

Chapter IV

Complex Integration

In this chapter results are derived which are fundamental in the study of analytic functions. The theorems presented here constitute one of the pillars of Mathematics and have far ranging applications.

§1. Riemann-Stieltjes integrals

We will begin by defining the Riemann-Stieltjes integral in order to define the integral of a function along a path in \mathbb{C} . The discussion of this integral is by no means complete, but is limited to those results essential to a cogent exposition of line integrals.

1.1 Definition. A function $\gamma: [a, b] \rightarrow \mathbb{C}$, for $[a, b] \subset \mathbb{R}$, is of *bounded variation* if there is a constant $M > 0$ such that for any partition $P = \{a = t_0 < t_1 < \dots < t_m = b\}$ of $[a, b]$

$$v(\gamma; P) = \sum_{k=1}^m |\gamma(t_k) - \gamma(t_{k-1})| \leq M.$$

The *total variation* of γ , $V(\gamma)$, is defined by

$$V(\gamma) = \sup \{v(\gamma; P) : P \text{ a partition of } [a, b]\}.$$

Clearly $V(\gamma) \leq M < \infty$.

It is easily shown that γ is of bounded variation if and only if $\operatorname{Re} \gamma$ and $\operatorname{Im} \gamma$ are of bounded variation. If γ is real valued and is non-decreasing then γ is of bounded variation and $V(\gamma) = \gamma(b) - \gamma(a)$. (Exercise 1) Other examples will be given, but first let us give some easily deduced properties of these functions.

1.2 Proposition. Let $\gamma: [a, b] \rightarrow \mathbb{C}$ be of bounded variation. Then:

- (a) If P and Q are partitions of $[a, b]$ and $P \subset Q$ then $v(\gamma; P) \leq v(\gamma; Q)$;
- (b) If $\sigma: [a, b] \rightarrow \mathbb{C}$ is also of bounded variation and $\alpha, \beta \in \mathbb{C}$ then $\alpha\gamma + \beta\sigma$ is of bounded variation and $V(\alpha\gamma + \beta\sigma) \leq |\alpha| V(\gamma) + |\beta| V(\sigma)$.

The proof is left to the reader.

The next proposition gives a wealthy collection of functions of bounded variation. In actuality this is the set of functions which is of principal concern to us.

1.3 Proposition. If $\gamma: [a, b] \rightarrow \mathbb{C}$ is piecewise smooth then γ is of bounded variation and

$$V(\gamma) = \int_a^b |\gamma'(t)| dt$$

Proof. Assume that γ is smooth (the complete proof is easily deduced from this). Recall that when we say that γ is smooth this means γ' is continuous.

Let $P = \{a = t_0 < t_1 < \dots < t_m = b\}$. Then, from the definition,

$$\begin{aligned} v(\gamma; P) &= \sum_{k=1}^m |\gamma(t_k) - \gamma(t_{k-1})| \\ &= \sum_{k=1}^m \left| \int_{t_{k-1}}^{t_k} \gamma'(t) dt \right| \\ &\leq \sum_{k=1}^m \int_{t_{k-1}}^{t_k} |\gamma'(t)| dt \\ &= \int_a^b |\gamma'(t)| dt \end{aligned}$$

Hence $V(\gamma) \leq \int_a^b |\gamma'(t)| dt$, so that γ is of bounded variation.

Since γ' is continuous it is uniformly continuous; so if $\epsilon > 0$ is given we can choose $\delta_1 > 0$ such that $|s - t| < \delta_1$ implies that $|\gamma'(s) - \gamma'(t)| < \epsilon$. Also, we may choose $\delta_2 > 0$ such that if $P = \{a = t_0 < t_1 < \dots < t_m = b\}$ and $\|P\| = \max \{(t_k - t_{k-1}) : 1 \leq k \leq m\} < \delta_2$ then

$$\left| \int_a^b |\gamma'(t)| dt - \sum_{k=1}^m |\gamma'(\tau_k)| (t_k - t_{k-1}) \right| < \epsilon$$

where τ_k is any point in $[t_{k-1}, t_k]$. Hence

$$\begin{aligned} \int_a^b |\gamma'(t)| dt &\leq \epsilon + \sum_{k=1}^m |\gamma'(\tau_k)| (t_k - t_{k-1}) \\ &= \epsilon + \sum_{k=1}^m \left| \int_{t_{k-1}}^{t_k} \gamma'(\tau_k) dt \right| \\ &\leq \epsilon + \sum_{k=1}^m \left| \int_{t_{k-1}}^{t_k} [\gamma'(\tau_k) - \gamma'(t)] dt \right| + \sum_{k=1}^m \left| \int_{t_{k-1}}^{t_k} \gamma'(t) dt \right| \end{aligned}$$

If $\|P\| < \delta = \min(\delta_1, \delta_2)$ then $|\gamma'(\tau_k) - \gamma'(t)| < \epsilon$ for t in $[t_{k-1}, t_k]$ and

$$\begin{aligned} \int_a^b |\gamma'(t)| dt &\leq \epsilon + \epsilon(b-a) + \sum_{k=1}^m |\gamma(t_k) - \gamma(t_{k-1})| \\ &= \epsilon[1 + (b-a)] + v(\gamma; P) \\ &\leq \epsilon[1 + (b-a)] + V(\gamma). \end{aligned}$$

Letting $\epsilon \rightarrow 0+$, gives

$$\int_a^b |\gamma'(t)| dt \leq V(\gamma),$$

which yields equality. ■

1.4 Theorem. Let $\gamma: [a, b] \rightarrow \mathbb{C}$ be of bounded variation and suppose that $f: [a, b] \rightarrow \mathbb{C}$ is continuous. Then there is a complex number I such that for every $\epsilon > 0$ there is a $\delta > 0$ such that when $P = \{t_0 < t_1 < \dots < t_m\}$ is a partition of $[a, b]$ with $\|P\| = \max \{(t_k - t_{k-1}): 1 \leq k \leq m\} < \delta$ then

$$\left| I - \sum_{k=1}^m f(\tau_k) [\gamma(t_k) - \gamma(t_{k-1})] \right| < \epsilon$$

for whatever choice of points $\tau_k, t_{k-1} \leq \tau_k \leq t_k$.

This number I is called the *integral of f with respect to γ* over $[a, b]$ and is designated by

$$I = \int_a^b f d\gamma = \int_a^b f(t) d\gamma(t).$$

Proof. Since f is continuous it is uniformly continuous; thus, we can find (inductively) positive numbers $\delta_1 > \delta_2 > \delta_3 > \dots$ such that if $|s - t| < \delta_m$, $|f(s) - f(t)| < \frac{1}{m}$. For each $m \geq 1$ let \mathcal{P}_m be the collection of all partitions P of $[a, b]$ with $\|P\| < \delta_m$; so $\mathcal{P}_1 \supset \mathcal{P}_2 \supset \dots$. Finally define F_m to be the closure of the set:

$$1.5 \quad \left\{ \sum_{k=1}^n f(\tau_k) [\gamma(t_k) - \gamma(t_{k-1})] : P \in \mathcal{P}_m \text{ and } t_{k-1} \leq \tau_k \leq t_k \right\}.$$

The following are claimed to hold:

$$1.6 \quad \begin{cases} F_1 \supset F_2 \supset \dots \text{ and} \\ \text{diam } F_m \leq \frac{2}{m} V(\gamma) \end{cases}$$

If this is done then, by Cantor's Theorem (II. 3.7), there is exactly one complex number I such that $I \in F_m$ for every $m \geq 1$. Let us show that this will complete the proof. If $\epsilon > 0$ let $m > (2/\epsilon) V(\gamma)$; then $\epsilon > (2/m) V(\gamma) \geq \text{diam } F_m$. Since $I \in F_m$, $F_m \subset B(I; \epsilon)$. Thus, if $\delta = \delta_m$ the theorem is proved.

Now to prove (1.6). The fact that $F_1 \supset F_2 \supset \dots$ follows trivially from the fact that $\mathcal{P}_1 \supset \mathcal{P}_2 \supset \dots$. To show that $\text{diam } F_m \leq \frac{2}{m} V(\gamma)$ it suffices

to show that the diameter of the set in (1.5) is $\leq \frac{2}{m} V(\gamma)$. This is done in two stages, each of which is easy although the first is tedious.

If $P = \{t_0, \dots, t_n\}$ is a partition we will denote by $S(P)$ a sum of the form $\sum f(\tau_k) [\gamma(t_k) - \gamma(t_{k-1})]$ where τ_k is any point with $t_{k-1} \leq \tau_k \leq t_k$.

Fix $m \geq 1$ and let $P \in \mathcal{P}_m$; the first step will be to show that if $P \subset Q$ (and hence $Q \in \mathcal{P}_m$) then $|S(P) - S(Q)| < \frac{1}{m} V(\gamma)$. We only give the proof for the case where Q is obtained by adding one extra partition point to P . Let $1 \leq p \leq m$ and let $t_{p-1} < t^* < t_p$; suppose that $P \cup \{t^*\} = Q$. If $t_{p-1} \leq \sigma \leq t^*$, $t^* \leq \sigma' \leq t_p$, and

$$\begin{aligned} S(Q) = & \sum_{k \neq p} f(\sigma_k) [\gamma(t_k) - \gamma(t_{k-1})] + f(\sigma) [\gamma(t^*) - \gamma(t_{p-1})] \\ & + f(\sigma') [\gamma(t_p) - \gamma(t^*)], \end{aligned}$$

then, using the fact that $|f(\tau) - f(\sigma)| < \frac{1}{m}$ for $|\tau - \sigma| < \delta_m$,

$$\begin{aligned} |S(P) - S(Q)| &= \left| \sum_{k \neq p} [f(\tau_k) - f(\sigma_k)] [\gamma(t_k) - \gamma(t_{k-1})] + f(\tau_p) [\gamma(t_p) - \gamma(t_{p-1})] \right. \\ &\quad \left. - f(\sigma) [\gamma(t^*) - \gamma(t_{p-1})] - f(\sigma') [\gamma(t_p) - \gamma(t^*)] \right| \\ &\leq \frac{1}{m} \sum_{k \neq p} |\gamma(t_k) - \gamma(t_{k-1})| + |[f(\tau_p) - f(\sigma)] [\gamma(t^*) - \gamma(t_{p-1})]| \\ &\quad + |[f(\tau_p) - f(\sigma')] [\gamma(t_p) - \gamma(t^*)]| \\ &\leq \frac{1}{m} \sum_{k \neq p} |\gamma(t_k) - \gamma(t_{k-1})| + \frac{1}{m} |\gamma(t^*) - \gamma(t_{p-1})| \\ &\quad + \frac{1}{m} |\gamma(t_p) - \gamma(t^*)| \\ &\leq \frac{1}{m} V(\gamma) \end{aligned}$$

For the second and final stage let P and R be any two partitions in \mathcal{P}_m . Then $Q = P \cup R$ is a partition and contains both P and R . Using the first part we get

$$|S(P) - S(R)| \leq |S(P) - S(Q)| + |S(Q) - S(R)| \leq \frac{2}{m} V(\gamma).$$

It now follows that the diameter of (1.5) is $\leq \frac{2}{m} V(\gamma)$. ■

The next result follows from the definitions by a routine $\epsilon - \delta$ argument.

1.7 Proposition. Let f and g be continuous functions on $[a, b]$ and let γ and σ be functions of bounded variation on $[a, b]$. Then for any scalars α and β :

$$\begin{aligned} \text{(a)} \quad \int_a^b (\alpha f + \beta g) d\gamma &= \alpha \int_a^b f d\gamma + \beta \int_a^b g d\gamma \\ \text{(b)} \quad \int_a^b f d(\alpha \gamma + \beta \sigma) &= \alpha \int_a^b f d\gamma + \beta \int_a^b f d\sigma. \end{aligned}$$

The following is a very useful result in calculating these integrals.

1.8 Proposition. Let $\gamma: [a, b] \rightarrow \mathbb{C}$ be of bounded variation and let $f: [a, b] \rightarrow \mathbb{C}$ be continuous. If $a = t_0 < t_1 < \dots < t_n = b$ then

$$\int_a^b f d\gamma = \sum_{k=1}^n \int_{t_{k-1}}^{t_k} f d\gamma.$$

The proof is left as an exercise.

As was mentioned before, we will mainly be concerned with those γ which are piecewise smooth. The following theorem says that in this case we can find $\int f d\gamma$ by the methods of integration learned in calculus.

1.9 Theorem. If γ is piecewise smooth and $f: [a, b] \rightarrow \mathbb{C}$ is continuous then

$$\int_a^b f d\gamma = \int_a^b f(t) \gamma'(t) dt.$$

Proof. Again we only consider the case where γ is smooth. Also, by looking at the real and imaginary parts of γ , we reduce the proof to the case where $\gamma([a, b]) \subset \mathbb{R}$. Let $\epsilon > 0$ and choose $\delta > 0$ such that if $P = \{a = t_0 < \dots < t_n = b\}$ has $\|P\| < \delta$ then

$$1.10 \quad \left| \int_a^b f d\gamma - \sum_{k=1}^n f(\tau_k) [\gamma(t_k) - \gamma(t_{k-1})] \right| < \frac{1}{2}\epsilon$$

and

$$1.11 \quad \left| \int_a^b f(t) \gamma'(t) dt - \sum_{k=1}^n f(\tau_k) \gamma'(\tau_k) (t_k - t_{k-1}) \right| < \frac{1}{2}\epsilon$$

for any choice of τ_k in $[t_{k-1}, t_k]$. If we apply the Mean Value Theorem for derivatives we get that there is a τ_k in $[t_{k-1}, t_k]$ with $\gamma'(\tau_k) = [\gamma(t_k) - \gamma(t_{k-1})] (t_k - t_{k-1})^{-1}$. (Note that the fact that γ is real valued is needed to apply the Mean Value Theorem.) Thus,

$$\sum_{k=1}^n f(\tau_k) [\gamma(t_k) - \gamma(t_{k-1})] = \sum_{k=1}^n f(\tau_k) \gamma'(\tau_k) (t_k - t_{k-1}).$$

Combining this with inequalities (1.10) and (1.11) gives

$$\left| \int_a^b f d\gamma - \int_a^b f(t) \gamma'(t) dt \right| < \epsilon.$$

Since $\epsilon > 0$ was arbitrary, this completes the proof of the theorem. ■

We have already defined a path as a continuous function $\gamma: [a, b] \rightarrow \mathbb{C}$. If $\gamma: [a, b] \rightarrow \mathbb{C}$ is a path then the set $\{\gamma(t): a \leq t \leq b\}$ is called the *trace* of γ and is denoted it by $\{\gamma\}$. Notice that the trace of a path is always a compact set. γ is a *rectifiable path* if γ is a function of bounded variation. If P is a partition of $[a, b]$ then $v(\gamma; P)$ is exactly the sum of lengths of line segments connecting points on the trace of γ . To say that γ is rectifiable is to say that γ has finite length and its length is $V(\gamma)$. In particular, if γ is piecewise smooth then γ is rectifiable and its length is $\int_a^b |\gamma'| dt$.

If $\gamma: [a, b] \rightarrow \mathbb{C}$ is a rectifiable path with $\{\gamma\} \subset E \subset \mathbb{C}$ and $f: E \rightarrow \mathbb{C}$ is a continuous function then $f \circ \gamma$ is a continuous function on $[a, b]$. With this in mind the following definition makes sense.

1.12 Definition. If $\gamma: [a, b] \rightarrow \mathbb{C}$ is a rectifiable path and f is a function defined and continuous on the trace of γ then the (line) *integral of f along γ* is

$$\int_a^b f(\gamma(t)) d\gamma(t).$$

This line integral is also denoted by $\int_\gamma f = \int_\gamma f(z) dz$.

As an example let us take $\gamma: [0, 2\pi] \rightarrow \mathbb{C}$ to be $\gamma(t) = e^{it}$ and define $f(z) = \frac{1}{z}$ for $z \neq 0$. Now γ is differentiable so, by Theorem 1.9 we have

$$\int_\gamma \frac{1}{z} dz = \int_0^{2\pi} e^{-it} (ie^{it}) dt = 2\pi i.$$

Using the same definition of γ and letting m be any integer ≥ 0 gives $\int_\gamma z^m dz = \int_0^{2\pi} e^{im t} (ie^{it}) dt = i \int_0^{2\pi} \exp(i(m+1)t) dt = i \int_0^{2\pi} \cos(m+1)t dt - \int_0^{2\pi} \sin(m+1)t dt = 0$.

Now let $a, b \in \mathbb{C}$ and put $\gamma(t) = tb + (1-t)a$ for $0 \leq t \leq 1$. Then $\gamma'(t) = b - a$, and using the Fundamental Theorem of Calculus we get that for $n \geq 0$, $\int_\gamma z^n dz = (b-a) \int_0^1 [tb + (1-t)a]^n dt = \frac{1}{n+1} (b^{n+1} - a^{n+1})$.

There are more examples in the exercises, but now we will prove a certain "invariance" result which, besides being useful in computations, forms the basis for our definition of a curve.

