos\theta = 1 \sin\left(\frac{\pi}{4}\right) \cos\left(-\frac{\pi}{2}\right) = \left(\frac{\sqrt{2}}{2}\right) \cdot 0 = 0,$$

$$y = \rho \sin\phi \sin\theta = 1 \sin\left(\frac{\pi}{4}\right) \sin\left(-\frac{\pi}{2}\right) = \left(\frac{\sqrt{2}}{2}\right)(-1) = -\frac{\sqrt{2}}{2},$$

$$z = \rho \cos\phi = 1 \cos\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}.$$

See Figure 1.4.7(b) ▲

EXAMPLE 3 Express (a) the surface $xz = 1$ and (b) the surface $x^2 + y^2 - z^2 = 1$ in spherical coordinates.

SOLUTION From formula (3), $x = \rho \sin\phi \cos\theta$, and $z = \rho\cos\phi$, and so the surface $xz = 1$ in (a) consists of all (ρ, θ, ϕ) such that

$$\rho^2 \sin\phi \cos\theta \cos\phi = 1, \qquad \text{that is,} \qquad \rho^2 \sin 2\phi \cos\theta = 2.$$

For part (b), we can write

$$x^2 + y^2 - z^2 = x^2 + y^2 + z^2 - 2z^2 = \rho^2 - 2\rho^2\cos^2\phi,$$

so that the surface is $\rho^2(1 - 2\cos^2\phi) = 1$; that is, $-\rho^2\cos(2\phi) = 1$. ▲

Associated with cylindrical and spherical coordinates are unit vectors that are the counterparts of $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ for rectangular coordinates. They are shown in Figure 1.4.8. For example, $\mathbf{e}_r$ is the unit vector parallel to the xy plane and in the radial direction, so that $\mathbf{e}_r = (\cos\theta)\mathbf{i} + (\sin\theta)\mathbf{j}$. Similarly, in spherical coordinates, $\mathbf{e}_\phi$ is the unit vector tangent to the curve parametrized by the variable ϕ with the variables ρ and θ held fixed. We shall use these unit vectors later when we use cylindrical and spherical coordinates in vector calculations.

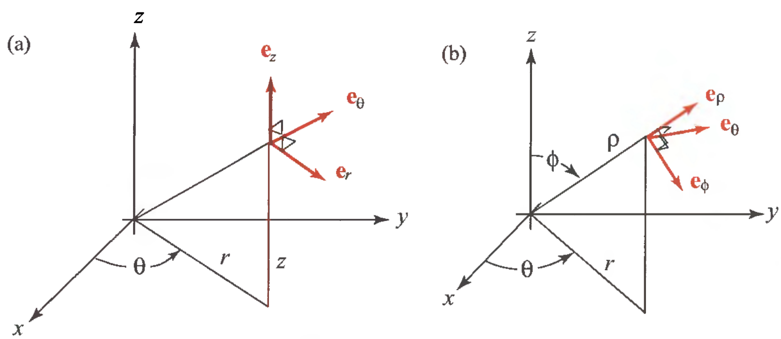


Figure 1.4.8 (a) Orthonormal vectors $\mathbf{e}_r$, $\mathbf{e}_\theta$, and $\mathbf{e}_z$ associated with cylindrical coordinates. The vector $\mathbf{e}_r$ is parallel to the line labeled r. (b) Orthonormal vectors $\mathbf{e}_\rho$, $\mathbf{e}_\theta$, and $\mathbf{e}_\phi$ associated with spherical coordinates.

EXERCISES

1. (a) The following points are given in cylindrical coordinates; express each in rectangular coordinates and spherical coordinates: $(1, 45°, 1)$, $(2, \pi/2, -4)$, $(0, 45°, 10)$, $(3, \pi/6, 4)$, $(1, \pi/6, 0)$, and $(2, 3\pi/4, -2)$. (Only the first point is solved in the Study Guide.)

(b) Change each of the following points from rectangular coordinates to spherical coordinates and to cylindrical coordinates: $(2, 1, -2)$, $(0, 3, 4)$, $(\sqrt{2}, 1, 1)$, $(-2\sqrt{3}, -2, 3)$. (Only the first point is solved in the Study Guide.)

2. Describe the geometric meaning of the following mappings in cylindrical coordinates:

(a) $(r, \theta, z) \mapsto (r, \theta, -z)$
(b) $(r, \theta, z) \mapsto (r, \theta + \pi, -z)$
(c) $(r, \theta, z) \mapsto (-r, \theta - \pi/4, z)$

3. Describe the geometric meaning of the following mappings in spherical coordinates:

(a) $(\rho, \theta, \phi) \mapsto (\rho, \theta + \pi, \phi)$
(b) $(\rho, \theta, \phi) \mapsto (\rho, \theta, \pi - \phi)$
(c) $(\rho, \theta, \phi) \mapsto (2\rho, \theta + \pi/2, \phi)$

4. (a) Describe the surfaces $r =$ constant, $\theta =$ constant, and $z =$ constant in the cylindrical coordinate system.

(b) Describe the surfaces $\rho =$ constant, $\theta =$ constant, and $\phi =$ constant in the spherical coordinate system.

5. Show that to represent each point in $\mathbb{R}^3$ by spherical coordinates it is necessary to take only values of θ between 0 and 2π, values of ϕ between 0 and π, and values of $\rho \geq 0$. Are coordinates unique if we allow $\rho \leq 0$?

6. Using cylindrical coordinates and the orthonormal (orthogonal normalized) vectors $\mathbf{e}_r$, $\mathbf{e}_\theta$, and $\mathbf{e}_z$ (see Figure 1.4.8),

(a) express each of $\mathbf{e}_r$, $\mathbf{e}_\theta$, and $\mathbf{e}_z$ in terms of $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ and (x, y, z); and
(b) calculate $\mathbf{e}_\theta \times \mathbf{j}$ both analytically, using part (a), and geometrically.

7. Using spherical coordinates and the orthonormal (orthogonal normalized) vectors $\mathbf{e}_\rho$, $\mathbf{e}_\theta$, and $\mathbf{e}_\phi$ (see Figure 1.4.8(b)),

(a) express each of $\mathbf{e}_\rho$, $\mathbf{e}_\theta$, and $\mathbf{e}_\phi$ in terms of $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ and (x, y, z); and
(b) calculate $\mathbf{e}_\theta \times \mathbf{j}$ and $\mathbf{e}_\phi \times \mathbf{j}$ both analytically and geometrically.

8. Express the plane $z = x$ in (a) cylindrical, and (b) spherical coordinates.

9. Show that in spherical coordinates:

(a) ρ is the length of $x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$.
(b) $\phi = \cos^{-1}(\mathbf{v} \cdot \mathbf{k}/\|\mathbf{v}\|)$, where $\mathbf{v} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$.
(c) $\theta = \cos^{-1}(\mathbf{u} \cdot \mathbf{i}/\|\mathbf{u}\|)$, where $\mathbf{u} = x\mathbf{i} + y\mathbf{j}$.

10. Two surfaces are described in spherical coordinates by the two equations $\rho = f(\theta, \phi)$ and $\rho = -2f(\theta, \phi)$, where $f(\theta, \phi)$ is a function of two variables. How is the second surface obtained geometrically from the first?

11. A circular membrane in space lies over the region $x^2 + y^2 \leq a^2$. The maximum z component of points in the membrane is b. Assume that (x, y, z) is a point on the membrane. Show that the corresponding point (r, θ, z) in cylindrical coordinates satisfies the conditions $0 \leq r \leq a$, $0 \leq \theta \leq 2\pi$, $|z| \leq b$.

12. A tank in the shape of a right-circular cylinder of radius 10 ft and height 16 ft is half filled and lying on its side. Describe the air space inside the tank by suitably chosen cylindrical coordinates.

13. A vibrometer is to be designed that withstands the heating effects of its spherical enclosure of diameter d, which is buried to a depth $d/3$ in the earth, the upper portion being heated by the sun (assume the surface is flat). Heat conduction analysis requires a description of the buried portion of the enclosure in spherical coordinates. Find it.

14. An oil filter cartridge is a porous right-circular cylinder inside which oil diffuses from the axis to the outer curved surface. Describe the cartridge in cylindrical coordinates, if the diameter of the filter is 4.5 inches, the height is 5.6 inches, and the center of the cartridge is drilled (all the way through) from the top to admit a $\frac{5}{8}$-inch-diameter bolt.

15. Describe the surface given in spherical coordinates by $\rho = \cos 2\theta$.

1.5 *n*-Dimensional Euclidean Space

Vectors in *n*-space

In Sections 1.1 and 1.2 we studied the spaces $\mathbb{R} = \mathbb{R}^1$, $\mathbb{R}^2$, and $\mathbb{R}^3$ and gave geometric interpretations to them. For example, a point (x, y, z) in $\mathbb{R}^3$ can be thought of as a geometric object, namely, the directed line segment or vector emanating from the origin and ending at the point (x, y, z). We can therefore think of $\mathbb{R}^3$ in either of two ways:

(i) Algebraically, as a set of triples (x, y, z) where x, y, and z are real numbers

(ii) Geometrically, as a set of directed line segments

These two ways of looking at $\mathbb{R}^3$ are equivalent. For generalization it is easier to use definition (i). Specifically, we can define $\mathbb{R}^n$, where n is a positive integer (possibly greater than 3), to be the set of all ordered n-tuples $(x_1, x_2, \dots, x_n)$, where the x_i are real numbers. For instance, $(1, \sqrt{5}, 2, \sqrt{3}) \in \mathbb{R}^4$.

The set $\mathbb{R}^n$ so defined is known as ***Euclidean n-space***, and its elements, which we write as $\mathbf{x} = (x_1, x_2, \dots, x_n)$, are known as ***vectors*** or ***n-vectors***. By setting $n = 1, 2$, or 3, we recover the line, the plane, and three-dimensional space, respectively.

We launch our study of Euclidean n-space by introducing several algebraic operations. These are analogous to those introduced in Section 1.1 for $\mathbb{R}^2$ and $\mathbb{R}^3$. The first two, addition and scalar multiplication, are defined as follows:

(i) $(x_1, x_2, \dots, x_n) + (y_1, y_2, \dots, y_n) = (x_1 + y_1, x_2 + y_2, \dots, x_n + y_n)$;

and

(ii) for any real number α,

$$\alpha(x_1, x_2, \dots, x_n) = (\alpha x_1, \alpha x_2, \dots, \alpha x_n).$$

The geometric significance of these operations for $\mathbb{R}^2$ and $\mathbb{R}^3$ was discussed in Section 1.1.

The n vectors

$$\mathbf{e}_1 = (1, 0, 0, \dots, 0), \mathbf{e}_2 = (0, 1, 0, \dots, 0), \dots, \mathbf{e}_n = (0, 0, \dots, 0, 1)$$

are called the ***standard basis vectors*** of $\mathbb{R}^n$, and they generalize the three mutually orthogonal unit vectors $\mathbf{i}, \mathbf{j}, \mathbf{k}$ of $\mathbb{R}^3$. The vector $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ can then be written as $\mathbf{x} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + \cdots + x_n\mathbf{e}_n$.

For two vectors $\mathbf{x} = (x_1, x_2, x_3)$ and $\mathbf{y} = (y_1, y_2, y_3)$ in $\mathbb{R}^3$, we defined the *dot* or *inner product* $\mathbf{x} \cdot \mathbf{y}$ to be the real number $\mathbf{x} \cdot \mathbf{y} = x_1y_1 + x_2y_2 + x_3y_3$. This definition easily extends to $\mathbb{R}^n$; specifically, for $\mathbf{x} = (x_1, x_2, \ldots, x_n)$, $\mathbf{y} = (y_1, y_2, \ldots, y_n)$, we define the ***inner product*** of $\mathbf{x}$ and $\mathbf{y}$ to be $\mathbf{x} \cdot \mathbf{y} = x_1y_1 + x_2y_2 + \cdots + x_ny_n$. In $\mathbb{R}^n$, the notation $\langle \mathbf{x}, \mathbf{y} \rangle$ is often used in place of $\mathbf{x} \cdot \mathbf{y}$ for the inner product.

Continuing the analogy with $\mathbb{R}^3$, we are led to define the notion of the ***length*** or ***norm*** of a vector $\mathbf{x}$ by the formula

$$\text{Length of } \mathbf{x} = \|\mathbf{x}\| = \sqrt{\mathbf{x} \cdot \mathbf{x}} = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}.$$

If $\mathbf{x}$ and $\mathbf{y}$ are two vectors in the plane ($\mathbb{R}^2$) or in space ($\mathbb{R}^3$), then we know that the angle θ between them is given by the formula

$$\cos\theta = \frac{\mathbf{x} \cdot \mathbf{y}}{\|\mathbf{x}\|\,\|\mathbf{y}\|}.$$

The right side of this equation can be defined in $\mathbb{R}^n$ as well as in $\mathbb{R}^2$ or $\mathbb{R}^3$. It still represents the cosine of the angle between $\mathbf{x}$ and $\mathbf{y}$; this angle is geometrically well defined, because $\mathbf{x}$ and $\mathbf{y}$ lie in a two-dimensional subspace of $\mathbb{R}^n$ (the plane determined by $\mathbf{x}$ and $\mathbf{y}$) and our usual geometry ideas apply to such planes.

It will be useful to have available some algebraic properties of the inner product. These are summarized in the next theorem [compare with properties (i), (ii), (iii), and (iv) of Section 1.2].

THEOREM 3 For $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{R}^n$ and α, β, real numbers, we have

(i) $(\alpha\mathbf{x} + \beta\mathbf{y}) \cdot \mathbf{z} = \alpha(\mathbf{x} \cdot \mathbf{z}) + \beta(\mathbf{y} \cdot \mathbf{z})$.

(ii) $\mathbf{x} \cdot \mathbf{y} = \mathbf{y} \cdot \mathbf{x}$.

(iii) $\mathbf{x} \cdot \mathbf{x} \geq 0$.

(iv) $\mathbf{x} \cdot \mathbf{x} = 0$ if and only if $\mathbf{x} = \mathbf{0}$.

PROOF Each of the four assertions can be proved by a simple computation. For example, to prove property (i) we write

$$\begin{aligned}(\alpha\mathbf{x} + \beta\mathbf{y}) \cdot \mathbf{z} &= (\alpha x_1 + \beta y_1, \alpha x_2 + \beta y_2, \ldots, \alpha x_n + \beta y_n) \cdot (z_1, z_2, \ldots, z_n)\\ &= (\alpha x_1 + \beta y_1)z_1 + (\alpha x_2 + \beta y_2)z_2 + \cdots + (\alpha x_n + \beta y_n)z_n\\ &= \alpha x_1 z_1 + \beta y_1 z_1 + \alpha x_2 z_2 + \beta y_2 z_2 + \cdots + \alpha x_n z_n + \beta y_n z_n\\ &= \alpha(\mathbf{x} \cdot \mathbf{z}) + \beta(\mathbf{y} \cdot \mathbf{z}).\end{aligned}$$

The other proofs are similar. ■

In Section 1.2, we proved an interesting property of dot products, called the Cauchy–Schwarz inequality.[4] For $\mathbb{R}^2$ our proof required the use of the law of cosines. For $\mathbb{R}^n$ we could also use this method, by confining our attention to a plane in $\mathbb{R}^n$. However, we can also give a direct, completely algebraic proof.