If $\gamma: [a, b] \rightarrow \mathbb{C}$ is a rectifiable path and $\varphi: [c, d] \rightarrow [a, b]$ is a continuous non-decreasing function whose image is all of $[a, b]$ (i.e., $\varphi(c) = a$ and $\varphi(d) = b$) then $\gamma \circ \varphi: [c, d] \rightarrow \mathbb{C}$ is a path with the same trace as γ . Moreover, $\gamma \circ \varphi$ is rectifiable because if $c = s_0 < s_1 < \dots < s_n = d$ then $a = \varphi(s_0) \leq \varphi(s_1) \leq \dots \leq \varphi(s_n) = b$ is a partition of $[a, b]$. Hence

$$\sum_{k=1}^n |\gamma(\varphi(s_k)) - \gamma(\varphi(s_{k-1}))| \leq V(\gamma)$$

so that $V(\gamma \circ \varphi) \leq V(\gamma) < \infty$. So if f is continuous on $\{\gamma\} = \{\gamma \circ \varphi\}$ then $\int_{\gamma \circ \varphi} f$ is well defined.

1.13 Proposition. If $\gamma: [a, b] \rightarrow \mathbb{C}$ is a rectifiable path and $\varphi: [c, d] \rightarrow [a, b]$ is a continuous non-decreasing function with $\varphi(c) = a$, $\varphi(d) = b$; then for any function f continuous on $\{\gamma\}$

$$\int_\gamma f = \int_{\gamma \circ \varphi} f$$

Proof. Let $\epsilon > 0$ and choose $\delta_1 > 0$ such that for $\{s_0 < s_1 < \dots < s_n\}$, a partition of $[c, d]$ with $(s_k - s_{k-1}) < \delta_1$, and $s_{k-1} \leq \sigma_k \leq s_k$ we have

$$1.14 \quad \left| \int_{\gamma \circ \varphi} f - \sum_{k=1}^n f(\gamma \circ \varphi(\sigma_k)) [\gamma \circ \varphi(s_k) - \gamma \circ \varphi(s_{k-1})] \right| < \frac{1}{2}\epsilon$$

Similarly choose $\delta_2 > 0$ such that if $\{t_0 < t_1 < \dots < t_n\}$ is a partition of $[a, b]$ with $(t_k - t_{k-1}) < \delta_2$ and $t_{k-1} \leq \tau_k \leq t_k$, then

$$1.15 \quad \left| \int_{\gamma} f - \sum_{k=1}^n f(\gamma(\tau_k)) [\gamma(t_k) - \gamma(t_{k-1})] \right| < \frac{1}{2}\epsilon.$$

But φ is uniformly continuous on $[c, d]$; hence there is a $\delta > 0$, which can be chosen with $\delta < \delta_1$, such that $|\varphi(s) - \varphi(s')| < \delta_2$ whenever $|s - s'| < \delta$. So if $\{s_0 < s_1 < \dots < s_n\}$ is a partition of $[c, d]$ with $(s_k - s_{k-1}) < \delta < \delta_1$ and $t_k = \varphi(s_k)$, then $\{t_0 \leq t_1 \leq \dots \leq t_n\}$ is a partition of $[a, b]$ with $(t_k - t_{k-1}) < \delta_2$. If $s_{k-1} \leq \sigma_k \leq s_k$ and $\tau_k = \varphi(\sigma_k)$ then both (1.14) and (1.15) hold. Moreover, the right hand parts of these two differences are equal! It follows that

$$\left| \int_{\gamma} f - \int_{\gamma \circ \varphi} f \right| < \epsilon$$

Since $\epsilon > 0$ was arbitrary, equality is proved. ■

We wish to define an equivalence relation on the collection of rectifiable paths so that each member of an equivalence class has the same trace and so that the line integral of a function continuous on this trace is the same for each path in the class. It would seem that we should define σ and γ to be equivalent if $\sigma = \gamma \circ \varphi$ for some function φ as above. However, this is not an equivalence relation!

1.16 Definition. Let $\sigma: [c, d] \rightarrow \mathbb{C}$ and $\gamma: [a, b] \rightarrow \mathbb{C}$ be rectifiable paths. The path σ is *equivalent* to γ if there is a function $\varphi: [c, d] \rightarrow [a, b]$ which is continuous, strictly increasing, and with $\varphi(c) = a$, $\varphi(d) = b$; such that $\sigma = \gamma \circ \varphi$. We call the function φ a *change of parameter*.

A *curve* is an equivalence class of paths. The trace of a curve is the trace of any one of its members. If f is continuous on the trace of the curve then the integral of f over the curve is the integral of f over any member of the curve.

A curve is smooth (piecewise smooth) if and only if some one of its representatives is smooth (piecewise smooth).

Henceforward, we will not make this distinction between a curve and its representative. In fact, expressions such as “let γ be the unit circle traversed once in the counter-clockwise direction” will be used to indicate a curve. The reader is asked to trust that a result for curves which is, in fact, a result only about paths will not be stated.

Let $\gamma: [a, b] \rightarrow \mathbb{C}$ be a rectifiable path and for $a \leq t \leq b$, let $|\gamma|(t)$ be $V(\gamma; [a, t])$. That is,

$$|\gamma|(t) = \sup \left\{ \sum_{k=1}^n |\gamma(t_k) - \gamma(t_{k-1})| : \{t_0, \dots, t_n\} \text{ is a partition of } [a, t] \right\}.$$

Clearly $|\gamma|(t)$ is increasing and so $|\gamma|: [a, b] \rightarrow \mathbb{C}$ is of bounded variation. If f is continuous on $\{\gamma\}$ define

$$\int_{\gamma} f|dz| = \int_a^b f(\gamma(t)) d|\gamma|(t).$$

If γ is a rectifiable curve then denote by $-\gamma$ the curve defined by $(-\gamma)(t)$

$\gamma(-t)$ for $-b \leq t \leq -a$. Another notation for this is γ^{-1} . Also if $c \in \mathbb{C}$ let $\gamma + c$ denote the curve defined by $(\gamma + c)(t) = \gamma(t) + c$. The following proposition gives many basic properties of the line integral.

1.17 Proposition. Let γ be a rectifiable curve and suppose that f is a function continuous on $\{\gamma\}$. Then:

- (a) $\int_{-\gamma} f = - \int_{\gamma} f$;
- (b) $|\int_{\gamma} f| \leq \int_{\gamma} |f| |dz| \leq V(\gamma) \sup \{|f(z)| : z \in \{\gamma\}\}$;
- (c) If $c \in \mathbb{C}$ then $\int_{\gamma} f(z) dz = \int_{\gamma+c} f(z-c) dz$.

The proof is left as an exercise.

The next result is the analogue of the Fundamental Theorem of Calculus for line integrals.

1.18 Theorem. Let G be open in \mathbb{C} and let γ be a rectifiable path in G with initial and end points α and β respectively. If $f: G \rightarrow \mathbb{C}$ is a continuous function with a primitive $F: G \rightarrow \mathbb{C}$, then

$$\int_{\gamma} f = F(\beta) - F(\alpha)$$

(Recall that F is a *primitive* of f when $F' = f$).

The following useful fact will be needed in the proof of this theorem.

1.19 Lemma. If G is an open set in \mathbb{C} , $\gamma: [a, b] \rightarrow G$ is a rectifiable path, and $f: G \rightarrow \mathbb{C}$ is continuous then for every $\epsilon > 0$ there is a polygonal path Γ in G such that $\Gamma(a) = \gamma(a)$, $\Gamma(b) = \gamma(b)$, and $|\int_{\gamma} f - \int_{\Gamma} f| < \epsilon$.

Proof. Case 1. Suppose G is an open disk. Since $\{\gamma\}$ is a compact set, $d = \text{dist}(\{\gamma\}, \partial G) > 0$. It follows that if $G = B(c; r)$ then $\{\gamma\} \subset B(c; \rho)$ where $\rho = r - \frac{1}{2}d$. The reason for passing to this smaller disk is that f is uniformly continuous in $\bar{B}(c; \rho) \subset G$. Hence without loss of generality it can be assumed that f is uniformly continuous on G . Choose $\delta > 0$ such that $|f(z) - f(w)| < \epsilon$ whenever $|z - w| < \delta$. If $\gamma: [a, b] \rightarrow \mathbb{C}$ then γ is uniformly continuous so there is a partition $\{t_0 < t_1 < \dots < t_n\}$ of $[a, b]$ such that

$$1.19a \quad |\gamma(s) - \gamma(t)| < \delta$$

if $t_{k-1} \leq s, t \leq t_k$; and such that for $t_{k-1} \leq \tau_k \leq t_k$ we have

$$1.20 \quad \left| \int_{\gamma} f - \sum_{k=1}^n f(\gamma(\tau_k)) [\gamma(t_k) - \gamma(t_{k-1})] \right| < \epsilon.$$

Define $\Gamma: [a, b] \rightarrow \mathbb{C}$ by

$$\Gamma(t) = \frac{1}{t_k - t_{k-1}} [(t_k - t)\gamma(t_{k-1}) + (t - t_{k-1})\gamma(t_k)]$$

if $t_{k-1} \leq t \leq t_k$. So on $[t_{k-1}, t_k]$, $\Gamma(t)$ traces out the straight line segment from $\gamma(t_{k-1})$ to $\gamma(t_k)$; that is, Γ is a polygonal path in G . From (1.19a)

$$1.21 \quad |\Gamma(t) - \gamma(\tau_k)| < \delta \quad \text{for } t_{k-1} \leq t \leq t_k.$$

Since $\int_{\Gamma} f = \int_a^b f(\Gamma(t))\Gamma'(t)dt$ it follows that

$$\int_{\Gamma} f = \sum_{k=1}^n \frac{\gamma(t_k) - \gamma(t_{k-1})}{t_k - t_{k-1}} \int_{t_{k-1}}^{t_k} f(\Gamma(t))dt.$$

Using (1.20) we obtain

$$\begin{aligned} \left| \int_{\gamma} f - \int_{\Gamma} f \right| &\leq \epsilon + \left| \sum_{k=1}^n f(\gamma(\tau_k))[\gamma(t_k) - \gamma(t_{k-1})] - \int_{\Gamma} f \right| \\ &\leq \epsilon + \sum_{k=1}^n |\gamma(t_k) - \gamma(t_{k-1})| (t_k - t_{k-1})^{-1} \int_{t_{k-1}}^{t_k} |f(\Gamma(t)) - f(\gamma(\tau_k))| dt. \end{aligned}$$

Applying (1.21) to the integrand gives

$$\left| \int_{\gamma} f - \int_{\Gamma} f \right| \leq \epsilon + \epsilon \sum_{k=1}^n |\gamma(t_k) - \gamma(t_{k-1})| \leq \epsilon(1 + V(\gamma)).$$

The proof of Case I now follows.

Case II. G is arbitrary. Since $\{\gamma\}$ is compact there is a number r with $0 < r < \text{dist}(\{\gamma\}, \partial G)$. Choose $\delta > 0$ such that $|\gamma(s) - \gamma(t)| < r$ when $|s - t| < \delta$. If $P = \{t_0 < t_1 < \dots < t_n\}$ is a partition of $[a, b]$ with $\|P\| < \delta$ then $|\gamma(t) - \gamma(t_{k-1})| < r$ for $t_{k-1} \leq t \leq t_k$. That is if $\gamma_k: [t_{k-1}, t_k] \rightarrow G$ is defined by $\gamma_k(t) = \gamma(t)$ then $\{\gamma_k\} \subset B(\gamma(t_{k-1}); r)$ for $1 \leq k \leq n$. By Case I there is a polygonal path $\Gamma_k: [t_{k-1}, t_k] \rightarrow B(\gamma(t_{k-1}); r)$ such that $\Gamma_k(t_{k-1}) = \gamma(t_{k-1})$, $\Gamma_k(t_k) = \gamma(t_k)$, and $|\int_{\gamma_k} f - \int_{\Gamma_k} f| < \epsilon/n$. If $\Gamma(t) = \Gamma_k(t)$ on $[t_{k-1}, t_k]$ then Γ has the required properties. ■

Proof of Theorem 1.18. Case I. $\gamma: [a, b] \rightarrow \mathbb{C}$ is piecewise smooth. Then $\int_{\gamma} f = \int_a^b f(\gamma(t))\gamma'(t)dt = \int_a^b F'(\gamma(t))\gamma'(t)dt = \int_a^b (F \circ \gamma)'(t)dt = F(\gamma(b)) - F(\gamma(a)) = F(\beta) - F(\alpha)$ by the Fundamental Theorem of Calculus.

Case II The General Case. If $\epsilon > 0$ then Lemma 1.19 implies there is a polygonal path Γ from α to β such that $|\int_{\gamma} f - \int_{\Gamma} f| < \epsilon$. But Γ is piecewise smooth, so by Case I $\int_{\Gamma} f = F(\beta) - F(\alpha)$. Hence $|\int_{\gamma} f - [F(\beta) - F(\alpha)]| < \epsilon$; since $\epsilon > 0$ is arbitrary, the desired equality follows. ■

The use of Lemma 1.19 in the proof of Theorem 1.18 to pass from the piecewise smooth case to the rectifiable case is typical of many proofs of results on line integrals. We shall see applications of Lemma 1.19 in the future.

A curve $\gamma: [a, b] \rightarrow \mathbb{C}$ is said to be *closed* if $\gamma(a) = \gamma(b)$.

1.22 Corollary. Let G , γ , and f satisfy the same hypothesis as in Theorem 1.18. If γ is a closed curve then

$$\int_{\gamma} f = 0$$

The Fundamental Theorem of Calculus says that each continuous function has a primitive. This is far from being true for functions of a complex variable. For example let $f(z) = |z|^2 = x^2 + y^2$. If F is a primitive of f then F is analytic. So if $F = U + iV$ then $x^2 + y^2 = F'(x + iy)$. Now, using the Cauchy-Riemann equations,

$$\frac{\partial U}{\partial x} = \frac{\partial V}{\partial y} = x^2 + y^2 \quad \text{and} \quad \frac{\partial U}{\partial y} = \frac{\partial V}{\partial x} = 0.$$

But $\frac{\partial U}{\partial y} = 0$ implies that $U(x, y) = u(x)$ for some differentiable function u .

But this gives $x^2 + y^2 = \frac{\partial U}{\partial x} = u'(x)$, a clear contradiction. Another way to see that $|z|^2$ does not have a primitive is to apply Theorem 1.18 (see Exercise 8).

Exercises

1. Let $\gamma: [a, b] \rightarrow \mathbb{R}$ be non decreasing. Show that γ is of bounded variation and $V(\gamma) = \gamma(b) - \gamma(a)$.
2. Prove Proposition 1.2.
3. Prove Proposition 1.7.
4. Prove Proposition 1.8 (Use induction).
5. Let $\gamma(t) = \exp((-1+i)t^{-1})$ for $0 < t \leq 1$ and $\gamma(0) = 0$. Show that γ is a rectifiable path and find $V(\gamma)$. Give a rough sketch of the trace of γ .
6. Show that if $\gamma: [a, b] \rightarrow \mathbb{C}$ is a Lipschitz function then γ is of bounded variation.
7. Show that $\gamma: [0, 1] \rightarrow \mathbb{C}$, defined by $\gamma(t) = t + it \sin \frac{1}{t}$ for $t \neq 0$ and $\gamma(0) = 0$, is a path but is not rectifiable. Sketch this path.
8. Let γ and σ be the two polygons $[1, i]$ and $[1, 1+i, i]$. Express γ and σ as paths and calculate $\int_{\gamma} f$ and $\int_{\sigma} f$ where $f(z) = |z|^2$.
9. Define $\gamma: [0, 2\pi] \rightarrow \mathbb{C}$ by $\gamma(t) = \exp(int)$ where n is some integer (positive, negative, or zero). Show that $\int_{\gamma} \frac{1}{z} dz = 2\pi in$.
10. Define $\gamma(t) = e^{it}$ for $0 \leq t \leq 2\pi$ and find $\int_{\gamma} z^n dz$ for every integer n .
11. Let γ be the closed polygon $[1-i, 1+i, -1+i, -1-i, 1-i]$. Find $\int_{\gamma} \frac{1}{z} dz$.
12. Let $I(r) = \int_{\gamma} \frac{e^{iz}}{z} dz$ where $\gamma: [0, \pi] \rightarrow \mathbb{C}$ is defined by $\gamma(t) = re^{it}$. Show that $\lim_{r \rightarrow \infty} I(r) = 0$.
13. Find $\int_{\gamma} z^{-1} dz$ where: (a) γ is the upper half of the unit circle from $+1$ to -1 ; (b) γ is the lower half of the unit circle from $+1$ to -1 .