THEOREM 4: Cauchy–Schwarz Inequality in $\mathbb{R}^n$ Let $\mathbf{x}, \mathbf{y}$ be vectors in $\mathbb{R}^n$. Then

$$|\mathbf{x}\cdot\mathbf{y}| \le \|\mathbf{x}\|\,\|\mathbf{y}\|.$$

PROOF Let $a = \mathbf{y}\cdot\mathbf{y}$ and $b = -\mathbf{x}\cdot\mathbf{y}$. If $a = 0$, the theorem is clearly valid, because then $\mathbf{y} = \mathbf{0}$ and both sides of the inequality reduce to 0. Thus, we may suppose $a \neq 0$. By Theorem 3 we have

$$\begin{aligned} 0 \le (a\mathbf{x} + b\mathbf{y})\cdot(a\mathbf{x} + b\mathbf{y}) &= a^2\mathbf{x}\cdot\mathbf{x} + 2ab\mathbf{x}\cdot\mathbf{y} + b^2\mathbf{y}\cdot\mathbf{y} \\ &= (\mathbf{y}\cdot\mathbf{y})^2\mathbf{x}\cdot\mathbf{x} - (\mathbf{y}\cdot\mathbf{y})(\mathbf{x}\cdot\mathbf{y})^2. \end{aligned}$$

Dividing by $\mathbf{y}\cdot\mathbf{y}$ gives $0 \le (\mathbf{y}\cdot\mathbf{y})(\mathbf{x}\cdot\mathbf{x}) - (\mathbf{x}\cdot\mathbf{y})^2$, that is, $(\mathbf{x}\cdot\mathbf{y})^2 \le (\mathbf{x}\cdot\mathbf{x})(\mathbf{y}\cdot\mathbf{y}) = \|\mathbf{x}\|^2\|\mathbf{y}\|^2$. Taking square roots on both sides of this inequality yields the desired result. ■

There is a useful consequence of the Cauchy–Schwarz inequality in terms of lengths. The triangle inequality is geometrically clear in $\mathbb{R}^3$ and was discussed in Section 1.2. The *analytic* proof of the triangle inequality that we gave in Section 1.2 works exactly the same in $\mathbb{R}^n$ and proves the following:

COROLLARY: Triangle Inequality in $\mathbb{R}^n$ Let $\mathbf{x}$, $\mathbf{y}$ be vectors in $\mathbb{R}^n$. Then

$$\|\mathbf{x} + \mathbf{y}\| \le \|\mathbf{x}\| + \|\mathbf{y}\|.$$

If Theorem 4 and its corollary are written out algebraically, they become the following useful inequalities:

$$\left|\sum_{i=1}^{n} x_i y_i\right| \le \left(\sum_{i=1}^{n} x_i^2\right)^{1/2}\left(\sum_{i=1}^{n} y_i^2\right)^{1/2};$$

$$\left(\sum_{i=1}^{n} (x_i + y_i)^2\right)^{1/2} \le \left(\sum_{i=1}^{n} x_i^2\right)^{1/2} + \left(\sum_{i=1}^{n} y_i^2\right)^{1/2}.$$

[4] Sometimes called the Cauchy–Bunyakovskii–Schwarz inequality, or simply the CBS inequality, because it was independently discovered in special cases by the French mathematician Cauchy, the Russian mathematician Bunyakovskii, and the German mathematician Schwarz.

EXAMPLE 1 Let $\mathbf{x} = (1, 2, 0, -1)$ and $\mathbf{y} = (-1, 1, 1, 0)$. Verify Theorem 4 and its corollary in this case.

SOLUTION

$$\begin{aligned}
\|\mathbf{x}\| &= \sqrt{1^2 + 2^2 + 0^2 + (-1)^2} = \sqrt{6} \\
\|\mathbf{y}\| &= \sqrt{(-1)^2 + 1^2 + 1^2 + 0^2} = \sqrt{3} \\
\mathbf{x} \cdot \mathbf{y} &= 1(-1) + 2 \cdot 1 + 0 \cdot 1 + (-1)0 = 1 \\
\mathbf{x} + \mathbf{y} &= (0, 3, 1, -1) \\
\|\mathbf{x} + \mathbf{y}\| &= \sqrt{0^2 + 3^2 + 1^2 + (-1)^2} = \sqrt{11}.
\end{aligned}$$

We compute $\mathbf{x} \cdot \mathbf{y} = 1 \leq 4.24 \approx \sqrt{6}\sqrt{3} = \|\mathbf{x}\| \, \|\mathbf{y}\|$, which verifies Theorem 4. Similarly, we can check its corollary:

$$\begin{aligned}
\|\mathbf{x} + \mathbf{y}\| &= \sqrt{11} \approx 3.32 \\
&\leq 4.18 = 2.45 + 1.73 \approx \sqrt{6} + \sqrt{3} = \|\mathbf{x}\| + \|\mathbf{y}\|. \quad \blacktriangle
\end{aligned}$$

By analogy with $\mathbb{R}^3$, we can define the notion of distance in $\mathbb{R}^n$; namely, if $\mathbf{x}$ and $\mathbf{y}$ are points in $\mathbb{R}^n$, the ***distance between* x *and* y** is defined to be $\|\mathbf{x} - \mathbf{y}\|$, or the length of the vector $\mathbf{x} - \mathbf{y}$. We do not attempt to define the cross product on $\mathbb{R}^n$ except for $n = 3$.

General Matrices

Generalizing 2×2 and 3×3 matrices (see Section 1.3), we can consider $m \times n$ matrices, which are arrays of mn numbers:

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}.$$

We shall also write A as $[a_{ij}]$. We define addition and multiplication by a scalar componentwise, just as we did for vectors. Given two $m \times n$ matrices A and B, we can add them to obtain a new $m \times n$ matrix $C = A + B$, whose ijth entry c_{ij} is the sum of a_{ij} and b_{ij}. It is clear that $A + B = B + A$.

EXAMPLE 2

(a) $\begin{bmatrix} 2 & 1 & 0 \\ 3 & 4 & 1 \end{bmatrix} + \begin{bmatrix} -1 & 1 & 3 \\ 0 & 0 & 7 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 4 & 8 \end{bmatrix}.$

(b) $[1 \quad 2] + [0 \quad -1] = [1 \quad 1]$.

(c) $\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$. ▲

Given a scalar λ and an $m \times n$ matrix A, we can multiply A by λ to obtain a new $m \times n$ matrix $\lambda A = C$, whose ijth entry c_{ij} is the product λa_{ij}.

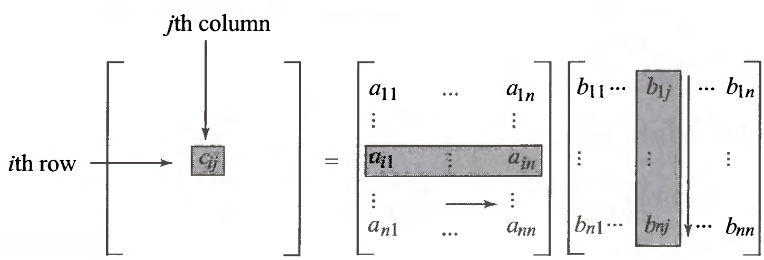
EXAMPLE 3

$$3 \begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & 5 \\ 1 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 3 & -3 & 6 \\ 0 & 3 & 15 \\ 3 & 0 & 9 \end{bmatrix}. \quad ▲$$

Next we turn to matrix multiplication. If $A = [a_{ij}]$, $B = [b_{ij}]$ are $n \times n$ matrices, then the product $AB = C$ has entries given by

$$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj},$$

which is the dot product of the ith row of A and the jth column of B:



EXAMPLE 4 Let

$$A = \begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}.$$

Then

$$AB = \begin{bmatrix} 0 & 4 & 3 \\ 1 & 2 & 0 \\ 0 & 1 & 0 \end{bmatrix} \quad \text{and} \quad BA = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 0 & 3 \\ 3 & 1 & 0 \end{bmatrix}.$$

Observe that $AB \neq BA$. ▲

Similarly, we can multiply an $m \times n$ matrix (m rows, n columns) by an $n \times p$ matrix (n rows, p columns) to obtain an $m \times p$ matrix (m rows, p columns) by the same rule. Note that for AB to be defined, the number of *columns* of A must equal the number of *rows* of B.

EXAMPLE 5 Let

$$A = \begin{bmatrix} 2 & 0 & 1 \\ 1 & 1 & 2 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

Then

$$AB = \begin{bmatrix} 3 & 1 & 5 \\ 3 & 4 & 5 \end{bmatrix},$$

and BA is not defined. ▲

EXAMPLE 6 Let

$$A = \begin{bmatrix} 1 \\ 2 \\ 1 \\ 3 \end{bmatrix} \quad \text{and} \quad B = [2 \quad 2 \quad 1 \quad 2].$$

Then

$$AB = \begin{bmatrix} 2 & 2 & 1 & 2 \\ 4 & 4 & 2 & 4 \\ 2 & 2 & 1 & 2 \\ 6 & 6 & 3 & 6 \end{bmatrix} \quad \text{and} \quad BA = [13]. \quad ▲$$

Any $m \times n$ matrix A determines a mapping of $\mathbb{R}^n$ to $\mathbb{R}^m$ defined as follows: Let $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n$; consider the $n \times 1$ column matrix associated with $\mathbf{x}$, which we shall *temporarily* denote $\mathbf{x}^T$

$$\mathbf{x}^T = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

and multiply A by $\mathbf{x}^T$ (considered to be an $n \times 1$ matrix) to get a new $m \times 1$ matrix:

$$A\mathbf{x}^T = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} y_1 \\ \vdots \\ y_m \end{bmatrix} = \mathbf{y}^T,$$

corresponding to the vector $\mathbf{y} = (y_1, \ldots, y_m)$.[5] Thus, although it may cause some confusion, we will write $\mathbf{x} = (x_1, \ldots, x_n)$ and $\mathbf{y} = (y_1, \ldots, y_m)$ as column matrices

$$\mathbf{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}, \qquad \mathbf{y} = \begin{bmatrix} y_1 \\ \vdots \\ y_m \end{bmatrix}$$

when dealing with matrix multiplication; that is, we will *identify* these two forms of writing vectors. Thus, we will delete the T on $\mathbf{x}^T$ and view $\mathbf{x}^T$ and $\mathbf{x}$ as the same.

Thus, $A\mathbf{x} = \mathbf{y}$ will "really" mean the following: Write $\mathbf{x}$ as a column matrix, multiply it by A, and let $\mathbf{y}$ be the vector whose components are those of the resulting column matrix. The rule $\mathbf{x} \mapsto A\mathbf{x}$ therefore defines a mapping of $\mathbb{R}^n$ to $\mathbb{R}^m$. This mapping is linear; that is, it satisfies

$$A(\mathbf{x} + \mathbf{y}) = A\mathbf{x} + A\mathbf{y}$$
$$A(\alpha\mathbf{x}) = \alpha(A\mathbf{x}), \qquad \alpha \text{ a scalar},$$

as may be easily verified. One learns in a linear algebra course that, conversely, any linear transformation of $\mathbb{R}^n$ to $\mathbb{R}^m$ is representable in this way by an $m \times n$ matrix.

If $A = [a_{ij}]$ is an $m \times n$ matrix and $\mathbf{e}_j$ is the jth standard basis vector of $\mathbb{R}^n$, then $A\mathbf{e}_j$ is a vector in $\mathbb{R}^m$ with components the same as the jth column of A. That is, the ith component of $A\mathbf{e}_j$ is a_{ij}. In symbols, $(A\mathbf{e}_j)_i = a_{ij}$.

EXAMPLE 7 If

$$A = \begin{bmatrix} 1 & 0 & 3 \\ -1 & 0 & 1 \\ 2 & 1 & 2 \\ -1 & 2 & 1 \end{bmatrix},$$

then $\mathbf{x} \mapsto A\mathbf{x}$ of $\mathbb{R}^3$ to $\mathbb{R}^4$ is the mapping defined by

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \mapsto \begin{bmatrix} x_1 + 3x_3 \\ -x_1 + x_3 \\ 2x_1 + x_2 + 2x_3 \\ -x_1 + 2x_2 + x_3 \end{bmatrix}. \quad \blacktriangle$$

[5] To use a matrix A to get a mapping from vectors $\mathbf{x} = (x_1, \ldots, x_n)$ to vectors $\mathbf{y} = (y_1, \ldots, y_n)$ according to the equation $A\mathbf{x}^T = \mathbf{y}^T$, we write the vectors in the column form $\mathbf{x}^T$ instead of the row form $(x_1, \ldots, x_n)$. This sudden switch from writing $\mathbf{x}$ as a row to writing $\mathbf{x}$ as a column is necessitated by standard conventions on matrix multiplication.

EXAMPLE 8 The following illustrates what happens to a specific point when mapped by a 4×3 matrix:

$$A\mathbf{e}_2 = \begin{bmatrix} 4 & 2 & 9 \\ 3 & 5 & 4 \\ 1 & 2 & 3 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \\ 2 \\ 1 \end{bmatrix} = \text{2nd column of } A. \quad \blacktriangle$$

Properties of Matrices

Matrix multiplication is not, in general, ***commutative***: If A and B are $n \times n$ matrices, then generally

$$AB \neq BA,$$

as Examples 4, 5, and 6 show.

An $n \times n$ matrix is said to be ***invertible*** if there is an $n \times n$ matrix B such that

$$AB = BA = I_n,$$

where

$$I_n = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}$$

is the $n \times n$ identity matrix: I_n has the property that $I_n C = C I_n = C$ for any $n \times n$ matrix C. We denote B by A^{-1} and call A^{-1} the ***inverse*** of A. The inverse, when it exists, is unique.

EXAMPLE 9 If

$$A = \begin{bmatrix} 2 & 4 & 0 \\ 0 & 2 & 1 \\ 3 & 0 & 2 \end{bmatrix}, \qquad \text{then} \qquad A^{-1} = \frac{1}{20} \begin{bmatrix} 4 & -8 & 4 \\ 3 & 4 & -2 \\ -6 & 12 & 4 \end{bmatrix},$$

because $AA^{-1} = I_3 = A^{-1}A$, as may be checked by matrix multiplication. ▲

Methods of computing inverses are learned in linear algebra; we won't require these methods in this book. If A is invertible, the equation $A\mathbf{x} = \mathbf{y}$ can be solved for the vector $\mathbf{x}$ by multiplying both sides by A^{-1} to obtain[6] $\mathbf{x} = A^{-1}\mathbf{y}$.