14. Prove that if $\varphi: [a, b] \rightarrow [c, d]$ is continuous and $\varphi(a) = c$, $\varphi(b) = d$ then φ is one-one iff φ is strictly increasing.
15. Show that the relation in Definition 1.16 is an equivalence relation.
16. Show that if γ and σ are equivalent rectifiable paths then $V(\gamma) = V(\sigma)$.
17. Show that if $\gamma: [a, b] \rightarrow \mathbb{C}$ is a path then there is an equivalent path $\sigma: [0, 1] \rightarrow \mathbb{C}$.
18. Prove Proposition 1.17.
19. In the proof of Case I of Lemma 1.19, where was the assumption that γ lies in a disk used?
20. Let $\gamma(t) = 1 + e^{it}$ for $0 \leq t \leq 2\pi$ and find $\int_{\gamma} (z^2 - 1)^{-1} dz$.
21. Let $\gamma(t) = 2e^{it}$ for $-\pi \leq t \leq \pi$ and find $\int_{\gamma} (z^2 - 1)^{-1} dz$.
22. Show that if F_1 and F_2 are primitives for $f: G \rightarrow \mathbb{C}$ and G is connected then there is a constant c such that $F_1(z) = c + F_2(z)$ for each z in G .
23. Let γ be a closed rectifiable curve in G and $a \notin G$. Show that for $n \geq 2$, $\int_{\gamma} (z - a)^{-n} dz = 0$.
24. Prove the following integration by parts formula. Let f and g be analytic in G and let γ be a rectifiable curve from a to b in G . Then $\int_{\gamma} f g' = f(b)g(b) - f(a)g(a) - \int_{\gamma} f' g$.

§2. Power series representation of analytic functions

In this section we will see that a function f , analytic in an open set G , has a power series expansion about each point of G . In particular, an analytic function is infinitely differentiable.

We begin by proving Leibniz's rule from Advanced Calculus.

2.1 Proposition. Let $\varphi: [a, b] \times [c, d] \rightarrow \mathbb{C}$ be a continuous function and define $g: [c, d] \rightarrow \mathbb{C}$ by

$$2.2 \quad g(t) = \int_a^b \varphi(s, t) ds.$$

Then g is continuous. Moreover, if $\frac{\partial \varphi}{\partial t}$ exists and is a continuous function on $[a, b] \times [c, d]$ then g is continuously differentiable and

$$2.3 \quad g'(t) = \int_a^b \frac{\partial \varphi}{\partial t}(s, t) ds.$$

Proof. The proof that g is continuous is left as an exercise. Notice that if we prove that g is differentiable with g' given by formula (2.3) then it will follow from the first part that g' is continuous since $\frac{\partial \varphi}{\partial t}$ is continuous. Hence, we need only verify formula (2.3).

Fix a point t_0 in $[c, d]$ and let $\epsilon > 0$. Denote $\frac{\partial \varphi}{\partial t}$ by φ_2 ; it follows that φ_2 must be uniformly continuous on $[a, b] \times [c, d]$. Thus, there is a $\delta > 0$ such that $|\varphi_2(s', t') - \varphi_2(s, t)| < \epsilon$ whenever $(s - s')^2 + (t - t')^2 < \delta^2$. In particular

$$2.4 \quad |\varphi_2(s, t) - \varphi_2(s, t_0)| < \epsilon$$

whenever $|t - t_0| < \delta$ and $a \leq s \leq b$. This gives that for $|t - t_0| < \delta$ and $a \leq s \leq b$,

$$2.5 \quad \left| \int_{t_0}^t [\varphi_2(s, \tau) - \varphi_2(s, t_0)] d\tau \right| \leq \epsilon |t - t_0|.$$

But for a fixed s in $[a, b]$ $\Phi(t) = \varphi(s, t) - t\varphi_2(s, t_0)$ is a primitive of $\varphi_2(s, t) - \varphi_2(s, t_0)$. By combining the Fundamental Theorem of Calculus with inequality (2.5), it follows that

$$|\varphi(s, t) - \varphi(s, t_0) - (t - t_0)\varphi_2(s, t_0)| \leq \epsilon |t - t_0|$$

for any s when $|t - t_0| < \delta$. But from the definition of g this gives

$$\left| \frac{g(t) - g(t_0)}{t - t_0} - \int_a^b \varphi_2(s, t_0) ds \right| \leq \epsilon(b - a)$$

when $0 < |t - t_0| < \delta$. ■

This result can be used to prove that

$$\int_0^{2\pi} \frac{e^{is}}{e^{is} - z} ds = 2\pi \quad \text{if } |z| < 1.$$

Actually, we will need this formula in the proof of the next proposition.

Let $\varphi(s, t) = \frac{e^{is}}{e^{is} - tz}$ for $0 \leq t \leq 1$, $0 \leq s \leq 2\pi$; (Note that φ is continuously differentiable because $|z| < 1$.) Hence $g(t) = \int_0^{2\pi} \varphi(s, t) ds$ is continuously differentiable. Also, $g(0) = 2\pi$; so if it can be shown that g is a constant, then $2\pi = g(1)$ and the desired result is obtained.

Now

$$g'(t) = \int_0^{2\pi} \frac{ze^{is}}{(e^{is} - tz)^2} ds;$$

but for t fixed, $\Phi(s) = zi(e^{is} - tz)^{-1}$ has $\Phi'(s) = -zi(e^{is} - tz)^{-2}(ie^{is}) = ze^{is}(e^{is} - tz)^{-2}$. Hence $g'(t) = \Phi(2\pi) - \Phi(0) = 0$, so g must be a constant.

The next result, although very important, is transitory. We will see a much more general result than this—Cauchy's Integral Formula; a formula which is one of the essential facts of the theory.

2.6 Proposition. Let $f: G \rightarrow \mathbb{C}$ be analytic and suppose $\bar{B}(a; r) \subset G$ ($r > 0$). If $\gamma(t) = a + re^{it}$, $0 \leq t \leq 2\pi$, then

$$f(z) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(w)}{w-z} dw$$

for $|z-a| < r$.

Proof. By considering $G_1 = \left\{ \frac{1}{r}(z-a): z \in G \right\}$ and the function $g(z) = f(a+rz)$ we see that, without loss of generality, it may be assumed that $a = 0$ and $r = 1$. That is we may assume that $\bar{B}(0; 1) \subset G$.

Fix z , $|z| < 1$; it must be shown that

$$\begin{aligned} f(z) &= \frac{1}{2\pi i} \int_{\gamma} \frac{f(w)}{w-z} dw \\ &= \frac{1}{2\pi} \int_0^{2\pi} \frac{f(e^{is})e^{is}}{e^{is}-z} ds; \end{aligned}$$

that is, we want to show that

$$\begin{aligned} 0 &= \int_0^{2\pi} \frac{f(e^{is})e^{is}}{e^{is}-z} ds - 2\pi f(z) \\ &= \int_0^{2\pi} \left[\frac{f(e^{is})e^{is}}{e^{is}-z} - f(z) \right] ds \end{aligned}$$

We will apply Leibniz's rule by letting

$$\varphi(s, t) = \frac{f(z + t(e^{is}-z))e^{is}}{e^{is}-z} - f(z),$$

for $0 \leq t \leq 1$ and $0 \leq s \leq 2\pi$. Since $|z + t(e^{is}-z)| = |z(1-t) + te^{is}| < 1$, φ is well defined and is continuously differentiable. Let $g(t) = \int_0^{2\pi} \varphi(s, t) ds$; so g has a continuous derivative.

The proposition will be proved if it can be shown that $g(1) = 0$; this is done by showing that $g(0) = 0$ and that g is constant. To see that $g(0) = 0$

Power series representation of analytic functions
compute:

$$\begin{aligned} g(0) &= \int_0^{2\pi} \varphi(s, 0) ds \\ &= \int_0^{2\pi} \left[\frac{f(z)e^{is}}{e^{is}-z} - f(z) \right] ds \\ &= f(z) \int_0^{2\pi} \frac{e^{is}}{e^{is}-z} ds - 2\pi f(z) \\ &= 0, \end{aligned}$$

since we showed that $\int_0^{2\pi} \frac{e^{is}}{e^{is}-z} ds = 2\pi$ prior to the statement of this proposition.

To show that g is constant compute g' . By Leibniz's rule, $g'(t) = \int_0^{2\pi} \varphi_2(s, t) ds$ where

$$\varphi_2(s, t) = e^{is} f'(z + t(e^{is}-z)).$$

However, for $0 < t \leq 1$ we have that $\Phi(s) = -it^{-1}f(z + t(e^{is}-z))$ is a primitive of $\varphi_2(s, t)$. So $g'(t) = \Phi(2\pi) - \Phi(0) = 0$ for $0 < t \leq 1$. Since g' is continuous we have $g' = 0$ and g must be a constant. ■

How is this result used to get the power series expansion? The answer is that we use a geometric series. Let $|z-a| < r$ and suppose that w is on the circle $|w-a| = r$. Then

$$\frac{1}{w-z} = \frac{1}{w-a} \cdot \frac{1}{1 - \left[\frac{z-a}{w-a} \right]} = \frac{1}{(w-a)} \sum_{n=0}^{\infty} \left(\frac{z-a}{w-a} \right)^n$$

since $|z-a| < r = |w-a|$. Now, multiplying both sides by $[f(w)/2\pi i]$ and integrating around the circle $\gamma: |w-a| = r$, the left hand side yields $f(z)$ by the preceding proposition. The right hand side becomes—what? To find the answer we must know that we can distribute the integral through the infinite sum.

2.7 Lemma. Let γ be a rectifiable curve in \mathbb{C} and suppose that F_n and F are continuous functions on $\{\gamma\}$. If $F = u\text{-}\lim F_n$ on $\{\gamma\}$ then

$$\int_{\gamma} F = \lim \int_{\gamma} F_n.$$

Proof. Let $\epsilon > 0$; then there is an integer N such that $|F_n(w) - F(w)| < \epsilon/V(\gamma)$ for all w on $\{\gamma\}$ and $n \geq N$. But this gives, by Proposition 1.17(b),

$$\begin{aligned} \left| \int_{\gamma} F - \int_{\gamma} F_n \right| &= \left| \int_{\gamma} (F - F_n) \right| \\ &\leq \int_{\gamma} |F(w) - F_n(w)| |dw| \\ &\leq \epsilon \end{aligned}$$

whenever $n \geq N$. ■

2.8 Theorem. Let f be analytic in $B(a; R)$; then $f(z) = \sum_{n=0}^{\infty} a_n(z-a)^n$ for $|z-a| < R$ where $a_n = \frac{1}{n!} f^{(n)}(a)$ and this series has radius of convergence $\geq R$.

Proof. Let $0 < r < R$ so that $\bar{B}(a; r) \subset B(a; R)$. If $\gamma(t) = a + re^{it}$, $0 \leq t \leq 2\pi$, then by Proposition 2.6,

$$f(z) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(w)}{w-z} dw \quad \text{for } |z-a| < r.$$

But, since $|z-a| < r$ and w is on the circle $\{\gamma\}$,

$$\frac{|f(w)| |z-a|^n}{|w-a|^{n+1}} \leq \frac{M}{r} \left(\frac{|z-a|}{r} \right)^n$$

where $M = \max \{|f(w)|; |w-a| = r\}$. Since $\frac{|z-a|}{r} < 1$, the Weierstrass M -test gives that $\sum f(w) (z-a)^n / (w-a)^{n+1}$ converges uniformly for w on $\{\gamma\}$. By Lemma 2.7 and the discussion preceding it

$$2.9 \quad f(z) = \sum_{n=0}^{\infty} \left[\frac{1}{2\pi i} \int_{\gamma} \frac{f(w)}{(w-a)^{n+1}} dw \right] (z-a)^n$$

If we set

$$a_n = \frac{1}{2\pi i} \int_{\gamma} \frac{f(w)}{(w-a)^{n+1}} dw,$$

then a_n is independent of z , and so (2.9) is a power series which converges for $|z-a| < r$.

It follows (Proposition III. 2.5) that $a_n = \frac{1}{n!} f^{(n)}(a)$, so that the value of a_n is independent of γ ; that is, it is independent of r . So

$$2.10 \quad f(z) = \sum_{n=0}^{\infty} a_n(z-a)^n$$

for $|z-a| < r$. Since r was chosen arbitrarily, $r < R$, we have that (2.10)

holds for $|z-a| < R$; giving that the radius of convergence of (2.10) must be at least R . ■

2.11 Corollary. If $f: G \rightarrow \mathbb{C}$ is analytic and $a \in G$ then $f(z) = \sum_{n=0}^{\infty} a_n(z-a)^n$ for $|z-a| < R$ where $R = d(a, \partial G)$.

Proof. Since $R = d(a, \partial G)$, $B(a; R) \subset G$ so that f is analytic on $B(a; R)$. The result now follows from the theorem. ■

2.12 Corollary. If $f: G \rightarrow \mathbb{C}$ is analytic then f is infinitely differentiable.

2.13 Corollary. If $f: G \rightarrow \mathbb{C}$ is analytic and $\bar{B}(a; r) \subset G$ then

$$f^{(n)}(a) = \frac{n!}{2\pi i} \int_{\gamma} \frac{f(w)}{(w-a)^{n+1}} dw$$

where $\gamma(t) = a + re^{it}$, $0 \leq t \leq 2\pi$.

2.14 Cauchy's Estimate. Let f be analytic in $B(a; R)$ and suppose $|f(z)| \leq M$ for all z in $B(a; R)$. Then

$$|f^{(n)}(a)| \leq \frac{n!M}{R^n}.$$

Proof. Since Corollary 2.13 applies with $r < R$, Proposition 1.17 implies that

$$|f^{(n)}(a)| \leq \left(\frac{n!}{2\pi} \right) \frac{M}{r^{n+1}} \cdot 2\pi r = \frac{n!M}{r^n}$$

Since $r < R$ is arbitrary, the result follows by letting $r \rightarrow R-$. ■

We will conclude this section by proving a proposition which is a special case of a more general result which will be presented later in this chapter.

2.15 Proposition. Let f be analytic in the disk $B(a; R)$ and suppose that γ is a closed rectifiable curve in $B(a; R)$. Then

$$\int_{\gamma} f = 0.$$

Proof. This is proved by showing that f has a primitive (Corollary 1.22). Now, by Theorem 2.8, $f(z) = \sum a_n(z-a)^n$ for $|z-a| < R$. Let

$$F(z) = \sum_{n=0}^{\infty} \left(\frac{a_n}{n+1} \right) (z-a)^{n+1} = (z-a) \sum_{n=0}^{\infty} \left(\frac{a_n}{n+1} \right) (z-a)^n.$$

Since $\lim (n+1)^{1/n} = 1$, it follows that this power series has the same radius of convergence as $\sum a_n(z-a)^n$. Hence, F is defined on $B(a; R)$. Moreover, $F'(z) = f(z)$ for $|z-a| < R$. ■

Exercises

1. Show that the function defined by (2.2) is continuous.
2. Prove the following analogue of Leibniz's rule (this exercise will be

frequently used in the later sections.) Let G be an open set and let γ be a rectifiable curve in G . Suppose that $\varphi: \{\gamma\} \times G \rightarrow \mathbb{C}$ is a continuous function and define $g: G \rightarrow \mathbb{C}$ by

$$g(z) = \int_{\gamma} \varphi(w, z) dw$$

then g is continuous. If $\frac{\partial \varphi}{\partial z}$ exists for each (w, z) in $\{\gamma\} \times G$ and is continuous then g is analytic and

$$g'(z) = \int_{\gamma} \frac{\partial \varphi}{\partial z}(w, z) dw.$$

3. Suppose that γ is a rectifiable curve in \mathbb{C} and φ is defined and continuous on $\{\gamma\}$. Use Exercise 2 to show that

$$g(z) = \int_{\gamma} \frac{\varphi(w)}{w-z} dw$$

is analytic on $\mathbb{C} - \{\gamma\}$ and

$$g^{(n)}(z) = n! \int_{\gamma} \frac{\varphi(w)}{(w-z)^{n+1}} dw.$$

4. (a) Prove Abel's Theorem: Let $\sum a_n (z-a)^n$ have radius of convergence 1 and suppose that $\sum a_n$ converges to A . Prove that

$$\lim_{r \rightarrow 1^-} \sum a_n r^n = A.$$

(Hint: Find a summation formula which is the analogue of integration by parts.)

(b) Use Abel's Theorem to prove that $\log 2 = 1 - \frac{1}{2} + \frac{1}{3} - \dots$

5. Give the power series expansion of $\log z$ about $z = i$ and find its radius of convergence.

6. Give the power series expansion of \sqrt{z} about $z = 1$ and find its radius of convergence.

7. Use the results of this section to evaluate the following integrals:

$$(a) \int_{\gamma} \frac{e^{iz}}{z^2} dz, \quad \gamma(t) = e^{it}, \quad 0 \leq t \leq 2\pi;$$

$$(b) \int_{\gamma} \frac{dz}{z-a}, \quad \gamma(t) = a + re^{it}, \quad 0 \leq t \leq 2\pi;$$

$$(c) \int_{\gamma} \frac{\sin z}{z^3} dz, \quad \gamma(t) = e^{it}, \quad 0 \leq t \leq 2\pi;$$

$$(d) \int_{\gamma} \frac{\log z}{z^n} dz, \quad \gamma(t) = 1 + \frac{1}{2}e^{it}, \quad 0 \leq t \leq 2\pi \text{ and } n > 0.$$

8. Use a Mobius transformation to show that Proposition 2.15 holds if the disk $B(a; R)$ is replaced by a half plane.

9. Use Corollary 2.13 to evaluate the following integrals:

$$(a) \int_{\gamma} \frac{e^z - e^{-z}}{z^n} dz \text{ where } n \text{ is a positive integer and } \gamma(t) = e^{it}, 0 \leq t \leq 2\pi;$$

$$(b) \int_{\gamma} \frac{dz}{(z-\frac{1}{2})^n} \text{ where } n \text{ is a positive integer and } \gamma(t) = e^{it}, 0 \leq t \leq 2\pi;$$

$$(c) \int_{\gamma} \frac{dz}{z^2+1} \text{ where } \gamma(t) = 2e^{it}, 0 \leq t \leq 2\pi. \text{ (Hint: expand } (z^2+1)^{-1} \text{ by means of partial fractions);}$$

$$(d) \int_{\gamma} \frac{\sin z}{z} dz \text{ where } \gamma(t) = e^{it}, 0 \leq t \leq 2\pi;$$

$$(e) \int_{\gamma} \frac{z^{1/m}}{(z-1)^m} dz \text{ where } \gamma(t) = 1 + \frac{1}{2}e^{it}, 0 \leq t \leq 2\pi.$$

10. Evaluate $\int_{\gamma} \frac{z^2+1}{z(z^2+4)} dz$ where $\gamma(t) = re^{it}$, $0 \leq t \leq 2\pi$, for all possible

values of r , $0 < r < 2$ and $2 < r < \infty$.