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[6]In fact, Cramer's rule from Section 1.3 provides one way to invert matrices. Numerically more efficient methods based on elimination methods are learned in linear algebra or computer science.

In Section 1.3, we defined the determinant of a 3×3 matrix. This can be generalized by induction to $n \times n$ determinants. We illustrate here how to write the determinant of a 4×4 matrix in terms of the determinants of 3×3 matrices:

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix} = a_{11} \begin{vmatrix} a_{22} & a_{23} & a_{24} \\ a_{32} & a_{33} & a_{34} \\ a_{42} & a_{43} & a_{44} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} & a_{24} \\ a_{31} & a_{33} & a_{34} \\ a_{41} & a_{43} & a_{44} \end{vmatrix}$$

$$+ a_{13} \begin{vmatrix} a_{21} & a_{22} & a_{24} \\ a_{31} & a_{32} & a_{34} \\ a_{41} & a_{42} & a_{44} \end{vmatrix} - a_{14} \begin{vmatrix} a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \\ a_{41} & a_{42} & a_{43} \end{vmatrix}$$

[see formula (2) of Section 1.3; the signs alternate $+, -, +, -$].

The basic properties of 3×3 determinants reviewed in Section 1.3 remain valid for $n \times n$ determinants. In particular, we note the fact that if A is an $n \times n$ matrix and B is the matrix formed by adding a scalar multiple of one row (or column) of A to another row (or, respectively, column) of A, then the determinant of A is equal to the determinant of B (see Example 10).

A basic theorem of linear algebra states that an $n \times n$ matrix A is invertible if and only if the determinant of A is not zero. Another basic property is that the determinant is multiplicative: $\det(AB) = (\det A)(\det B)$. In this text, we shall not make use of many details of linear algebra, and so we shall leave these assertions unproved.

EXAMPLE 10 Let

$$A = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 \\ 2 & 1 & 0 & 1 \\ 1 & 1 & 0 & 2 \end{bmatrix}.$$

Find det A. Does A have an inverse?

SOLUTION Adding $(-1) \times$ first column to the third column, we get

$$\det A = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 \\ 2 & 1 & -2 & 1 \\ 1 & 1 & -1 & 2 \end{vmatrix} = 1 \begin{vmatrix} 1 & 0 & 1 \\ 1 & -2 & 1 \\ 1 & -1 & 2 \end{vmatrix}.$$

Adding $(-1) \times$ first column to the third column of this 3×3 determinant gives

$$\det A = \begin{vmatrix} 1 & 0 & 0 \\ 1 & -2 & 0 \\ 1 & -1 & 1 \end{vmatrix} = \begin{vmatrix} -2 & 0 \\ -1 & 1 \end{vmatrix} = -2.$$

Thus, $\det A = -2 \neq 0$, and so A has an inverse. ▲

If we have three matrices A, B, and C such that the products AB and BC are defined, then the products $(AB)C$ and $A(BC)$ are defined and are in fact equal (that is, matrix multiplication is *associative*). We call this the *triple product* of matrices and denote it by ABC.

EXAMPLE 11 Let

$$A = \begin{bmatrix} 3 \\ 5 \end{bmatrix}, \qquad B = [1 \quad 1], \qquad \text{and} \qquad C = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

Then

$$ABC = A(BC) = \begin{bmatrix} 3 \\ 5 \end{bmatrix} [3] = \begin{bmatrix} 9 \\ 15 \end{bmatrix}. \quad \blacktriangle$$

EXAMPLE 12

$$\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 1 & 0 \end{bmatrix}. \quad \blacktriangle$$

Historical Note

The founder of modern (coordinate) geometry was René Descartes (see Figure 1.5.1), a great physicist, philosopher, and mathematician, as well as a founder of modern biology.

Born in Touraine, France, in 1596, Descartes had a fascinating life. After studying law, he settled in Paris, where he developed an interest in mathematics. In 1628, he moved to Holland, where he wrote his only mathematical work, *La Geometrie*, one of the origins of modern coordinate geometry.

Descartes had been highly critical of the geometry of the ancient Greeks, with all their undefined terms and with their proofs requiring ever newer and more ingenious approaches. For Descartes, this geometry was so tied to geometrical figures "that it can exercise the understanding only on condition of greatly fatiguing the imagination." He undertook to exploit, in geometry, the use of algebra, which had recently been developed. The result was *La Geometrie*, which made possible analytic or computational methods in geometry.

Remember that the Greeks were, like Descartes, philosophers as well as mathematicians and physicists. Their answer to the question of the meaning of space was "Euclidean geometry." Descartes had therefore succeeded in "algebrizing"the Greek model of space.

Gottfried Wilhelm Leibniz, cofounder (with Isaac Newton) of calculus, was also interested in "space analysis," but he did not think that Descartes'

Figure 1.5.1 René Descartes (1596–1650).

algebra went far enough. Leibniz called for a direct method of space analysis (*analysis situs*) that could be interpreted as a call for the development of vector analysis.

On September 8, 1679, Leibniz outlined his ideas in a letter to Christian Huygens:

> I am still not satisfied with algebra, because it does not give the shortest methods or the most beautiful constructions in geometry. This is why I believe that, so far as geometry is concerned, we need still another analysis which is distinctly geometrical or linear and which will express situation (situs) directly as algebra expresses magnitude directly. And I believe that I have found the way and that we can represent figures and even machines and movements by characters, as algebra represents numbers or magnitudes. I am sending you an essay which seems to me to be important.

In the essay, Leibniz described his ideas in greater detail:

> I have discovered certain elements of a new characteristic which is entirely different from algebra and which will have great advantages in representing to the mind, exactly and in a way faithful to its nature, even without figures, everything which depends on sense perception. Algebra is the characteristic for undetermined numbers of magnitudes only, but it does not express situation, angles, and

motion directly. Hence it is often difficult to analyze the properties of a figure by calculation, and still more difficult to find very convenient geometrical demonstrations and constructions, even when the algebraic calculation is completed. But this new characteristic, which follows the visual figures, cannot fail to give the solution, the construction, and the geometric demonstration all at the same time, and in a natural way and in one analysis, that is, through determined procedure.

Leibniz's ideas influenced Hamilton and others. In the middle of the nineteenth century, Bolyai and Lobachevsky developed their "non-Euclidean" geometry, and Gauss studied and developed a theory of curved surfaces in three-dimensional space. Gauss developed two measures of curvature, the *mean curvature* and the *Gauss curvature*. For example, soap bubbles and soap films have constant mean curvature, whereas only soap bubbles have constant Gauss curvature. We discuss these ideas further in Section 7.7.

Bernhard Riemann (see Figure 1.5.2), possibly the greatest mathematical genius of all time, gave an inaugural address in 1854 before the faculty of Göttingen University, entitled "On the Hypotheses Which Lie at the Foundation of Geometry." It was this monumental work that would lay the foundation, 50 years later, of Einstein's general theory of relativity. Riemann, like Leibniz and the early Greeks, was interested in space, especially its metric (or distance) properties.