11. Find the domain of analyticity of

$$f(z) = \frac{1}{2i} \log \left(\frac{1+iz}{1-iz} \right);$$

also, show that $\tan f(z) = z$ (i.e., f is a branch of $\arctan z$). Show that

$$f(z) = \sum_{k=0}^{\infty} (-1)^k \frac{z^{2k+1}}{2k+1} \text{ for } |z| < 1$$

(Hint: see Exercise III. 3.19.)

12. Show that

$$\sec z = 1 + \sum_{k=1}^{\infty} \frac{E_{2k}}{(2k)!} z^{2k}$$

for some constants E_2, E_4, \dots . These numbers are called Euler's constants. What is the radius of convergence of this series? Use the fact that $1 = \cos z \sec z$ to show that

$$E_{2n} - \binom{2n}{2n-2} E_{2n-2} + \binom{2n}{2n-4} E_{2n-4} - \dots + (-1)^n \binom{2n}{2} E_2 + (-1)^n = 0.$$

Evaluate $E_2, E_4, E_6, E_8, (E_{10} = 50521 \text{ and } E_{12} = 2702765).$

13. Find the series expansion of $\frac{e^z - 1}{z}$ about zero and determine its radius of convergence. Consider $f(z) = \frac{z}{e^z - 1}$ and let

$$f(z) = \sum_{k=0}^{\infty} \frac{a_k}{k!} z^k$$

be its power series expansion about zero. What is the radius of convergence? Show that

$$0 = a_0 + \binom{n+1}{1} a_1 + \cdots + \binom{n+1}{n} a_n.$$

Using the fact that $f(z) + \frac{1}{2}z$ is an even function show that $a_k = 0$ for k odd and $k > 1$. The numbers $B_{2n} = (-1)^{n-1} a_{2n}$ are called the Bernoulli numbers for $n \geq 1$. Calculate B_2, B_4, \dots, B_{10} .

14. Find the power series expansion of $\tan z$ about $z = 0$, expressing the coefficients in terms of Bernoulli numbers. (Hint: use Exercise 13 and the formula $\cot 2z = \frac{1}{2} \cot z - \frac{1}{2} \tan z$.)

§3. Zeros of an analytic function

If $p(z)$ and $q(z)$ are two polynomials then it is well known that $p(z) = s(z)q(z) + r(z)$ where $s(z)$ and $r(z)$ are also polynomials and the degree of $r(z)$ is less than the degree of $q(z)$. In particular, if a is such that $p(a) = 0$ then choose $(z-a)$ for $q(z)$. Hence, $p(z) = (z-a)s(z) + r(z)$ and $r(z)$ must be a constant polynomial. But letting $z = a$ gives $0 = p(a) = r(a)$. Thus, $p(z) = (z-a)s(z)$. If we also have that $s(a) = 0$ we can factor $(z-a)$ from $s(z)$. Continuing we get $p(z) = (z-a)^m t(z)$ where $1 \leq m \leq \text{degree of } p(z)$, and $t(z)$ is a polynomial such that $t(a) \neq 0$. Also, $\text{degree } t(z) = \text{degree } p(z) - m$.

3.1 Definition. If $f: G \rightarrow \mathbb{C}$ is analytic and a in G satisfies $f(a) = 0$ then a is a zero of f of multiplicity $m \geq 1$ if there is an analytic function $g: G \rightarrow \mathbb{C}$ such that $f(z) = (z-a)^m g(z)$ where $g(a) \neq 0$.

Returning to the discussion of polynomials, we have that the multiplicity of a zero of a polynomial must be less than the degree of the polynomial. If $n = \text{degree of the polynomial } p(z)$ and a_1, \dots, a_k are all the distinct zeros of $p(z)$ then $p(z) = (z-a_1)^{m_1} \cdots (z-a_k)^{m_k} s(z)$ where $s(z)$ is a polynomial with no zeros. Now the Fundamental Theorem of Algebra says that a polynomial with no zeros is constant. Hence, if we can prove this result we will have succeeded in completely factoring $p(z)$ into the product of first degree polynomials. The reader might be pleasantly surprised to know that after many years of studying Mathematics he is right now on the threshold of proving the Fundamental Theorem of Algebra. But first it is necessary to prove a famous result about analytic functions. It is also convenient to introduce some new terminology.

3.2 Definition. An entire function is a function which is defined and analytic in the whole complex plane \mathbb{C} . (The term “integral function” is also used.)

The following result follows from Theorem 2.8 and the fact that \mathbb{C} contains $B(0; R)$ for arbitrarily large R .

3.3 Proposition. If f is an entire function then f has a power series expansion

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

with infinite radius of convergence.

In light of the preceding proposition, entire functions can be considered as polynomials of “infinite degree”. So the question arises: can the theory of polynomials be generalized to entire functions? For example, can an entire function be factored? The answer to this is difficult and is postponed to Section VII. 5. Another property of polynomials is that no non constant polynomial is bounded. Indeed, if $p(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_0$ then $\lim_{z \rightarrow \infty} p(z) = \lim_{z \rightarrow \infty} z^n [1 + a_{n-1}z^{-1} + \cdots + a_0z^{-n}] = \infty$. The fact that this also holds for entire functions is an extremely useful result.

3.4 Liouville's Theorem. If f is a bounded entire function then f is constant.

Proof. Suppose $|f(z)| \leq M$ for all z in \mathbb{C} . We will show that $f'(z) = 0$ for all z in \mathbb{C} . To do this use Cauchy's Estimate (Corollary 2.14). Since f is analytic in any disk $B(z; R)$ we have that $|f'(z)| \leq M/R$. Since R was arbitrary, it follows that $f'(z) = 0$ for each z in \mathbb{C} . ■

The reader should not be deceived into thinking that this theorem is insignificant because it has such a short proof. We have expended a great deal of effort building up machinery and increasing our knowledge of analytic functions. We have plowed, planted, and fertilized; we shouldn't be surprised if, occasionally, something is available for easy picking.

Liouville's Theorem will be better appreciated in the following application.

3.5 Fundamental Theorem of Algebra. If $p(z)$ is a non constant polynomial then there is a complex number a with $p(a) = 0$.

Proof. Suppose $p(z) \neq 0$ for all z and let $f(z) = [p(z)]^{-1}$; then f is an entire function. If p is not constant then, as was shown above, $\lim_{z \rightarrow \infty} p(z) = \infty$; so $\lim_{z \rightarrow \infty} f(z) = 0$. In particular, there is a number $R > 0$ such that $|f(z)| < 1$ if $|z| > R$. But f is continuous on $\bar{B}(0; R)$ so there is a constant M such that $|f(z)| \leq M$ for $|z| \leq R$. Hence f is bounded and, therefore, must be constant by Liouville's theorem. It follows that p must be constant, contradicting our assumption. ■

3.6 Corollary. If $p(z)$ is a polynomial and a_1, \dots, a_m are its zeros with a_j having multiplicity k_j then $p(z) = c(z-a_1)^{k_1} \cdots (z-a_m)^{k_m}$ for some constant c and $k_1 + \cdots + k_m$ is the degree of p .

Returning to the analogy between entire functions and polynomials, the

reader should be warned that this cannot be taken too far. For example, if p is a polynomial and $a \in \mathbb{C}$ then there is a number z with $p(z) = a$. In fact, this follows from the Fundamental Theorem of Algebra by considering the polynomial $p(z) - a$. However the exponential function fails to have this property since it does not assume the value zero. (Nevertheless, we are able to show that this is the worst that can happen. That is, a function analytic in \mathbb{C} omits at most one value. This is known as Picard's Little Theorem and will be proved later.) Moreover, no one should begin to make an analogy between analytic functions in an open set G and a polynomial p defined on \mathbb{C} ; rather, you should only think of the polynomials as defined on G .

For example, let

$$f(z) = \cos\left(\frac{1+z}{1-z}\right), \quad |z| < 1.$$

Notice that $\frac{1+z}{1-z}$ maps $D = \{z: |z| < 1\}$ onto $G = \{z: \operatorname{Re} z > 0\}$. The zeros

of f are the points $\left\{\frac{n\pi-2}{n\pi+2}: n \text{ is odd}\right\}$; so f has infinitely many zeros.

However, as $n \rightarrow \infty$ the zeros approach 1 which is not in the domain of analyticity D . This is the story for the most general case.

3.7 Theorem. Let G be a connected open set and let $f: G \rightarrow \mathbb{C}$ be an analytic function. Then the following are equivalent statements:

- (a) $f \equiv 0$;
- (b) there is a point a in G such that $f^{(n)}(a) = 0$ for each $n \geq 0$;
- (c) $\{z \in G: f(z) = 0\}$ has a limit point in G .

Proof. Clearly (a) implies both (b) and (c). (c) implies (b): Let $a \in G$ and a limit point of $Z = \{z \in G: f(z) = 0\}$, and let $R > 0$ be such that $B(a; R) \subset G$. Since a is a limit point of Z and f is continuous it follows that $f(a) = 0$. Suppose there is an integer $n \geq 1$ such that $f(a) = f'(a) = \cdots = f^{(n-1)}(a) = 0$ and $f^{(n)}(a) \neq 0$. Expanding f in power series about a gives that

$$f(z) = \sum_{k=n}^{\infty} a_k(z-a)^k$$

for $|z-a| < R$. If

$$g(z) = \sum_{k=n}^{\infty} a_k(z-a)^{k-n}$$

then g is analytic in $B(a; R)$, $f(z) = (z-a)^n g(z)$, and $g(a) = a_n \neq 0$. Since g is analytic (and therefore continuous) in $B(a; R)$ we can find an r , $0 < r < R$, such that $g(z) \neq 0$ for $|z-a| < r$. But since a is a limit point of Z there is a point b with $f(b) = 0$ and $0 < |b-a| < r$. This gives $0 = (b-a)^n g(b)$ and so $g(b) = 0$, a contradiction. Hence no such integer n can be found; this proves part (b).

(b) implies (a): Let $A = \{z \in G: f^{(n)}(z) = 0 \text{ for all } n \geq 0\}$. From the hypothesis of (b) we have that $A \neq \emptyset$. We will show that A is both open

and closed in G ; by the connectedness of G it will follow that A must be G and so $f \equiv 0$. To see that A is closed let $z \in A^-$ and let z_k be a sequence in A such that $z = \lim a_k$. Since each $f^{(n)}$ is continuous it follows that $f^{(n)}(z) = \lim f^{(n)}(z_k) = 0$. So $z \in A$ and A is closed.

To see that A is open, let $a \in A$ and let $R > 0$ be such that $B(a; R) \subset G$.

Then $f(z) = \sum a_n(z-a)^n$ for $|z-a| < R$ where $a_n = \frac{1}{n!} f^{(n)}(a) = 0$ for each $n \geq 0$. Hence $f(z) = 0$ for all z in $B(a; R)$ and, consequently, $B(a; R) \subset A$. Thus A is open and this completes the proof of the theorem. ■

3.8 Corollary. If f and g are analytic on a region G then $f \equiv g$ iff $\{z \in G: f(z) = g(z)\}$ has a limit point in G .

This follows by applying the preceding theorem to the analytic function $f-g$.

3.9 Corollary. If f is analytic on an open connected set G and f is not identically zero then for each a in G with $f(a) = 0$ there is an integer $n \geq 1$ and an analytic function $g: G \rightarrow \mathbb{C}$ such that $g(a) \neq 0$ and $f(z) = (z-a)^n g(z)$ for all z in G . That is, each zero of f has finite multiplicity.

Proof. Let n be the largest integer (≥ 1) such that $f^{(n-1)}(a) = 0$ and define $g(z) = (z-a)^{-n} f(z)$ for $z \neq a$ and $g(a) = \frac{1}{n!} f^{(n)}(a)$. Then g is clearly analytic in $G - \{a\}$; to see that g is analytic in G it need only be shown to be analytic in a neighborhood of a . This is accomplished by using the method of the proof that (c) implies (b) in the theorem. ■

3.10 Corollary. If $f: G \rightarrow \mathbb{C}$ is analytic and not constant, $a \in G$, and $f(a) = 0$ then there is an $R > 0$ such that $B(a; R) \subset G$ and $f(z) \neq 0$ for $0 < |z-a| < R$.

Proof. By the above theorem the zeros of f are isolated. ■

There is one instance where the analogy between polynomials and analytic functions works in reverse. That is, there is a property of analytic functions which is not so transparent for polynomials.

3.11 Maximum Modulus Theorem. If G is a region and $f: G \rightarrow \mathbb{C}$ is an analytic function such that there is a point a in G with $|f(a)| \geq |f(z)|$ for all z in G , then f is constant.

Proof. Let $\bar{B}(a; r) \subset G$, $\gamma(t) = a + re^{it}$ for $0 \leq t \leq 2\pi$; according to Proposition 2.6

$$\begin{aligned} f(a) &= \frac{1}{2\pi i} \int_{\gamma} \frac{f(w)}{w-a} dw \\ &= \frac{1}{2\pi} \int_0^{2\pi} f(a + re^{it}) dt \end{aligned}$$

Hence

$$|f(a)| \leq \frac{1}{2\pi} \int_0^{2\pi} |f(a+re^{it})| dt \leq |f(a)|$$

since $|f(a+re^{it})| \leq |f(a)|$ for all t . This gives that

$$0 = \int_0^{2\pi} [|f(a)| - |f(a+re^{it})|] dt;$$

but since the integrand is non-negative it follows that $|f(a)| = |f(a+re^{it})|$ for all t . Moreover, since r was arbitrary, we have that f maps any disk $B(a; R) \subset G$ into the circle $|z| = |\alpha|$ where $\alpha = f(a)$. But this implies that f is constant on $B(a; R)$ (Exercise III. 3.17). In particular $f(z) = \alpha$ for $|z-a| < R$. According to Corollary 3.8, $f \equiv \alpha$. ■

According to the Maximum Modulus Theorem, a non-constant analytic function on a region cannot assume its maximum modulus; this fact is far from obvious even in the case of polynomials. The consequences of this theorem are far reaching; some of these, along with a closer examination of the Maximum Modulus Theorem, are presented in Chapter VI. (Actually, the reader at this point can proceed to Sections VI. 1 and VI. 2.)

Exercises

1. Let f be an entire function and suppose there is a constant M , an $R > 0$, and an integer $n \geq 1$ such that $|f(z)| \leq M|z|^n$ for $|z| > R$. Show that f is a polynomial of degree $\leq n$.
2. Give an example to show that G must be assumed to be connected in Theorem 3.7.
3. Find all entire functions f such that $f(x) = e^x$ for x in \mathbb{R} .
4. Prove that $e^{z+a} = e^z e^a$ by applying Corollary 3.8.
5. Prove that $\cos(a+b) = \cos a \cos b - \sin a \sin b$ by applying Corollary 3.8.
6. Let G be a region and suppose that $f: G \rightarrow \mathbb{C}$ is analytic and $a \in G$ such that $|f(a)| \leq |f(z)|$ for all z in G . Show that either $f(a) = 0$ or f is constant.
7. Give an elementary proof of the Maximum Modulus Theorem for polynomials.
8. Let G be a region and let f and g be analytic functions on G such that $f(z)g(z) = 0$ for all a in G . Show that either $f \equiv 0$ or $g \equiv 0$.
9. Let $U: \mathbb{C} \rightarrow \mathbb{R}$ be a harmonic function such that $U(z) \geq 0$ for all z in \mathbb{C} ; prove that U is constant.
10. Show that if f and g are analytic functions on a region G such that $\bar{f}g$ is analytic then either f is constant or $g \equiv 0$.

§4. The index of a closed curve

We have already shown that $\int_{\gamma} (z-a)^{-1} dz = 2\pi in$ if $\gamma(t) = a + e^{2\pi i n t}$.

4.1 Proposition. If $\gamma: [0, 1] \rightarrow \mathbb{C}$ is a closed rectifiable curve and $a \notin \{\gamma\}$ then

$$\frac{1}{2\pi i} \int_{\gamma} \frac{dz}{z-a}$$

is an integer.

Proof. This is only proved under the hypothesis that γ is smooth. In this case define $g: [0, 1] \rightarrow \mathbb{C}$ by

$$g(t) = \int_0^t \frac{\gamma'(s)}{\gamma(s)-a} ds$$

Hence, $g(0)=0$ and $g(1) = \int_{\gamma} (z-a)^{-1} dz$. We also have that

$$g'(t) = \frac{\gamma'(t)}{\gamma(t)-a} \quad \text{for } 0 \leq t \leq 1.$$

But this gives

$$\begin{aligned} \frac{d}{dt} e^{-g}(\gamma-a) &= e^{-g} \gamma' - g' e^{-g}(\gamma-a) \\ &= e^{-g} [\gamma' - \gamma'(\gamma-a)^{-1}(\gamma-a)] \\ &= 0 \end{aligned}$$

So $e^{-g}(\gamma-a)$ is the constant function $e^{-g(0)}(\gamma(0)-a) = \gamma(0)-a = e^{-g(1)}(\gamma(1)-a)$. Since $\gamma(0)=\gamma(1)$ we have that $e^{-g(1)}=1$ or that $g(1)=2\pi i k$ for some integer k . ■

4.2 Definition. If γ is a closed rectifiable curve in \mathbb{C} then for $a \notin \{\gamma\}$

$$n(\gamma; a) = \frac{1}{2\pi i} \int_{\gamma} (z-a)^{-1} dz$$

is called the *index* of γ with respect to the point a . It is also sometimes called the *winding number* of γ around a .

Recall that if $\gamma: [0, 1] \rightarrow \mathbb{C}$ is a curve, $-\gamma$ or γ^{-1} is the curve defined by $(-\gamma)(t) = \gamma(1-t)$ (this is actually a reparametrization of the original definition). Also if γ and σ are curves defined on $[0, 1]$ with $\gamma(1) = \sigma(0)$ then $\gamma + \sigma$ is the curve $(\gamma + \sigma)(t) = \gamma(2t)$ if $0 \leq t \leq \frac{1}{2}$ and $(\gamma + \sigma)(t) = \sigma(2t-1)$ if $\frac{1}{2} \leq t \leq 1$. The proof of the following proposition is left to the reader.

4.3 Proposition. If γ and σ are closed rectifiable curves having the same initial points then

- (a) $n(\gamma; a) = -n(-\gamma; a)$ for every $a \notin \{\gamma\}$;
- (b) $n(\gamma + \sigma; a) = n(\gamma; a) + n(\sigma; a)$ for every $a \notin \{\gamma\} \cup \{\sigma\}$.

Why is $n(\gamma; a)$ called the winding number of γ about a ? As was said before if $\gamma(t) = a + e^{2\pi i n t}$ for $0 \leq t \leq 1$ then $n(\gamma; a) = n$. In fact if $(b-a) < 1$ then $n(\gamma; b) = n$ and if $|b-a| \geq 1$ then $n(\gamma; b) = 0$. This can be shown directly or one can invoke Theorem 4.4 below. So at least in this case $n(\gamma; b)$ measures the number of times γ wraps around b - with the minus sign indicating that the curve wraps around b in the clockwise direction.

The following discussion is intuitive and mathematically imprecise. Actually, with a little more sophistication this discussion can be corrected and gives insight into the Argument Principle (V.3).

If γ is smooth then

$$\int_{\gamma} (z-a)^{-1} dz = \int_0^1 \frac{\gamma'(t)}{\gamma(t)-a} dt.$$

Taking inspiration from calculus one is tempted to write $\int_{\gamma} (z-a)^{-1} dx = \log[\gamma(t)-a]|_{t=0}^1$. Since $\gamma(1)=\gamma(0)$, this would always give zero. The difficulty lies in the fact that $\gamma(t)-a$ is complex valued and unless $\gamma(t)-a$ lies in a region on which a branch of the logarithm can be defined, the above inspiration turns out to be only so much hot air. In fact if γ wraps around the point a then we cannot define $\log(\gamma(t)-a)$ since there is no analytic branch of the logarithm defined on $\mathbb{C}-\{a\}$.

Nevertheless there is a correct interpretation of the preceding discussion. If we think of $\log z = \log|z| + i \arg z$ as defined then

$$\int_{\gamma} (z-a)^{-1} dz = \log[\gamma(1)-a] - \log[\gamma(0)-a] = \{\log|\gamma(1)-a| - \log|\gamma(0)-a|\} + i\{\arg[\gamma(1)-a] - \arg[\gamma(0)-a]\}.$$

Since the difficulty in defining $\log z$ is in choosing the correct value of $\arg z$, we can think of the real part of the last expression as equal to zero. Since $\gamma(1)=\gamma(0)$ it must be that even with the ambiguity in defining $\arg z$, $\arg[\gamma(1)-a] - \arg[\gamma(0)-a]$ must equal an integral multiple of 2π , and furthermore this integer counts the number of times γ wraps around a .

Let γ be a closed rectifiable curve and consider the open set $G = \mathbb{C} - \{\gamma\}$. Since $\{\gamma\}$ is compact $\{z: |z| > R\} \subset G$ for some sufficiently large R . This says that G has one, and only one, unbounded component.

4.4 Theorem. Let γ be a closed rectifiable curve in \mathbb{C} . Then $n(\gamma; a)$ is constant for a belonging to a component of $G = \mathbb{C} - \{\gamma\}$. Also, $n(\gamma; a) = 0$ for a belonging to the unbounded component of G .

Proof. Define $f: G \rightarrow \mathbb{C}$ by $f(a) = n(\gamma; a)$. It will be shown that f is continuous. If this is done then it follows that $f(D)$ is connected for each component D of G . But since $f(G)$ is contained in the set of integers it follows that $f(D)$ reduces to a single point. That is, f is constant on D .

To show that f is continuous recall that the components of G are open (Theorem II. 2.9). Fix a in G and let $r = d(a, \{\gamma\})$. If $|a-b| < \delta < \frac{1}{2}r$ then

$$\begin{aligned} |f(a) - f(b)| &= \frac{1}{2\pi} \left| \int_{\gamma} [(z-a)^{-1} - (z-b)^{-1}] dz \right| \\ &= \frac{1}{2\pi} \left| \int_{\gamma} \frac{(a-b)}{(z-a)(z-b)} dz \right| \\ &\leq \frac{|a-b|}{2\pi} \int_{\gamma} \frac{|dz|}{|z-a||z-b|} \end{aligned}$$

But for $|a-b| < \frac{1}{2}r$ and z on $\{\gamma\}$ we have that $|z-a| \geq r > \frac{1}{2}r$ and $|z-b| > \frac{1}{2}r$. It follows that $|f(a) - f(b)| < \frac{2\delta}{\pi r^2} V(\gamma)$. So if $\epsilon > 0$ is given then, by choosing δ to be smaller than $\frac{1}{2}r$ and $(\pi r^2 \epsilon) / 2\delta V(\gamma)$, we see that f must be continuous. (Also, see Exercise 2.3.)

Now let U be the unbounded component of G . As was mentioned before the theorem there is an $R > 0$ such that $U \supset \{z: |z| > R\}$. If $\epsilon > 0$ choose a with $|a| > R$ and $|z-a| > (2\pi)^{-1} V(\gamma)$ uniformly for z on $\{\gamma\}$; then $|n(\gamma; a)| < \epsilon$. That is, $n(\gamma; a) \rightarrow 0$ as $a \rightarrow \infty$. Since $n(\gamma; a)$ is constant on U , it must be zero. ■

Exercises

1. Prove Proposition 4.3.
2. Give an example of a closed rectifiable curve γ in \mathbb{C} such that for any integer k there is a point $a \notin \{\gamma\}$ with $n(\gamma; a) = k$.
3. Let $p(z)$ be a polynomial of degree n and let $R > 0$ be sufficiently large so that p never vanishes in $\{z: |z| > R\}$. If $\gamma(t) = Re^{it}$, $0 \leq t \leq 2\pi$, show that $\int_{\gamma} \frac{p'(z)}{p(z)} dz = 2\pi i n$.
4. Fix $w = re^{i\theta} \neq 0$ and let γ be a rectifiable path in $\mathbb{C} - \{0\}$ from 1 to w . Show that there is an integer k such that $\int_{\gamma} z^{-1} dz = \log r + i\theta + 2\pi i k$.

§5. Cauchy's Theorem and Integral Formula

We have already proved Cauchy's Theorem for functions analytic in a disk: if G is an open disk then $\int_{\gamma} f = 0$ for any analytic function f on G and any closed rectifiable curve γ in G (Proposition 2.15). For which regions G does this result remain valid? There are regions for which the result is false. For example, if $G = \mathbb{C} - \{0\}$ and $f(z) = z^{-1}$ then $\gamma(t) = e^{it}$ for $0 \leq t \leq 2\pi$ gives that $\int_{\gamma} f = 2\pi i$. The difficulty with $\mathbb{C} - \{0\}$ is the presence of a hole (namely $\{0\}$). In the next section it will be shown that $\int_{\gamma} f = 0$ for every analytic function f and every closed rectifiable curve γ in regions G that have no "holes."

In the present section we adopt a different approach. Fix a region G and an analytic function f on G . Is there a condition on a closed rectifiable curve γ such that $\int_{\gamma} f = 0$? The answer is furnished by the index of γ with respect to points outside G . Before presenting this result we need the following lemma. (This has already been seen in Exercise 2.3.)

5.1 Lemma. Let γ be a rectifiable curve and suppose φ is a function defined and continuous on $\{\gamma\}$. For each $m \geq 1$ let $F_m(z) = \int_{\gamma} \varphi(w)(w-z)^{-m} dw$ for $z \notin \{\gamma\}$. Then each F_m is analytic on $\mathbb{C} - \{\gamma\}$ and $F'_m(z) = mF_{m+1}(z)$.

Proof. We first claim that each F_m is continuous. In fact, this follows in the same way that we showed that the index was continuous (see the proof of Theorem 4.4). One need only observe that, since $\{\gamma\}$ is compact, φ is

bounded there; and use the factorization.

$$\begin{aligned} \frac{1}{(w-z)^m} - \frac{1}{(w-a)^m} &= \left[\frac{1}{w-z} - \frac{1}{w-a} \right] \sum_{k=1}^m \frac{1}{(w-z)^{m-k}} \frac{1}{(w-a)^{k-1}} \\ 5.2 \quad &= (z-a) \left[\frac{1}{(w-z)^m(w-a)} + \frac{1}{(w-z)^{m-1}(w-a)^2} + \cdots \right. \\ &\quad \left. + \frac{1}{(w-z)(w-a)^m} \right] \end{aligned}$$

The details are left to the reader.

Now fix a in $G = \mathbb{C} - \{\gamma\}$ and let $z \in G$, $z \neq a$. It follows from (5.2) that

$$5.3 \quad \frac{F_m(z) - F_m(a)}{z-a} = \int_{\gamma} \frac{\varphi(w)(w-a)^{-1}}{(w-z)^m} dw + \cdots + \int_{\gamma} \frac{\varphi(w)(w-a)^{-m}}{w-z} dw$$

Since $a \notin \{\gamma\}$, $\varphi(w)(w-a)^{-k}$ is continuous on $\{\gamma\}$ for each k . By the first part of this proof (the part left to the reader), each integral on the right hand side of (5.3) defines a continuous function of z , z in G . Hence letting $z \rightarrow a$, (5.3) gives that the limit exists and

$$\begin{aligned} F'_m(a) &= \int_{\gamma} \frac{\varphi(w)}{(w-a)^{m+1}} dw + \cdots + \int_{\gamma} \frac{\varphi(w)}{(w-a)^{m+1}} dw \\ &= mF_{m+1}(a). \quad \blacksquare \end{aligned}$$

5.4 Cauchy's Integral Formula (First Version). Let G be an open subset of the plane and $f: G \rightarrow \mathbb{C}$ an analytic function. If γ is a closed rectifiable curve in G such that $n(\gamma; w) = 0$ for all w in $\mathbb{C} - G$, then for a in $G - \{\gamma\}$

$$n(\gamma; a)f(a) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(z)}{z-a} dz.$$

Proof. Define $\varphi: G \times G \rightarrow \mathbb{C}$ by $\varphi(z, w) = [f(z) - f(w)]/(z-w)$ if $z \neq w$ and $\varphi(z, z) = f'(z)$. It follows that φ is continuous; and for each w in G , $z \rightarrow \varphi(z, w)$ is analytic (Exercise 1). Let $H = \{w \in \mathbb{C} : n(\gamma; w) = 0\}$. Since $n(\gamma; w)$ is a continuous integer-valued function of w , H is open. Moreover $H \cup G = \mathbb{C}$ by the hypothesis.

Define $g: \mathbb{C} \rightarrow \mathbb{C}$ by $g(z) = \int_{\gamma} \varphi(z, w) dw$ if $z \in G$ and $g(z) = \int_{\gamma} (w-z)^{-1} f(w) dw$ if $z \in H$. If $z \in G \cap H$ then

$$\begin{aligned} \int_{\gamma} \varphi(z, w) dw &= \int_{\gamma} \frac{f(w) - f(z)}{w-z} dw \\ &= \int_{\gamma} \frac{f(w)}{w-z} dw - f(z) n(\gamma; z) 2\pi i \\ &= \int_{\gamma} \frac{f(w)}{w-z} dw. \end{aligned}$$

Hence g is a well-defined function.

By Lemma 5.1 g is analytic on \mathbb{C} ; that is, g is an entire function. But Theorem 4.4 implies that H contains a neighborhood of ∞ in \mathbb{C}_{∞} . Since f is bounded on $\{\gamma\}$ and $\lim_{z \rightarrow \infty} (w-z)^{-1} = 0$ uniformly for w in $\{\gamma\}$,

$$5.5 \quad \lim_{z \rightarrow \infty} g(z) = \lim_{z \rightarrow \infty} \int_{\gamma} \frac{f(w)}{w-z} dw = 0.$$

In particular (5.5) implies there is an $R > 0$ such that $|g(z)| \leq 1$ for $|z| \geq R$. Since g is bounded on $\bar{B}(0; R)$ it follows that g is a bounded entire function. Hence g is constant by Liouville's Theorem. But then (5.5) says that $g \equiv 0$. That is, if $a \in G - \{\gamma\}$ then

$$\begin{aligned} 0 &= \int_{\gamma} \frac{f(z) - f(a)}{z-a} dz \\ &= \int_{\gamma} \frac{f(z)}{z-a} dz - f(a) \int_{\gamma} \frac{dz}{z-a}. \end{aligned}$$

This proves the theorem. \blacksquare

Often there is a need for a more general version of Cauchy's Integral Formula that involves more than one curve. For example in dealing with an annulus one needs a formula involving two curves.

5.6 Cauchy's Integral Formula (Second Version). Let G be an open subset of the plane and $f: G \rightarrow \mathbb{C}$ an analytic function. If $\gamma_1, \dots, \gamma_m$ are closed rectifiable curves in G such that $n(\gamma_1; w) + \cdots + n(\gamma_m; w) = 0$ for all w in $\mathbb{C} - G$, then for a in $G - \{\gamma\}$

$$f(a) \sum_{k=1}^m n(\gamma_k; a) = \sum_{k=1}^m \frac{1}{2\pi i} \int_{\gamma_k} \frac{f(z)}{z-a} dz.$$

Proof. The proof follows the lines of Theorem 5.4. Define $g(z, w)$ as it was done there and let $H = \{w : n(\gamma_1; w) + \cdots + n(\gamma_m; w) = 0\}$. Now $g(z)$ is defined as in the proof of (5.4) except that the sum of the integrals over $\gamma_1, \dots, \gamma_m$ is used. The remaining details of the proof are left to the reader. \blacksquare

Though an easy corollary of the preceding theorem, the next result is very important in the development of the theory of analytic functions.

5.7 Cauchy's Theorem (First Version). Let G be an open subset of the plane and $f: G \rightarrow \mathbb{C}$ an analytic function. If $\gamma_1, \dots, \gamma_m$ are closed rectifiable curves in G such that $n(\gamma_1; w) + \cdots + n(\gamma_m; w) = 0$ for all w in $\mathbb{C} - G$ then

$$\sum_{k=1}^m \int_{\gamma_k} f = 0.$$

Proof. Substitute $f(z)(z-a)$ for f in Theorem 5.6. \blacksquare

Let $G = \{z : R_1 < |z| < R_2\}$ and define curves γ_1 and γ_2 in G by $\gamma_1(t) = r_1 e^{it}$, $\gamma_2(t) = r_2 e^{it}$ for $0 \leq t \leq 2\pi$, where $R_1 < r_1 < r_2 < R_2$. If $|w| < R_1$,

$n(\gamma_1; w) = 1 = -n(\gamma_2; w)$; if $|w| \geq R_2$ then $n(\gamma_1; w) = n(\gamma_2; w) = 0$. So $n(\gamma_1; w) + n(\gamma_2; w) = 0$ for all w in $\mathbb{C} - G$.