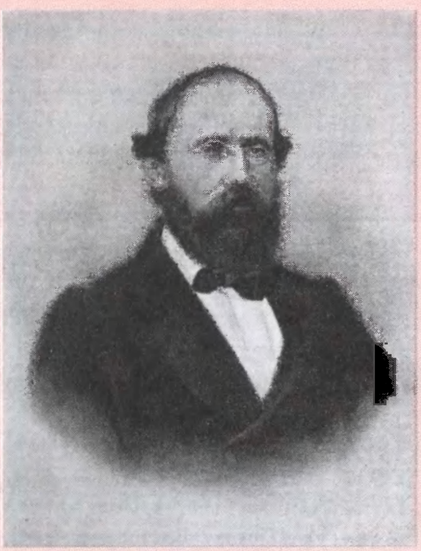

Figure 1.5.2 Bernhard Riemann (1826–1866).

Riemann called for the study of n-dimensional spaces and surfaces. He showed how to measure the curvature of three-, four-, ..., n-dimensional surfaces and (incredibly) showed that in order to be called "curved," a surface need not be "curving" inside anything else; curvature was simply a consequence of the intrinsic "metric properties of space." Once Riemann demonstrated that mathematical models permitted us to think of spaces of any dimension, the question naturally arose as to why our space is three-dimensional and not four-, five-, or more dimensional. Surprisingly, no one has yet put forth a convincing explanation why, at the moment of creation, space became three-dimensional.

Around 1910, Albert Einstein realized that gravity could be explained as a consequence of the curvature of a four-dimensional space–time (matter and energy curve space and time), and, thanks to Riemann, Einstein's space–time need not be enclosed in an ambient universe. Exactly how matter and energy curve space–time is the essence of Einstein's field equations in general relativity. In Section 7.7, we will discuss the ideas of curvature in greater depth and will indicate some of the ideas behind general relativity. The idea of n-dimensions also began to creep into mathematics from another, very different direction—from matrices.

The definition of a matrix, as an isolated abstract object, is due to the English mathematician Arthur Cayley. Cayley was born in 1821, and in 1863 was appointed Sedlesian Professor of Mathematics at Cambridge University. Around 1855, one year after Riemann's inaugural address, Cayley, in an effort to simplify notation for his study of linear equations (as we saw in Section 1.5), introduced the abstract idea of a matrix of m columns and n rows. Naturally, a $1 \times n$ matrix could also be viewed as a vector in an "n-dimensional space."

After this concept took hold, mathematicians realized that they lost little in working in general dimensions, and the subject of modern linear algebra was off and running. Again, physics was to be a major impetus. Modern, abstract, linear algebra, including abstract vector spaces, began to turn up in textbooks after the appearance in 1918 of Hermann Weyl's *Space–Time–Matter*.

EXERCISES

1. Calculate the dot product of $\mathbf{x} = (1, -1, 0, 2) \in \mathbb{R}^4$ and $\mathbf{y} = (1, 2, 3, 4) \in \mathbb{R}^4$.

2. In $\mathbb{R}^n$ show that

   (a) $2\|\mathbf{x}\|^2 + 2\|\mathbf{y}\|^2 = \|\mathbf{x}+\mathbf{y}\|^2 + \|\mathbf{x}-\mathbf{y}\|^2$ (This is known as the ***parallelogram law***.)
   (b) $\|\mathbf{x}-\mathbf{y}\|\|\mathbf{x}+\mathbf{y}\| \le \|\mathbf{x}\|^2 + \|\mathbf{y}\|^2$
   (c) $4\langle\mathbf{x}, \mathbf{y}\rangle = \|\mathbf{x}+\mathbf{y}\|^2 - \|\mathbf{x}-\mathbf{y}\|^2$ (This is called the ***polarization identity***.)

Interpret these results geometrically in terms of the parallelogram formed by $\mathbf{x}$ and $\mathbf{y}$.

Verify the Cauchy–Schwarz inequality and the triangle inequality for the vectors in Exercises 3 to 6.

3. $\mathbf{x} = (2, 0, -1)$, $\mathbf{y} = (4, 0, -2)$

4. $\mathbf{x} = (1, 0, 2, 6)$, $\mathbf{y} = (3, 8, 4, 1)$

5. $\mathbf{x} = (1, -1, 1, -1, 1)$, $\mathbf{y} = (3, 0, 0, 0, 2)$

6. $\mathbf{x} = (1, 0, 0, 1)$, $\mathbf{y} = (-1, 0, 0, 1)$

7. Compute AB, $\det A$, $\det B$, $\det (AB)$, and $\det (A + B)$ for

$$A = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 3 & 2 \\ 3 & 1 & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} -2 & 0 & 2 \\ -1 & 1 & -1 \\ 1 & 4 & 3 \end{bmatrix}.$$

8. Compute AB, $\det A$, $\det B$, $\det (AB)$, and $\det (A + B)$ for

$$A = \begin{bmatrix} 3 & 0 & 1 \\ 1 & 2 & -1 \\ 1 & 0 & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$

9. Use induction on k to prove that if $\mathbf{x}_1, \dots, \mathbf{x}_k \in \mathbb{R}^n$, then

$$\|\mathbf{x}_1 + \cdots + \mathbf{x}_k\| \le \|\mathbf{x}_1\| + \cdots + \|\mathbf{x}_k\|.$$

10. Prove using algebra, the ***identity of Lagrange***: For real numbers $x_1, \dots, x_n$ and $y_1, \dots, y_n$.

$$\left(\sum_{i=1}^{n} x_i y_i\right)^2 = \left(\sum_{i=1}^{n} x_i^2\right)\left(\sum_{i=1}^{n} y_i^2\right) - \sum_{i<j} (x_i y_j - x_j y_i)^2.$$

Use this to give another proof of the Cauchy–Schwarz inequality in $\mathbb{R}^n$,

11. Prove that if A is an $n \times n$ matrix, then

(a) $\det (\lambda A) = \lambda^n \det A$; and
(b) if B is a matrix obtained from A by multiplying any row or column by a scalar λ, then $\det B = \lambda \det A$.

In Exercises 12 to 14, A, B, and C denote $n \times n$ matrices.

12. Is $\det (A + B) = \det A + \det B$? Give a proof or counterexample.

13. Does $(A + B)(A - B) = A^2 - B^2$?

14. Assuming the law $\det (AB) = (\det A)(\det B)$, prove that $\det (ABC) = (\det A)(\det B)(\det C)$.

15. (This exercise assumes a knowledge of integration of continuous functions of one variable.) Note that the proof of the Cauchy–Schwarz inequality (Theorem 4) depends only on the properties of the inner product listed in Theorem 1. Use this observation to establish the following inequality for continuous functions $f, g\colon [0, 1] \to \mathbb{R}$:

$$\left| \int_0^1 f(x)g(x)\,dx \right| \leq \sqrt{\int_0^1 [f(x)]^2\,dx}\sqrt{\int_0^1 [g(x)]^2\,dx}.$$

Do this by

(a) verifying that the space of continuous functions from [0, 1] to $\mathbb{R}$ forms a vector space; that is, we may think of functions f, g abstractly as "vectors" that can be added to each other and multiplied by scalars.

(b) introducing the inner product of functions

$$f \cdot g = \int_0^1 f(x)g(x)\,dx$$

and verifying that it satisfies conditions (i) to (iv) of Theorem 3.

16. Define the transpose A^T of an $n \times n$ matrix A as follows: the ijth element of A^T is a_{ji} where a_{ij} is the ijth entry of A. Show that A^T is characterized by the following property: For all $\mathbf{x}$, $\mathbf{y}$ in $\mathbb{R}^n$,

$$(A^T\mathbf{x}) \cdot \mathbf{y} = \mathbf{x} \cdot (A\mathbf{y}).$$

17. Verify that the inverse of

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \text{is} \quad \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

18. Use your answer in Exercise 17 to show that the solution of the system

$$ax + by = e$$
$$cx + dy = f$$

is

$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}\begin{bmatrix} e \\ f \end{bmatrix}.$$

19. Assuming the law $\det(AB) = (\det A)(\det B)$, verify that $(\det A)(\det A^{-1}) = 1$ and conclude that if A has an inverse, then $\det A \neq 0$.