5.8 Theorem. Let G be an open subset of the plane and $f: G \rightarrow \mathbb{C}$ an analytic function. If $\gamma_1, \dots, \gamma_m$ are closed rectifiable curves in G such that $n(\gamma_1; w) + \dots + n(\gamma_m; w) = 0$ for all w in $\mathbb{C} - G$ then for a in $G - \{\gamma\}$ and $k \geq 1$

$$f^{(k)}(a) \sum_{j=1}^m n(\gamma_j; a) = k! \sum_{j=1}^m \frac{1}{2\pi i} \int_{\gamma_j} \frac{f(z)}{(z-a)^{k+1}} dz.$$

Proof. This follows immediately by differentiating both sides of the formula in Theorem 5.6 and applying Lemma 5.1. ■

5.9 Corollary. Let G be an open set and $f: G \rightarrow \mathbb{C}$ an analytic function. If γ is a closed rectifiable curve in G such that $n(\gamma; w) = 0$ for all w in $\mathbb{C} - G$ then for a in $G - \{\gamma\}$

$$f^{(k)}(a) n(\gamma; a) = \frac{k!}{2\pi i} \int_{\gamma} \frac{f(z)}{(z-a)^{k+1}} dz.$$

Cauchy's Theorem and Integral Formula is the basic result of complex analysis. With a result that is so fundamental to an entire theory it is usual in mathematics to seek the outer limits of the theorem's validity. Are there other functions that satisfy $\int_{\gamma} f = 0$ for all closed curves γ ? The answer is no as the following converse to Cauchy's Theorem shows.

A closed path T is said to be *triangular* if it is polygonal and has three sides.

5.10 Morera's Theorem. Let G be a region and let $f: G \rightarrow \mathbb{C}$ be a continuous function such that $\int_T f = 0$ for every triangular path T in G ; then f is analytic in G .

Proof. First observe that f will be shown to be analytic if it can be proved that f is analytic on each open disk contained in G . Hence, without loss of generality, we may assume G to be an open disk; suppose $G = B(a; R)$.

Use the hypothesis to prove that f has a primitive. For z in G define $F(z) = \int_{[a, z]} f$. Fix z_0 in G ; then for any point z in G the hypothesis gives that $F(z) = \int_{[a, z_0]} f + \int_{[z_0, z]} f$. Hence

$$\frac{F(z) - F(z_0)}{z - z_0} = \frac{1}{z - z_0} \int_{[z_0, z]} f.$$

This gives

$$\begin{aligned} \frac{F(z) - F(z_0)}{z - z_0} - f(z_0) &= \frac{1}{(z - z_0)} \int_{[z_0, z]} f - f(z_0) \\ &= \frac{1}{(z - z_0)} \int_{[z_0, z]} [f(w) - f(z_0)] dw. \end{aligned}$$

But by taking absolute values

$$\left| \frac{F(z) - F(z_0)}{z - z_0} - f(z_0) \right| \leq |f(z) - f(z_0)|.$$

which shows that

$$\lim_{z \rightarrow z_0} \frac{F(z) - F(z_0)}{z - z_0} = f(z_0). \blacksquare$$

Exercises

1. Suppose $f: G \rightarrow \mathbb{C}$ is analytic and define $\varphi: G \times G \rightarrow \mathbb{C}$ by $\varphi(z, w) = [f(z) - f(w)](z - w)^{-1}$ if $z \neq w$ and $\varphi(z, z) = f'(z)$. Prove that φ is continuous and for each fixed w , $z \rightarrow \varphi(z, w)$ is analytic.
2. Give the details of the proof of Theorem 5.6.
3. Let $B_{\pm} = B(\pm 1; \frac{1}{2})$, $G = B(0; 3) - (B_+ \cup B_-)$. Let $\gamma_1, \gamma_2, \gamma_3$ be curves whose traces are $|z - 1| = 1$, $|z + 1| = 1$, and $|z| = 2$, respectively. Give γ_1, γ_2 , and γ_3 orientations such that $n(\gamma_1; w) + n(\gamma_2; w) + n(\gamma_3; w) = 0$ for all w in $\mathbb{C} - G$.
4. Show that the Integral Formula follows from Cauchy's Theorem.
5. Let γ be a closed rectifiable curve in \mathbb{C} and $a \notin \{\gamma\}$. Show that for $n \geq 2$ $\int_{\gamma} (z - a)^{-n} dz = 0$.
6. Let f be analytic on $D = B(0; 1)$ and suppose $|f(z)| \leq 1$ for $|z| < 1$. Show $|f'(0)| \leq 1$.
7. Let $\gamma(t) = 1 + e^{it}$ for $0 \leq t \leq 2\pi$. Find $\int_{\gamma} \left(\frac{z}{z-1}\right)^n dz$ for all positive integers n .
8. Let G be a region and suppose $f_n: G \rightarrow \mathbb{C}$ is analytic for each $n \geq 1$. Suppose that $\{f_n\}$ converges uniformly to a function $f: G \rightarrow \mathbb{C}$. Show that f is analytic.
9. Show that if $f: \mathbb{C} \rightarrow \mathbb{C}$ is a continuous function such that f is analytic off $[-1, 1]$ then f is an entire function.
10. Use Cauchy's Integral Formula to prove the Cayley-Hamilton Theorem: If A is an $n \times n$ matrix over \mathbb{C} and $f(z) = \det(z - A)$ is the characteristic polynomial of A then $f(A) = 0$. (This exercise was taken from a paper by C. A. McCarthy, *Amer. Math. Monthly*, **82** (1975), 390-391).

§6 The homotopic version of Cauchy's Theorem and simple connectivity

This section presents a condition on a closed curve γ such that $\int_{\gamma} f = 0$ for an analytic function. This condition is less general but more geometric than the winding number condition of Theorem 5.7. This condition is also used to introduce the concept of a simply connected region; in a simply connected region Cauchy's Theorem is valid for every analytic function and every closed rectifiable curve. Let us illustrate this condition by

considering a closed rectifiable curve in a disk, a region where Cauchy's Theorem is always valid (Proposition 2.15).

Let $G = B(a; R)$ and let $\gamma: [0, 1] \rightarrow G$ be a closed rectifiable curve. If $0 \leq t \leq 1$ and $0 \leq s \leq 1$, and we put $z = ta + (1-t)\gamma(s)$; then z lies on the straight line segment from a to $\gamma(s)$. Hence, z must lie in G . Let $\gamma_t(s) = ta + (1-t)\gamma(s)$ for $0 \leq s \leq 1$ and $0 \leq t \leq 1$. So, $\gamma_0 = \gamma$ and γ_1 is the curve constantly equal to a ; the curves γ_t are somewhere in between. We were able to draw γ down to a because there were no holes. If a point inside γ were missing from G (imagine a stick protruding up from the disk with its base inside γ), then as γ shrinks it would get caught on the hole and could not go to the constant curve.

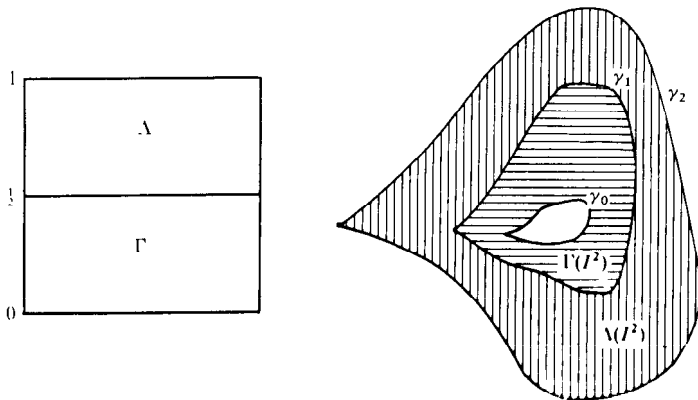
6.1 Definition. Let $\gamma_0, \gamma_1: [0, 1] \rightarrow G$ be two closed rectifiable curves in a region G ; then γ_0 is *homotopic* to γ_1 in G if there is a continuous function $\Gamma: [0, 1] \times [0, 1] \rightarrow G$ such that

$$\text{6.2} \quad \begin{cases} \Gamma(s, 0) = \gamma_0(s) & \text{and} & \Gamma(s, 1) = \gamma_1(s) & (0 \leq s \leq 1) \\ \Gamma(0, t) = \Gamma(1, t) & (0 \leq t \leq 1) \end{cases}$$

So if we define $\gamma_t: [0, 1] \rightarrow G$ by $\gamma_t(s) = \Gamma(s, t)$ then each γ_t is a closed curve and they form a continuous family of curves which start at γ_0 and go to γ_1 . Notice however that there is no requirement that each γ_t be rectifiable. In practice when γ_0 and γ_1 are rectifiable (or smooth) each of the γ_t will also be rectifiable (or smooth).

If γ_0 is homotopic to γ_1 in G write $\gamma_0 \sim \gamma_1$. Actually a notation such as $\gamma_0 \sim \gamma_1(G)$ should be used because of the role of G . If the range of Γ is not required to be in G then, as we shall see shortly, all curves would be homotopic. However, unless there is the possibility of confusion, we will only write $\gamma_0 \sim \gamma_1$.

It is easy to show that " \sim " is an equivalence relation. Clearly any curve is homotopic to itself. If $\gamma_0 \sim \gamma_1$ and $\Gamma: [0, 1] \times [0, 1] \rightarrow G$ satisfies (6.2) then define $\Lambda(s, t) = \Gamma(s, 1-t)$ to see that $\gamma_1 \sim \gamma_0$. Finally, if $\gamma_0 \sim \gamma_1$ and $\gamma_1 \sim \gamma_2$ with Γ satisfying (6.2) and $\Lambda: [0, 1] \times [0, 1] \rightarrow G$ satisfying $\Lambda(s, 0) =$



$\gamma_1(s)$, $\Lambda(s, 1) = \gamma_2(s)$, and $\Lambda(0, t) = \Lambda(1, t)$ for all s and t ; define $\Phi[0, 1] \times [0, 1] \rightarrow G$ by

$$\Phi(s, t) = \begin{cases} \Gamma(s, 2t) & \text{if } 0 \leq t \leq \frac{1}{2} \\ \Lambda(s, 2t-1) & \text{if } \frac{1}{2} \leq t \leq 1 \end{cases}$$

Then Φ is continuous and shows that $\gamma_0 \sim \gamma_2$.

6.3 Definition. A set G is *convex* if given any two points a and b in G the line segment joining a and b , $[a, b]$, lies entirely in G . The set G is *star shaped* if there is a point a in G such that for each z in G , the line segment $[a, z]$ lies entirely in G . Clearly each convex set is star shaped but the converse is just as clearly false.

We will say that G is *a-star shaped* if $[a, z] \subset G$ whenever $z \in G$. If G is *a-star shaped* and z and w are points in G then $[z, a, w]$ is a polygon in G connecting z and w . Hence, each star shaped set is connected.

6.4 Proposition. Let G be an open set which is *a-star shaped*. If γ_0 is the curve which is constantly equal to a then every closed rectifiable curve in G is homotopic to γ_0 .

Proof. Let γ_1 be a closed rectifiable curve in G and put $\Gamma(s, t) = t\gamma_1(s) + (1-t)a$. Because G is *a-star shaped*, $\Gamma(s, t) \in G$ for $0 \leq s, t \leq 1$. It is easy to see that Γ satisfies (6.2). ■

The situation in which a curve is homotopic to a constant curve is one that we will often encounter. Hence it is convenient to introduce some new terminology.

6.5 Definition. If γ is a closed rectifiable curve in G then γ is *homotopic to zero* ($\gamma \sim 0$) if γ is homotopic to a constant curve.

6.6 Cauchy's Theorem (Second Version). If $f: G \rightarrow \mathbb{C}$ is an analytic function and γ is a closed rectifiable curve in G such that $\gamma \sim 0$, then

$$\int_{\gamma} f = 0.$$

This version of Cauchy's Theorem would follow immediately from the first version if it could be shown that $n(\gamma; w) = 0$ for all w in $\mathbb{C} - G$ whenever $\gamma \sim 0$. This can be done. A plausible argument proceeds as follows.

Let $\gamma_1 = \gamma$ and let γ_0 be a constant curve such that $\gamma_1 \sim \gamma_0$. Let Γ satisfy (6.2) and define $h(t) = n(\gamma_t; w)$, where $\gamma_t(s) = \Gamma(s, t)$ for $0 \leq s, t \leq 1$ and w is fixed in $\mathbb{C} - G$. Now show that h is continuous on $[0, 1]$. Since h is integer valued and $h(0) = 0$ it must be that $h(t) \equiv 0$. In particular, $n(\gamma; w) = 0$ for all w in $\mathbb{C} - G$.

The only point of difficulty with this argument is that for $0 < t < 1$ it may be that γ_t is not rectifiable.

As was stated after Definition 6.1, in practice each of the curves γ_t will not only be rectifiable but also smooth. So there is justification in making

this assumption and providing the details to transform the preceding paragraph into a legitimate proof (Exercise 9). Indeed, in a course designed for physicists and engineers this is probably preferable. But this is not desirable for the training of mathematicians.

The statement of a theorem is not as important as its proof. Proofs are important in mathematics for several reasons, not the least of which is that a proof deepens our insight into the meaning of the theorems and gives a natural delineation of the extent of the theorem's validity. Most important for the education of a mathematician, it is essential to examine other proofs because they reveal methods.

A good method is worth a thousand theorems. Not only is this statement valid as a value judgement, but also in a literal sense. An important method can be reused in other situations to obtain further results.

With this in mind a complete proof of Theorem 6.6 will be presented. In fact, we will prove a somewhat more general fact since the proof of this new result necessitates only a little more effort than the proof that $n(\gamma; w) = 0$ for w in $\mathbb{C} - G$ whenever $\gamma \sim 0$. In fact, the proof of the next result more clearly reveals the usefulness of the method.

6.7 Cauchy's Theorem (Third Version). *If γ_0 and γ_1 are two closed rectifiable curves in G and $\gamma_0 \sim \gamma_1$ then*

$$\int_{\gamma_0} f = \int_{\gamma_1} f$$

for every function f analytic on G .

Before proceeding let us consider a special case. Suppose Γ satisfies (6.2) and also suppose Γ has continuous second partial derivatives. Hence

$$\frac{\partial^2 \Gamma}{\partial s \partial t} = \frac{\partial^2 \Gamma}{\partial t \partial s}$$

throughout the square $I^2 = [0, 1] \times [0, 1]$. Define

$$g(t) = \int_0^1 f(\Gamma(s, t)) \frac{\partial \Gamma}{\partial s}(s, t) ds;$$

then $g(0) = \int_{\gamma_0} f$ and $g(1) = \int_{\gamma_1} f$. By Leibniz's rule g has a continuous derivative,

$$g'(t) = \int_0^1 \left[f'(\Gamma(s, t)) \frac{\partial \Gamma}{\partial s} \frac{\partial \Gamma}{\partial t} + f(\Gamma(s, t)) \frac{\partial^2 \Gamma}{\partial t \partial s} \right] ds$$

But

$$\frac{\partial}{\partial s} \left[(f' \circ \Gamma) \frac{\partial \Gamma}{\partial t} \right] = (f' \circ \Gamma) \frac{\partial^2 \Gamma}{\partial s \partial t} + f' \circ \Gamma \frac{\partial^2 \Gamma}{\partial s \partial t};$$

hence

$$g'(t) = f'(\Gamma(1, t)) \frac{\partial \Gamma}{\partial t}(1, t) - f'(\Gamma(0, t)) \frac{\partial \Gamma}{\partial t}(0, t).$$

Since $\Gamma(1, t) = \Gamma(0, t)$ for all t we get $g'(t) = 0$ for all t . So g is a constant; in particular $\int_{\gamma_0} f = \int_{\gamma_1} f$.

Proof of Theorem 6.7. Let $\Gamma: I^2 \rightarrow G$ satisfy (6.2). Since Γ is continuous and I^2 is compact, Γ is uniformly continuous and $\Gamma(I^2)$ is a compact subset of G . Thus

$$r = d(\Gamma(I^2), \mathbb{C} - G) > 0$$

and there is an integer n such that if $(s-s')^2 + (t-t')^2 < 4/n^2$ then

$$|\Gamma(s, t) - \Gamma(s', t')| < r.$$

Let

$$Z_{jk} = \Gamma\left(\frac{j}{n}, \frac{k}{n}\right), 0 \leq j, k \leq n$$

and put

$$J_{jk} = \left[\frac{j}{n}, \frac{j+1}{n}\right] \times \left[\frac{k}{n}, \frac{k+1}{n}\right]$$

for $0 \leq j, k \leq n-1$. Since the diameter of the square J_{jk} is $\frac{\sqrt{2}}{n}$, it follows that $\Gamma(J_{jk}) \subset B(Z_{jk}, r)$. So if we let P_{jk} be the closed polygon $[Z_{jk}, Z_{j+1, k}, Z_{j+1, k+1}, Z_{j, k+1}, Z_{jk}]$; then, because disks are convex, $P_{jk} \subset B(Z_{jk}, r)$. But from Proposition 2.15 it is known that

$$\oint_{P_{jk}} f = 0 \quad (6.8)$$

for any function f analytic in G .