REVIEW EXERCISES FOR CHAPTER 1

1. Let $\mathbf{v} = 3\mathbf{i} + 4\mathbf{j} + 5\mathbf{k}$ and $\mathbf{w} = \mathbf{i} - \mathbf{j} + \mathbf{k}$. Compute $\mathbf{v} + \mathbf{w}$, $3\mathbf{v}$, $6\mathbf{v} + 8\mathbf{w}$, $-2\mathbf{v}$, $\mathbf{v} \cdot \mathbf{w}$, $\mathbf{v} \times \mathbf{w}$. Interpret each operation geometrically by graphing the vectors.

2. Repeat Exercise 1 with $\mathbf{v} = 2\mathbf{j} + \mathbf{k}$ and $\mathbf{w} = -\mathbf{i} - \mathbf{k}$.

3. (a) Find the equation of the line through $(-1, 2, -1)$ in the direction of $\mathbf{j}$.
(b) Find the equation of the line passing through $(0, 2, -1)$ and $(-3, 1, 0)$.
(c) Find the equation for the plane perpendicular to the vector $(-2, 1, 2)$ and passing through the point $(-1, 1, 3)$.

4. (a) Find the equation of the line through $(0, 1, 0)$ in the direction of $3\mathbf{i} + \mathbf{k}$.
(b) Find the equation of the line passing through $(0, 1, 1)$ and $(0, 1, 0)$.
(c) Find an equation for the plane perpendicular to the vector $(-1, 1, -1)$ and passing through the point $(1, 1, 1)$.

5. Compute $\mathbf{v} \cdot \mathbf{w}$ for the following sets of vectors:

(a) $\mathbf{v} = -\mathbf{i} + \mathbf{j}; \mathbf{w} = \mathbf{k}$.
(b) $\mathbf{v} = \mathbf{i} + 2\mathbf{j} - \mathbf{k}; \mathbf{w} = 3\mathbf{i} + \mathbf{j}$.
(c) $\mathbf{v} = -2\mathbf{i} - \mathbf{j} + \mathbf{k}; \mathbf{w} = 3\mathbf{i} + 2\mathbf{j} - 2\mathbf{k}$.

6. Compute $\mathbf{v} \times \mathbf{w}$ for the vectors in Exercise 5. [Only part (b) is solved in the Study Guide.]

7. Find the cosine of the angle between the vectors in Exercise 5. [Only part (b) is solved in the Study Guide.]

8. Find the area of the parallelogram spanned by the vectors in Exercise 5. [Only part (b) is solved in the Study Guide.]

9. Use vector notation to describe the triangle in space whose vertices are the origin and the endpoints of vectors $\mathbf{a}$ and $\mathbf{b}$.

10. Show that three vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$ lie in the same plane through the origin if and only if there are three scalars α, β, γ, not all zero, such that $\alpha\mathbf{a} + \beta\mathbf{b} + \gamma\mathbf{c} = \mathbf{0}$.

11. For real numbers $a_1, a_2, a_3, b_1, b_2, b_3$, show that

$$(a_1 b_1 + a_2 b_2 + a_3 b_3)^2 \leq (a_1^2 + a_2^2 + a_3^2)(b_1^2 + b_2^2 + b_3^2).$$

12. Let $\mathbf{u}, \mathbf{v}, \mathbf{w}$ be unit vectors that are orthogonal to each other. If $\mathbf{a} = \alpha\mathbf{u} + \beta\mathbf{v} + \gamma\mathbf{w}$, show that

$$\alpha = \mathbf{a} \cdot \mathbf{u}, \qquad \beta = \mathbf{a} \cdot \mathbf{v}, \qquad \gamma = \mathbf{a} \cdot \mathbf{w}.$$

Interpret the results geometrically.

13. Let $\mathbf{a}, \mathbf{b}$ be two vectors in the plane, $\mathbf{a} = (a_1, a_2)$. $\mathbf{b} = (b_1, b_2)$, and let λ be a real number. Show that the area of the parallelogram determined by $\mathbf{a}$ and $\mathbf{b} + \lambda\mathbf{a}$ is the same as that determined by $\mathbf{a}$ and $\mathbf{b}$. Sketch. Relate this result to a known property of determinants.

14. Find the volume of the parallelepiped determined by the vertices $(0, 1, 0)$, $(1, 1, 1)$, $(0, 2, 0)$, $(3, 1, 2)$.

15. Given nonzero vectors $\mathbf{a}$ and $\mathbf{b}$ in $\mathbb{R}^3$, show that the vector $\mathbf{v} = \|\mathbf{a}\|\mathbf{b} + \|\mathbf{b}\|\mathbf{a}$ bisects the angle between $\mathbf{a}$ and $\mathbf{b}$.

16. Use vector methods to prove that the distance from the point (x_1, y_1) to the line $ax + by = c$ is

$$\frac{|ax_1 + by_1 - c|}{\sqrt{a^2 + b^2}}.$$

17. Verify that the direction of $\mathbf{b} \times \mathbf{c}$ is given by the right-hand rule, by choosing $\mathbf{b}, \mathbf{c}$ to be two of the vectors $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$.

18. (a) Suppose $\mathbf{a} \cdot \mathbf{b} = \mathbf{a}' \cdot \mathbf{b}$ for all $\mathbf{b}$. Show that $\mathbf{a} = \mathbf{a}'$.
(b) Suppose $\mathbf{a} \times \mathbf{b} = \mathbf{a}' \times \mathbf{b}$ for all $\mathbf{b}$. Is it true that $\mathbf{a} = \mathbf{a}'$?

19. (a) Using vector methods, show that the distance between two nonparallel lines l_1 and l_2 is given by

$$d = \frac{|(\mathbf{v}_2 - \mathbf{v}_1) \cdot (\mathbf{a}_1 \times \mathbf{a}_2)|}{\|\mathbf{a}_1 \times \mathbf{a}_2\|},$$

where $\mathbf{v}_1, \mathbf{v}_2$ are any two points on l_1 and l_2, respectively, and $\mathbf{a}_1$ and $\mathbf{a}_2$ are the directions of l_1 and l_2. [HINT: Consider the plane through l_2 that is parallel to l_1. Show that the vector $(\mathbf{a}_1 \times \mathbf{a}_2)/\|\mathbf{a}_1 \times \mathbf{a}_2\|$ is a unit normal for this plane; now project $\mathbf{v}_2 - \mathbf{v}_1$ onto this normal direction.]

(b) Find the distance between the line l_1 determined by the points $(-1, -1, 1)$ and $(0, 0, 0)$ and the line l_2 determined by the points $(0, -2, 0)$ and $(2, 0, 5)$.

20. Show that two planes given by the equations $Ax + By + Cz + D_1 = 0$ and $Ax + By + Cz + D_2 = 0$ are parallel, and that the distance between them is

$$\frac{|D_1 - D_2|}{\sqrt{A^2 + B^2 + C^2}}.$$

21. (a) Prove that the area of the triangle in the plane with vertices (x_1, y_1), (x_2, y_2), (x_3, y_3) is the absolute value of

$$\frac{1}{2}\begin{vmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix}.$$

(b) Find the area of the triangle with vertices $(1, 2)$, $(0, 1)$, $(-1, 1)$.

22. Convert the following points from Cartesian to cylindrical and spherical coordinates and plot:

(a) $(0, 3, 4)$
(b) $(-\sqrt{2}, 1, 0)$
(c) $(0, 0, 0)$
(d) $(-1, 0, 1)$
(e) $(-2\sqrt{3}, -2, 3)$

23. Convert the following points from cylindrical to Cartesian and spherical coordinates and plot:

(a) $(1, \pi/4, 1)$
(b) $(3, \pi/6, -4)$

(c) $(0, \pi/4, 1)$
(d) $(2, -\pi/2, 1)$
(e) $(-2, -\pi/2, 1)$

24. Convert the following points from spherical to Cartesian and cylindrical coordinates and plot:

(a) $(1, \pi/2, \pi)$
(b) $(2, -\pi/2, \pi/6)$
(c) $(0, \pi/8, \pi/35)$
(d) $(2, -\pi/2, -\pi)$
(e) $(-1, \pi, \pi/6)$

25. Rewrite the equation $z = x^2 - y^2$ using cylindrical and spherical coordinates.

26. Using spherical coordinates, show that

$$\phi = \cos^{-1}\left(\frac{\mathbf{u}\cdot\mathbf{k}}{\|\mathbf{u}\|}\right)$$

where $\mathbf{u} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$. Interpret geometrically.

27. Verify the Cauchy–Schwarz and triangle inequalities for

$$\mathbf{x} = (3, 2, 1, 0) \quad \text{and} \quad \mathbf{y} = (1, 1, 1, 2).$$

28. Multiply the matrices

$$A = \begin{bmatrix} 3 & 0 & 1 \\ 2 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

Does $AB = BA$?

29. (a) Show that for two $n \times n$ matrices A and B, and $\mathbf{x} \in \mathbb{R}^n$,

$$(AB)\mathbf{x} = A(B\mathbf{x}).$$

(b) What does the equality in part (a) imply about the relationship between the composition of the mappings $\mathbf{x} \mapsto B\mathbf{x}$, $\mathbf{y} \mapsto A\mathbf{y}$, and matrix multiplication?

30. Find the volume of the parallelepiped spanned by the vectors

$$(1, 0, 1), \quad (1, 1, 1), \quad \text{and} \quad (-3, 2, 0).$$

31. (For students with some knowledge of linear algebra.) Verify that a linear mapping T of $\mathbb{R}^n$ to $\mathbb{R}^n$ is determined by an $n \times n$ matrix.

32. Find an equation for the plane that contains $(3, -1, 2)$ and the line with equation $\mathbf{v} = (2, -1, 0) + t(2, 3, 0)$.

33. The work W done in moving an object from $(0, 0)$ to $(7, 2)$ subject to a constant force $\mathbf{F}$ is $W = \mathbf{F}\cdot\mathbf{r}$, where $\mathbf{r}$ is the vector with its head at $(7, 2)$ and tail at $(0, 0)$. The units are feet and pounds.

(a) Suppose the force $\mathbf{F} = 10 \cos \theta \mathbf{i} + 10 \sin \theta \mathbf{j}$. Find W in terms of θ.

(b) Suppose the force $\mathbf{F}$ has magnitude of 6 lb and makes an angle of $\pi/6$ rad with the horizontal, pointing right. Find W in foot-pounds.

34. If a particle with mass m moves with velocity $\mathbf{v}$, its *momentum* is $\mathbf{p} = m\mathbf{v}$. In a game of marbles, a marble with mass 2 grams (g) is shot with velocity 2 meters per second (m/s), hits two marbles with mass 1 g each, and comes to a dead halt. One of the marbles flies off with a velocity of 3 m/s at an angle of 45° to the incident direction of the larger marble as in Figure 1.R.1. Assuming that the total momentum before and after the collision is the same (according to the law of conservation of momentum), at what angle and speed does the second marble move?

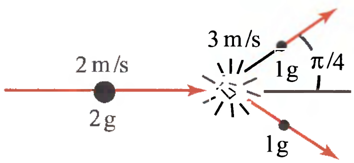

Figure 1.R.1 Momentum and marbles.

35. Show that for all x, y, z,

$$\begin{vmatrix} x+2 & y & z \\ z & y+1 & 10 \\ 5 & 5 & 2 \end{vmatrix} = -\begin{vmatrix} y & x+2 & z \\ 1 & z-x-2 & 10-z \\ 5 & 5 & 2 \end{vmatrix}.$$

36. Show that

$$\begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix} \neq 0$$

if x, y, and z are all different.

37. Show that

$$\begin{vmatrix} 66 & 628 & 246 \\ 88 & 435 & 24 \\ 2 & -1 & 1 \end{vmatrix} = \begin{vmatrix} 68 & 627 & 247 \\ 86 & 436 & 23 \\ 2 & -1 & 1 \end{vmatrix}.$$

38. Show that

$$\begin{vmatrix} n & n+1 & n+2 \\ n+3 & n+4 & n+5 \\ n+6 & n+7 & n+8 \end{vmatrix}$$

has the same value no matter what n is. What is this value?

39. The volume of a tetrahedron with concurrent edges **a**, **b**, **c** is given by $V = \frac{1}{6}\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$.

(a) Express the volume as a determinant.
(b) Evaluate V when $\mathbf{a} = \mathbf{i} + \mathbf{j} + \mathbf{k}$, $\mathbf{b} = \mathbf{i} - \mathbf{j} + \mathbf{k}$, $\mathbf{c} = \mathbf{i} + \mathbf{j}$.

*Use the following definition for Exercises 40 and 41: Let $\mathbf{r}_1, \dots, \mathbf{r}_n$ be vectors in $\mathbb{R}^3$ from 0 to the masses $m_1, \dots, m_n$. The **center of mass** is the vector*

$$\mathbf{c} = \frac{\sum_{i=1}^{n} m_i \mathbf{r}_i}{\sum_{i=1}^{n} m_i}.$$

40. A tetrahedron sits in xyz coordinates with one vertex at (0, 0, 0), and the three edges concurrent at (0, 0, 0) are coincident with the vectors **a**, **b**, **c**.

(a) Draw a figure and label the heads of the vectors **a**, **b**, **c**.
(b) Find the center of mass of each of the four triangular faces of the tetrahedron if a unit mass is placed at each vertex.

41. Show that for any vector **r**, the center of mass of a system satisfies

$$\sum_{i=1}^{n} m_i \|\mathbf{r} - \mathbf{r}_i\|^2 = \sum_{i=1}^{n} m_i \|\mathbf{r}_i - \mathbf{c}\|^2 + m\|\mathbf{r} - \mathbf{c}\|^2,$$

where $m = \sum_{i=1}^{n} m_i$ is the total mass of the system.

In Exercises 42 to 47, find a unit vector that has the given property.

42. Parallel to the line $x = 3t + 1, y = 16t - 2, z = -(t + 2)$.

43. Orthogonal to the plane $x - 6y + z = 12$.

44. Parallel to both the planes $8x + y + z = 1$ and $x - y - z = 0$.

45. Orthogonal to $\mathbf{i} + 2\mathbf{j} - \mathbf{k}$ and to $\mathbf{k}$.

46. Orthogonal to the line $x = 2t - 1, y = -t - 1, z = t + 2$, and the vector $\mathbf{i} - \mathbf{j}$.

47. At an angle of 30° to **i** and making equal angles with **j** and **k**.