It can now be shown that $\int_{\gamma_0} f = \int_{\gamma_1} f$ by going up the ladder we have constructed, one rung at a time. That is, let Q_k be the closed polygon $[Z_{0, k}, Z_{1, k}, \dots, Z_{n, k}]$. We will show that $\int_{\gamma_0} f = \int_{Q_0} f = \int_{Q_1} f = \dots = \int_{Q_n} f = \int_{\gamma_1} f$ (one rung at a time!). To see that $\int_{\gamma_0} f = \int_{Q_0} f$ observe that if $\sigma_j(t) = \gamma_0(t)$ for

$$\frac{j}{n} \leq t \leq \frac{j+1}{n}$$

then $\sigma_j + [Z_{j+1, 0}, Z_{j0}]$ (the $+$ indicating that σ_j is to be followed by the polygon) is a closed rectifiable curve in the disk $B(Z_{j0}; r) \subset G$. Hence

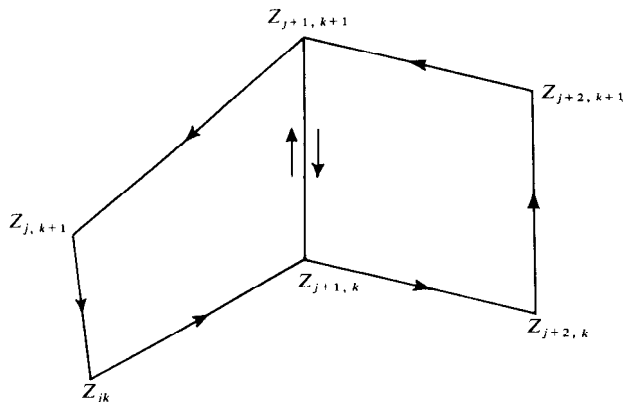
$$\int_{\sigma_j} f = - \int_{[Z_{j+1, 0}, Z_{j0}]} f = \int_{[Z_{j0}, Z_{j+1, 0}]} f.$$

Adding both sides of this equation for $0 \leq j \leq n$ yields $\int_{\gamma_0} f = \int_{Q_0} f$. Similarly $\int_{\gamma_1} f = \int_{Q_n} f$.

To see that $\int_{Q_k} f = \int_{Q_{k+1}} f$ use equation (6.8); this gives

6.9

$$0 = \sum_{j=0}^{n-1} \int_{P_{jk}} f$$



However, notice that the integral $\int_{P_{jk}} f$ includes the integral over $[Z_{j+1,k}, Z_{j+1,k+1}]$, which is the negative of the integral over $[Z_{j+1,k+1}, Z_{j+1,k}]$, which is part of the integral $\int_{P_{j+1,k}} f$. Also,

$$Z_{0,k} = \Gamma\left(0, \frac{k}{n}\right) = \Gamma\left(1, \frac{k}{n}\right) = Z_{1,k}$$

so that $[Z_{0,k+1}, Z_{0,k}] = -[Z_{1k}, Z_{1,k+1}]$. Hence, taking these cancellations into consideration, equation (6.9) becomes

$$0 = \int_{Q_k} f - \int_{Q_{k+1}} f$$

This completes the proof of the theorem. ■

The second version of Cauchy's Theorem immediately follows by letting γ_1 be a constant path in (6.7).

6.10 Corollary. If γ is a closed rectifiable curve in G such that $\gamma \sim 0$ then $n(\gamma; w) = 0$ for all w in $\mathbb{C} - G$.

The converse of the above corollary is not valid. That is, there is a closed rectifiable curve γ in a region G such that $n(\gamma; w) = 0$ for all w in $\mathbb{C} - G$ but γ is not homotopic to a constant curve (Exercise 8).

If G is an open set and γ_0 and γ_1 are closed rectifiable curves in G then $n(\gamma_0; a) = n(\gamma_1; a)$ for each a in $\mathbb{C} - G$ provided $\gamma_0 \sim \gamma_1(G)$. Let $\gamma_0(t) = e^{2\pi i t}$ and $\gamma_1(t) = e^{-2\pi i t}$ for $0 \leq t \leq 1$. Then $n(\gamma_0; 0) = 1$ and $n(\gamma_1; 0) = -1$ so that γ_0 and γ_1 are not homotopic in $\mathbb{C} - \{0\}$.

6.11 Definition. If $\gamma_0, \gamma_1: [0, 1] \rightarrow G$ are two rectifiable curves in G such that $\gamma_0(0) = \gamma_1(0) = a$ and $\gamma_0(1) = \gamma_1(1) = b$ then γ_0 and γ_1 are *fixed-end-point*

homotopic (FEP homotopic) if there is a continuous map $\Gamma: I^2 \rightarrow G$ such that

6.12

$$\Gamma(s, 0) = \gamma_0(s) \quad \Gamma(s, 1) = \gamma_1(s)$$

$$\Gamma(0, t) = a \quad \Gamma(1, t) = b$$

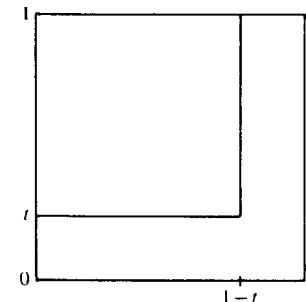
for $0 \leq s, t \leq 1$.

Again the relation of FEP homotopic is an equivalence relationship on curves from one given point to another (Exercise 3).

Notice that if γ_0 and γ_1 are rectifiable curves from a to b then $\gamma_0 - \gamma_1$ is a closed rectifiable curve. Suppose Γ satisfies (6.12) and define $\gamma: [0, 1] \rightarrow G$ by $\gamma(s) = \gamma_0(3s)$ for $0 \leq s \leq \frac{1}{3}$; $\gamma(s) = b$ for $\frac{1}{3} \leq s \leq \frac{2}{3}$; and $\gamma(s) = \gamma_1(3-3s)$ for $\frac{2}{3} \leq s \leq 1$. We will show that $\gamma \sim 0$. In fact, define $\Lambda: I^2 \rightarrow G$ by

$$\Lambda(s, t) = \begin{cases} \Gamma(3s(1-t), t) & \text{if } 0 \leq s \leq \frac{1}{3} \\ \Gamma(1-t, 3s-1+2t-3st) & \text{if } \frac{1}{3} \leq s \leq \frac{2}{3} \\ \gamma_1((3-3s)(1-t)) & \text{if } \frac{2}{3} \leq s \leq 1. \end{cases}$$

Although this formula may appear mysterious it can easily be understood by seeing that for a given value of t , Λ is the restriction of Γ to the boundary of the square $[0, 1-t] \times [t, 1]$ (see the figure). It is left to the reader to check that Λ shows $\gamma \sim 0$.



Hence, for f analytic on G the second version of Cauchy's Theorem gives

$$0 = \int_{\gamma} f = \int_{\gamma_0} f - \int_{\gamma_1} f.$$

This is summarized in the following.

6.13 Independence of Path Theorem. If γ_0 and γ_1 are two rectifiable curves in G from a to b and γ_0 and γ_1 are FEP homotopic then

$$\int_{\gamma_0} f = \int_{\gamma_1} f$$

for any function f analytic in G .

Those regions G for which the integral of an analytic function around a closed curve is always zero can be characterized.

6.14 Definition. An open set G is *simply connected* if G is connected and every closed curve in G is homotopic to zero.

6.15 Cauchy's Theorem (Fourth Version). If G is simply connected then $\int_\gamma f = 0$ for every closed rectifiable curve and every analytic function f .

Let us now take a few moments to digest the concept of simple connectedness. Clearly every star shaped region is simply connected. Also, examine the complement of the spiral $r = \theta$. That is, let $G = \mathbb{C} - \{\theta e^{i\theta} : 0 \leq \theta < \infty\}$; then G is simply connected. In fact, it is easily seen that G is open and connected. If one argues in an intuitive way it is not difficult to become convinced that every curve in G is homotopic to zero. A rigorous proof will be postponed until we have proved the following: *A region G is simply connected iff $\mathbb{C}_\infty - G$, its complement in the extended plane, is connected in \mathbb{C}_∞ .* This will not be proved until Chapter VIII. If this criterion is applied to the region G above then G is simply connected since $\mathbb{C}_\infty - G$ consists of the spiral $r = \theta$ and the point at infinity.

Notice that for $G = \mathbb{C} - \{0\}$, $\mathbb{C} - G = \{0\}$ is connected but $\mathbb{C}_\infty - G = \{0, \infty\}$ is not. Also, the domain of the principal branch of the logarithm is simply connected.

It was shown earlier in this chapter (Corollary 1.22) that if an analytic function f has a primitive in a region G then the integral of f around every closed rectifiable curve in G is zero. The next result should not be too surprising in light of this.

6.16 Corollary. If G is simply connected and $f: G \rightarrow \mathbb{C}$ is analytic in G then f has a primitive in G .

Proof. Fix a point a in G and let γ_1, γ_2 be any two rectifiable curves in G from a to a point z in G . (Since G is open and connected there is always a path from a to any other point of G .) Then, by Theorem 6.15, $0 = \int_{\gamma_1 - \gamma_2} f = \int_{\gamma_1} f - \int_{\gamma_2} f$ (where $\gamma_1 - \gamma_2$ is the curve which goes from a to z along γ_1 and then back to a along $-\gamma_2$). Hence we can get a well defined function $F: G \rightarrow \mathbb{C}$ by setting $F(z) = \int_\gamma f$ where γ is any rectifiable curve from a to z . We claim that F is a primitive of f .

If $z_0 \in G$ and $r > 0$ is such that $B(z_0; r) \subset G$, then let γ be a path from a to z_0 . For z in $B(z_0; r)$ let $\gamma_z = \gamma + [z_0, z]$; that is, γ_z is the path γ followed by the straight line segment from z_0 to z . Hence

$$\frac{F(z) - F(z_0)}{z - z_0} = \frac{1}{(z - z_0)} \int_{[z_0, z]} f.$$

Now proceed as in the proof of Morera's Theorem to show that $F'(z_0) = f(z_0)$. ■

Perhaps a somewhat less expected consequence of simple connectedness is the fact that a branch of $\log f(z)$ where f is analytic and never vanishes, can be defined on a simply connected region. Nevertheless this is a direct consequence of the preceding corollary.

6.17 Corollary. Let G be simply connected and let $f: G \rightarrow \mathbb{C}$ be an analytic function such that $f(z) \neq 0$ for any z in G . Then there is an analytic function

$g: G \rightarrow \mathbb{C}$ such that $f(z) = \exp g(z)$. If $z_0 \in G$ and $e^{w_0} = f(z_0)$, we may choose g such that $g(z_0) = w_0$.

Proof. Since f never vanishes, $\frac{f'}{f}$ is analytic on G ; so, by the preceding corollary, it must have a primitive g_1 . If $h(z) = \exp g_1(z)$ then h is analytic and never vanishes. So, $\frac{f}{h}$ is analytic and its derivative is

$$\frac{h(z)f'(z) - h'(z)f(z)}{h(z)^2}$$

But $h' = g_1' h$ so that $hf' - fh' = 0$. Hence f/h is a constant c for all z in G . That is $f(z) = c \exp g_1(z) = \exp [g_1(z) + c']$ for some c' . By letting $g(z) = g_1(z) + c' + 2\pi i k$ for an appropriate k , $g(z_0) = w_0$ and the theorem is proved. ■

Let us emphasize that the hypothesis of simple connectedness is a topological one and this was used to obtain some basic results of analysis. Not only are these last three theorems (6.15, 6.16, and 6.17) consequences of simple connectivity, but they are equivalent to it. It will be shown in Chapter VIII that if a region G has the conclusion of each of these theorems satisfied for every function analytic on G , then G must be simply connected.

We close this section with a definition.

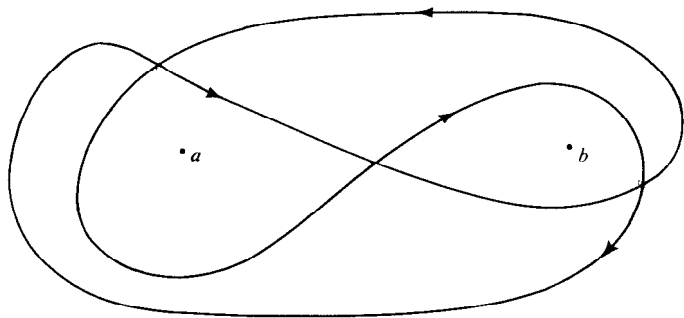
6.18 Definition. If G is an open set then γ is homologous to zero, in symbols $\gamma \approx 0$, if $n(\gamma; w) = 0$ for all w in $\mathbb{C} - G$.

Using this notation, Corollary 6.10 says that $\gamma \sim 0$ implies $\gamma \approx 0$. This result appears in Algebraic Topology when it is shown that the first homology group of a space is isomorphic to the abelianization of the fundamental group. In fact, those familiar with homology theory will recognize in the proof of Theorem 6.7 the elements of simplicial approximation.

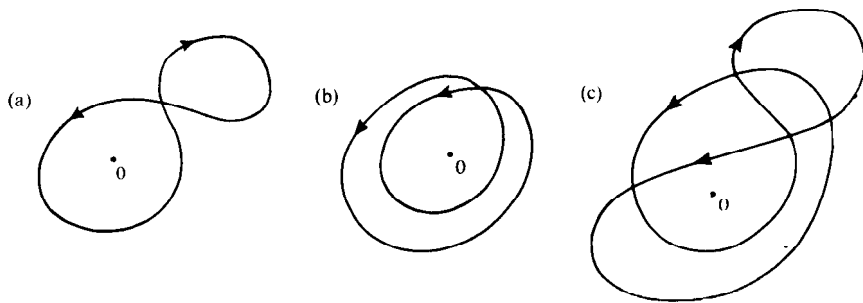
Exercises

1. Let G be a region and let $\sigma_1, \sigma_2: [0, 1] \rightarrow G$ be the constant curves $\sigma_1(t) \equiv a, \sigma_2(t) \equiv b$. Show that if γ is closed rectifiable curve in G and $\gamma \sim \sigma_1$ then $\gamma \sim \sigma_2$. (Hint: connect a and b by a curve.)
2. Show that if we remove the requirement " $\Gamma(0, t) = \Gamma(1, t)$ for all t " from Definition 6.1 then the curve $\gamma_0(t) = e^{2\pi i t}, 0 \leq t \leq 1$, is homotopic to the constant curve $\gamma_1(t) \equiv 1$ in the region $G = \mathbb{C} - \{0\}$.
3. Let \mathcal{C} = all rectifiable curves in G joining a to b and show that Definition 6.11 gives an equivalence relation on \mathcal{C} .
4. Let $G = \mathbb{C} - \{0\}$ and show that every closed curve in G is homotopic to a closed curve whose trace is contained in $\{z: |z| = 1\}$.

5. Evaluate the integral $\int_{\gamma} \frac{dz}{z^2+1}$ where $\gamma(\theta) = 2|\cos 2\theta| e^{i\theta}$ for $0 \leq \theta \leq 2\pi$.
6. Let $\gamma(\theta) = \theta e^{i\theta}$ for $0 \leq \theta \leq 2\pi$ and $\gamma(\theta) = 4\pi - \theta$ for $2\pi \leq \theta \leq 4\pi$. Evaluate $\int_{\gamma} \frac{dz}{z^2 + \pi^2}$.
7. Let $f(z) = [(z - \frac{1}{2} - i) \cdot (z - 1 - \frac{3}{2}i) \cdot (z - 1 - \frac{i}{2}) \cdot (z - \frac{3}{2} - i)]^{-1}$ and let γ be the polygon $[0, 2, 2+2i, 2i, 0]$. Find $\int_{\gamma} f$.
8. Let $G = \mathbb{C} - \{a, b\}$, $a \neq b$, and let γ be the curve in the figure below.



- (a) Show that $n(\gamma; a) = n(\gamma; b) = 0$.
- (b) Convince yourself that γ is not homotopic to zero. (Notice that the word is “convince” and not “prove”. Can you prove it?) Notice that this example shows that it is possible to have a closed curve γ in a region such that $n(\gamma; z) = 0$ for all z not in G without γ being homotopic to zero. That is, the converse to Corollary 6.10 is false.
9. Let G be a region and let γ_0 and γ_1 be two closed smooth curves in G . Suppose $\gamma_0 \sim \gamma_1$ and Γ satisfies (6.2). Also suppose that $\gamma_t(s) = \Gamma(s, t)$ is smooth for each t . If $w \in \mathbb{C} - G$ define $h(t) = n(\gamma_t; w)$ and show that $h: [0, 1] \rightarrow \mathbb{Z}$ is continuous.
10. Find all possible values of $\int_{\gamma} \frac{dz}{1+z^2}$ where γ is any closed rectifiable curve in \mathbb{C} not passing through $\pm i$.
11. Evaluate $\int_{\gamma} \frac{e^z - e^{-z}}{z^4} dz$ where γ is one of the curves depicted below. (Justify your answer.)



§7. Counting zeros; the Open Mapping Theorem

In this section some applications of Cauchy's Integral Theorem are given. It is shown how to count the number of zeros inside a curve; also, using some information on the existence of roots of an analytic equation, it will be proved that a non-constant analytic function on a region maps open sets onto open sets.

In section 3 it was shown that if an analytic function f had a zero at $z = a$ we could write $f(z) = (z-a)^m g(z)$ where g is analytic and $g(a) \neq 0$. Suppose G is a region and let f be analytic in G with zeros at a_1, \dots, a_m . (Where some of the a_k may be repeated according to the multiplicity of the zero.) So we can write $f(z) = (z-a_1)(z-a_2) \dots (z-a_m)g(z)$ where g is analytic on G and $g(z) \neq 0$ for any z in G . Applying the formula for differentiating a product gives

$$7.1 \quad \frac{f'(z)}{f(z)} = \frac{1}{z-a_1} + \frac{1}{z-a_2} + \dots + \frac{1}{z-a_m} + \frac{g'(z)}{g(z)}$$

for $z \neq a_1, \dots, a_m$. Now that this is done, the proof of the following theorem is straightforward.

7.2 Theorem. Let G be a region and let f be an analytic function on G with zeros a_1, \dots, a_m (repeated according to multiplicity). If γ is a closed rectifiable curve in G which does not pass through any point a_k and if $\gamma \approx 0$ then

$$\frac{1}{2\pi i} \int_{\gamma} \frac{f'(z)}{f(z)} dz = \sum_{k=1}^m n(\gamma; a_k)$$

Proof. If $g(z) \neq 0$ for any z in G then $g'(z)/g(z)$ is analytic in G ; since $\gamma \approx 0$, Cauchy's Theorem gives $\int_{\gamma} \frac{g'(z)}{g(z)} dz = 0$. So, using (7.1) and the definition of the index, the proof of the theorem is finished. ■

7.3 Corollary. Let f , G , and γ be as in the preceding theorem except that a_1, \dots, a_m are the points in G that satisfy the equation $f(z) = \alpha$; then

$$\frac{1}{2\pi i} \int_{\gamma} \frac{f'(z)}{f(z) - \alpha} dz = \sum_{k=1}^m n(\gamma; a_k)$$

As an illustration, let us calculate $\int_{\gamma} \frac{(2z+1)}{z^2+z+1} dz$ where γ is the circle $|z| = 2$. Since the denominator of the integrand factors into $(z-\omega_1)(z-\omega_2)$, where ω_1 and ω_2 are the non-real cubic roots of 1, Theorem 7.2 gives

$$\int_{\gamma} \frac{2z+1}{z^2+z+1} dz = 4\pi i.$$

As another illustration, let $\gamma: [0, 1] \rightarrow G$ be a closed rectifiable curve in \mathbb{C} , $\gamma \approx 0$. Suppose that f is analytic in G . Then $f \circ \gamma = \sigma$ is a closed rectifiable curve in \mathbb{C} (Exercise 1). Suppose that α is some complex number with $\alpha \notin \{\sigma\} = f(\{\gamma\})$, and let us calculate $n(\sigma; \alpha)$. We get

$$\begin{aligned} n(\sigma; \alpha) &= \frac{1}{2\pi i} \int_{\sigma} \frac{dw}{w - \alpha} \\ &= \frac{1}{2\pi i} \int_{\gamma} \frac{f'(z)}{f(z) - \alpha} dz \\ &= \sum_{k=1}^m n(\gamma; a_k) \end{aligned}$$

where a_k are the points in G with $f(a_k) = \alpha$. (To show the second equality above takes a little effort, although for γ smooth it is easy. The details are left to the reader.)

Note. It may be that there are infinitely many points in G that satisfy the equation $f(z) = \alpha$. However, from what we have proved, this sequence must converge to the boundary of G . It follows that $n(\gamma; z) \neq 0$ for at most a finite number of solutions of $f(z) = \alpha$. (See Exercise 2.)

Now if β in $\mathbb{C} - \{\sigma\}$ belongs to the same component of $\mathbb{C} - \{\sigma\}$ as does α , then $n(\sigma; \alpha) = n(\sigma; \beta)$; or,

$$\sum_k n(\gamma; z_k(\alpha)) = \sum_j n(\gamma; z_j(\beta))$$

where $z_k(\alpha)$ and $z_j(\beta)$ are the points in G that satisfy $f(z) = \alpha$ and $f(z) = \beta$ respectively. If we had chosen γ so that $n(\gamma; z_k(\alpha)) = 1$ for each k , we would have that $f(G)$ contains the component of $\mathbb{C} - f(\{\gamma\})$ that contains α . We would also have some information about the number of solutions of $f(z) = \beta$. This procedure is used to prove the following result which, in addition to being of interest in itself, will yield the Open Mapping Theorem as a consequence.

7.4 Theorem. Suppose f is analytic in $B(a; R)$ and let $\alpha = f(a)$. If $f(z) - \alpha$ has a zero of order m at $z = a$ then there is an $\epsilon > 0$ and $\delta > 0$ such that for $|z - a| < \delta$, the equation $f(z) = \zeta$ has exactly m simple roots in $B(a; \epsilon)$.

A simple root of $f(z) = \zeta$ is a zero of $f(z) - \zeta$ of multiplicity 1. Notice that this theorem says that $f(B(a; \epsilon)) \supset B(\alpha; \delta)$. Also, the condition that $f(z) - \alpha$ have a zero of finite multiplicity guarantees that f is not constant.

Proof of Theorem. Since the zeros of an analytic function are isolated we can choose $\epsilon > 0$ such that $\epsilon < \frac{1}{2}R$, $f(z) = \alpha$ has no solutions with $0 < |z - a| < 2\epsilon$, and $f'(z) \neq 0$ if $0 < |z - a| < 2\epsilon$. (If $m \geq 2$ then $f'(a) = 0$.) Let $\gamma(t) = a + \epsilon \exp(2\pi i t)$, $0 \leq t \leq 1$, and put $\sigma = f \circ \gamma$. Now $\alpha \notin \{\sigma\}$; so there is a $\delta > 0$ such that $B(\alpha; \delta) \cap \{\sigma\} = \emptyset$. Thus, $B(\alpha; \delta)$ is contained in the same component of $\mathbb{C} - \{\sigma\}$; that is, $|\alpha - \zeta| < \delta$ implies $n(\sigma; \alpha) = n(\sigma; \zeta) = \sum_{k=1}^p n(\gamma; z_k(\zeta))$. But since $n(\gamma; z)$ must be either zero or one, we have that there are exactly m solutions to the equation $f(z) = \zeta$ inside $B(a; \epsilon)$. Since $f'(z) \neq 0$ for $0 < |z - a| < \epsilon$, each of these roots (for $\zeta \neq \alpha$) must be simple (Exercise 3). ■

7.5 Open Mapping Theorem. Let G be a region and suppose that f is a non constant analytic function on G . Then for any open set U in G , $f(U)$ is open.

Proof. If $U \subset G$ is open, then we will have shown that $f(U)$ is open if we can show that for each a in U there is a $\delta > 0$ such that $B(\alpha; \delta) \subset f(U)$, where $\alpha = f(a)$. But only part of the strength of the preceding theorem is needed to find an $\epsilon > 0$ and a $\delta > 0$ such that $B(a; \epsilon) \subset U$ and $f(B(a; \epsilon)) \supset B(\alpha; \delta)$. ■

If X and Ω are metric spaces and $f: X \rightarrow \Omega$ has the property that $f(U)$ is open in Ω whenever U is open in X , then f is called an *open map*. If f is a one-one and onto map then we can define the inverse map $f^{-1}: \Omega \rightarrow X$ by $f^{-1}(\omega) = x$ where $f(x) = \omega$. It follows that f^{-1} is continuous exactly when f is open; in fact, for $U \subset X$, $(f^{-1})^{-1}(U) = f(U)$.

7.6 Corollary. Suppose $f: G \rightarrow \mathbb{C}$ is one-one, analytic and $f(G) = \Omega$. Then $f^{-1}: \Omega \rightarrow \mathbb{C}$ is analytic and $(f^{-1})'(\omega) = [f'(z)]^{-1}$ where $\omega = f(z)$.

Proof. By the Open Mapping Theorem, f^{-1} is continuous and Ω is open. Since $z = f^{-1}(f(z))$ for each $z \in \Omega$, the result follows from Proposition III. 2.20. ■

Exercises

1. Show that if $f: G \rightarrow \mathbb{C}$ is analytic and γ is a rectifiable curve in G then $f \circ \gamma$ is also a rectifiable curve. (First assume G is a disk.)
2. Let G be open and suppose that γ is a closed rectifiable curve in G such that $\gamma \approx 0$. Set $r = d(\{\gamma\}, \partial G)$ and $H = \{z \in \mathbb{C}: n(\gamma; z) = 0\}$. (a) Show that $\{z: d(z, \partial G) < \frac{1}{2}r\} \subset H$. (b) Use part (a) to show that if $f: G \rightarrow \mathbb{C}$ is analytic then $f(z) = \alpha$ has at most a finite number of solutions z such that $n(\gamma; z) \neq 0$.
3. Let f be analytic in $B(a; R)$ and suppose that $f(a) = 0$. Show that a is a zero of multiplicity m iff $f^{(m-1)}(a) = \dots = f'(a) = 0$ and $f^{(m)}(a) \neq 0$.

4. Suppose that $f: G \rightarrow \mathbb{C}$ is analytic and one-one; show that $f'(z) \neq 0$ for any z in G .
5. Let X and Ω be metric spaces and suppose that $f: X \rightarrow \Omega$ is one-one and onto. Show that f is an open map iff f is a closed map. (A function f is a closed map if it takes closed sets onto closed sets.)
6. Let $P: \mathbb{C} \rightarrow \mathbb{R}$ be defined by $P(z) = \operatorname{Re} z$; show that P is an open map but is not a closed map. (Hint: Consider the set $F = \{z: \operatorname{Im} z = (\operatorname{Re} z)^{-1} \text{ and } \operatorname{Re} z \neq 0\}$.)
7. Use Theorem 7.2 to give another proof of the Fundamental Theorem of Algebra.

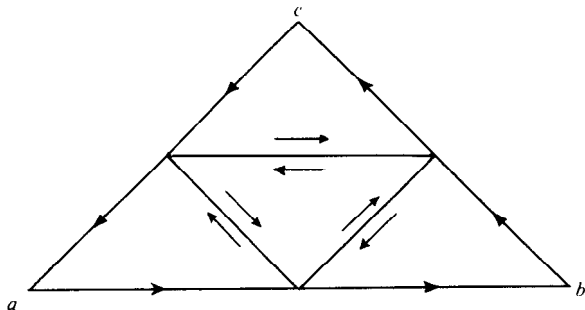
§8. Goursat's Theorem

Most modern books define an analytic function as one which is differentiable on an open set (not assuming the continuity of the derivative). In this section it is shown that this definition is the same as ours.

Goursat's Theorem. Let G be an open set and let $f: G \rightarrow \mathbb{C}$ be a differentiable function; then f is analytic on G .

Proof. We need only show that f' is continuous on each open disk contained in G ; so, we may assume that G is itself an open disk. It will be shown that f is analytic by an application of Morera's Theorem (5.7). That is, we must show that $\int_T f = 0$ for each triangular path T in G .

Let $T = [a, b, c, a]$ and let Δ be the closed set formed by T and its inside. Notice that $T = \partial\Delta$. Now using the midpoints of the sides of Δ form four triangles $\Delta_1, \Delta_2, \Delta_3, \Delta_4$ inside Δ and, by giving the boundaries appropriate



directions, we have that each $T_j = \partial\Delta_j$ is a triangle path and

$$8.1 \quad \int_T f = \sum_{j=1}^4 \int_{T_j} f$$

Among these four paths there is one, call it $T^{(1)}$, such that $|\int_{T^{(1)}} f| \geq |\int_{T_j} f|$ for $j = 1, 2, 3, 4$. Note that the length of each T_j (perimeter of Δ_j)—denoted by $\ell(T_j)$ —is $\frac{1}{2}\ell(T)$. Also $\operatorname{diam} T_j = \frac{1}{2} \operatorname{diam} T$; finally, using (8.1)

$$\left| \int_T f \right| \leq 4 \left| \int_{T^{(1)}} f \right|.$$

Now perform the same process on $T^{(1)}$, getting a triangle $T^{(2)}$ with the analogous properties. By induction we get a sequence $\{T^{(n)}\}$ of closed triangular paths such that if $\Delta^{(n)}$ is the inside of $T^{(n)}$ along with $T^{(n)}$ then;

$$8.2 \quad \Delta^{(1)} \supset \Delta^{(2)} \supset \dots;$$

$$8.3 \quad \left| \int_{T^{(n)}} f \right| \leq 4 \left| \int_{T^{(n+1)}} f \right|;$$

$$8.4 \quad \ell(T^{(n+1)}) = \frac{1}{2} \ell(T^{(n)});$$

$$8.5 \quad \operatorname{diam} \Delta^{(n+1)} = \frac{1}{2} \operatorname{diam} \Delta^{(n)}.$$

These equations imply:

$$8.6 \quad \left| \int_T f \right| \leq 4^n \left| \int_{T^{(n)}} f \right|;$$

$$8.7 \quad \ell(T^{(n)}) = \left(\frac{1}{2}\right)^n \ell \quad \text{where} \quad \ell = \ell(T);$$

$$8.8 \quad \operatorname{diam} \Delta^{(n)} = \left(\frac{1}{2}\right)^n d \quad \text{where} \quad d = \operatorname{diam} \Delta.$$

Since each $\Delta^{(n)}$ is closed, (8.2) and (8.8) allow us to apply Cantor's Theorem (II. 3.6), and get that $\bigcap_{n=1}^{\infty} \Delta^{(n)}$ consists of a single point z_0 .

Let $\epsilon > 0$; since f has a derivative at z_0 we can find a $\delta > 0$ such that $B(z_0; \delta) \subset G$ and

$$\left| \frac{f(z) - f(z_0)}{z - z_0} - f'(z_0) \right| < \epsilon$$

whenever $0 < |z - z_0| < \delta$. Alternately,

$$8.9 \quad |f(z) - f(z_0) - f'(z_0)(z - z_0)| < \epsilon |z - z_0|$$

whenever $|z - z_0| < \delta$. Choose n such that $\operatorname{diam} \Delta^{(n)} = \left(\frac{1}{2}\right)^n d < \delta$. Since $z_0 \in \Delta^{(n)}$ this gives $\Delta^{(n)} \subset B(z_0; \delta)$. Now Cauchy's Theorem implies that $0 = \int_{T^{(n)}} dz = \int_{T^{(n)}} z dz$. Hence

$$\begin{aligned} \left| \int_{T^{(n)}} f \right| &= \left| \int_{T^{(n)}} [f(z) - f(z_0) - f'(z_0)(z - z_0)] dz \right| \\ &\leq \epsilon \int_{T^{(n)}} |z - z_0| |dz| \\ &\leq \epsilon [\operatorname{diam} \Delta^{(n)}] [\ell(T^{(n)})] \\ &= \epsilon d \left(\frac{1}{2}\right)^n \end{aligned}$$

But using (8.6) this gives

$$\left| \int_T f \right| \leq 4^n \epsilon d \ell \left(\frac{1}{4}\right)^n = \epsilon d \ell.$$

Since ϵ was arbitrary and d and ℓ are fixed, $\int_T f = 0$. ■

Chapter V

Singularities

In this chapter functions which are analytic in a punctured disk (an open disk with the center removed) are examined. From information about the behavior of the function near the center of the disk, a number of interesting and useful results will be derived. In particular, we will use these results to evaluate certain definite integrals over the real line which cannot be evaluated by the methods of calculus.

§1. Classification of singularities

This section begins by studying the best behaved singularity—the removable kind.

1.1 Definition. A function f has an *isolated singularity* at $z = a$ if there is an $R > 0$ such that f is defined and analytic in $B(a; R) - \{a\}$ but not in $B(a; R)$. The point a is called a *removable singularity* if there is an analytic function $g: B(a; R) \rightarrow \mathbb{C}$ such that $g(z) = f(z)$ for $0 < |z - a| < R$.

The functions $\frac{\sin z}{z}$, $\frac{1}{z}$, and $\exp \frac{1}{z}$ all have isolated singularities at $z = 0$.

However, only $\frac{\sin z}{z}$ has a removable singularity (see Exercise 1). It is left to the reader to see that the other two functions do not have removable singularities.

How can we determine when a singularity is removable? Since the function has an analytic extension to $B(a; R)$, $\int_\gamma f = 0$ for any closed curve in the punctured disk; but this may be difficult to apply. Also it must happen that $\lim_{z \rightarrow a} f(z)$ exists. This is easier to verify, but a much weaker criterion is available.

1.2 Theorem. If f has an isolated singularity at a then the point $z = a$ is a removable singularity iff

$$\lim_{z \rightarrow a} (z - a)f(z) = 0$$

Proof. Suppose f is analytic in $\{z: 0 < |z - a| < R\}$, and define $g(z) = (z - a)f(z)$ for $z \neq a$ and $g(a) = 0$. Suppose $\lim_{z \rightarrow a} (z - a)f(z) = 0$; then g is clearly a continuous function. If we can show that g is analytic then it follows that a is a removable singularity. In fact, if g is analytic we have $g(z) = (z - a)h(z)$ for some analytic function defined on $B(a; R)$ because $g(a) = 0$ (IV. 3.9). But then $h(z)$ and $f(z)$ must agree for $0 < |z - a| < R$, so that a is, by definition, a removable singularity.