

ON SOME TRANSFORMS OF TRIGONOMETRIC SERIES

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ABSTRACT. We give a transform of convergent trigonometric series into equivalent convergent series and sufficient conditions for the transformed series to converge faster than the original one.

1. INTRODUCTION

Let

$$\sum_{n=1}^{\infty} a_n \cos(\alpha n + \beta)x \quad (1)$$

be a convergent real or complex trigonometric series. A method of accelerating the convergence of (1) is given in [5]. It consists of as follows:

Let $r \neq 1$ be a real or complex number and Δ_r^k a linear operator defined by

$$\begin{aligned} \Delta_r(a_n) &= a_{n+1} - ra_n \\ \Delta_r^{k+1}(a_n) &= \Delta_r(\Delta_r^k(a_n)) \quad (k = 1, 2, \dots). \end{aligned}$$

If $\lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = r$, $\lim_{n \rightarrow \infty} \frac{\Delta_r^k(a_{n+1})}{\Delta_r^k(a_n)} = r$ ($k = 1, 2, \dots, p$), then

$$\begin{aligned} \sum_{n=1}^{\infty} a_n \cos(\alpha n + \beta)x &= \frac{a_1 C_r^1(0)}{1 - 2r \cos \alpha x + r^2} + \sum_{k=1}^{p-1} \frac{\Delta_r^k(a_1) C_r^{k+1}(0)}{(1 - 2r \cos \alpha x + r^2)^{k+1}} \\ &\quad + \frac{1}{(1 - 2r \cos \alpha x + r^2)^p} \sum_{n=1}^{\infty} \Delta_r^p(a_n) \Delta_r^p \cos(\alpha n + \beta)x, \quad (2) \end{aligned}$$

where are $C_r^k(n) = \Delta_r^k \cos(\alpha n + \beta)x$ ($n = 0, 1, 2, \dots$).

A generalisation for number series is given in [3] and for power series in [4]. More detailed approach on these issues is given in [2]. In this paper we obtain a generalisation of transform (2) for cosine and sine series and give sufficient conditions for the modified transform to converge faster than (1).

For a sequence of real or complex numbers $\{a_n\}_{n=1}^{\infty}$ and a given sequence $\{r_n\}_{n=1}^{\infty}$ we define a linear operator $L_{r_1 \dots r_p}$ by

$$\begin{aligned} L_{r_1}(a_n) &= a_{n+1} - r_1 a_n \\ L_{r_1 \dots r_{p+1}}(a_n) &= L_{r_1 \dots r_p}(a_{n+1}) - r_{p+1} L_{r_1 \dots r_p}(a_n) \quad (p = 1, 2, \dots). \end{aligned} \quad (3)$$

In particular, for the sequence $\{\cos(\alpha n + \beta)x\}_{n=1}^{\infty}$ we put

$$C_{r_1 \dots r_p}(n) = L_{r_1 \dots r_p} \cos(\alpha n + \beta)x \quad (n = 0, 1, 2, \dots).$$

For fixed p put

$$\begin{aligned} E_0 &= 1, \quad E_1 = \sum_{i=1}^p r_i, \quad E_2 = \sum_{1 \leq i < j \leq p} r_i r_j, \\ E_3 &= \sum_{1 \leq i < j < k \leq p} r_i r_j r_k, \quad \dots, \quad E_p = r_1 r_2 \dots r_p \end{aligned}$$

(where the summation for E_m is performed over all combinations of distinct indices between 1 and p taken m at a time); we note that

$$L_{r_1 \dots r_p}(a_n) = \sum_{k=0}^p (-1)^k E_k a_{n+p-k}. \quad (4)$$

In order to establish the modified transform, we use the following

Lemma 1. *Suppose that the coefficients t_{nm} ($0 \leq m \leq n$) of the infinite triangular matrix (t_{nm}) satisfy the following conditions:*

- (1) $\lim_{n \rightarrow \infty} t_{nm} = 0$ for each fixed m ;
- (2) there exists a constant K such that $\sum_{k=0}^p |t_{pk}| \leq K$ for each nonnegative p .

Let $\{x_n\}_{n=1}^{\infty}$ be a sequence and define the sequence $\{x'_n\}_{n=1}^{\infty}$ by

$$x'_n = \sum_{k=0}^n t_{nk} x_k \quad (n = 0, 1, 2, \dots).$$

Then we have: if $\lim_{n \rightarrow \infty} x_n = 0$, then $\lim_{n \rightarrow \infty} x'_n = 0$.

The proof of the lemma is due to Toeplitz [1, p. 325].

2. A MODIFIED TRANSFORM OF TRIGONOMETRIC SERIES

The following theorem gives a generalisation of transform (2) for trigonometric series (1).

Theorem 1. *Let (1) be a convergent real or complex trigonometric series ($\alpha \neq 0$), r_1, \dots, r_p ($r_j e^{\pm \alpha x_i} \neq 1, j = 1, \dots, p$) arbitrary real or complex numbers. Then*

$$\begin{aligned} \sum_{n=1}^{\infty} a_n \cos(\alpha n + \beta)x &= \frac{a_1 C_{r_1}(0)}{1 - 2r_1 \cos \alpha x + r_1^2} \\ &+ \sum_{k=1}^{p-1} \frac{L_{r_1 \dots r_k}(a_1) C_{r_1 \dots r_{k+1}}(0)}{(1 - 2r_1 \cos \alpha x + r_1^2) \dots (1 - 2r_{k+1} \cos \alpha x + r_{k+1}^2)} \\ &+ \left(\prod_{j=1}^p (1 - 2r_j \cos \alpha x + r_j^2) \right)^{-1} \sum_{n=1}^{\infty} L_{r_1 \dots r_p}(a_n) L_{r_1 \dots r_p} \cos(\alpha n + \beta)x. \quad (5) \end{aligned}$$

Proof. Considering the Euler's formula for $\cos(\alpha n + \beta)x$ we have

$$\sum_{n=1}^{\infty} a_n \cos(\alpha n + \beta)x = \frac{1}{2} \sum_{n=1}^{\infty} a_n e^{(\alpha n + \beta)x i} + \frac{1}{2} \sum_{n=1}^{\infty} a_n e^{-(\alpha n + \beta)x i}.$$

Let $f_1(x) = \sum_{n=1}^{\infty} a_n e^{(\alpha n + \beta)x i}$, $f_2(x) = \sum_{n=1}^{\infty} a_n e^{-(\alpha n + \beta)x i}$. Then

$$f_1(x) = a_1 e^{(\alpha + \beta)x i} + e^{\alpha x i} \Phi_1(x),$$

where $\Phi_1(x) = \sum_{k=1}^{\infty} a_{k+1} e^{(\alpha k + \beta)x i}$. So

$$(1 - r_1 e^{\alpha x i}) \Phi_1(x) = \sum_{k=1}^{\infty} (a_{k+1} - r_1 a_k) e^{(\alpha k + \beta)x i} + r_1 a_1 e^{(\alpha + \beta)x i}.$$

Thus

$$\Phi_1(x) = \frac{r_1 a_1 e^{(\alpha+\beta)xi}}{1 - r_1 e^{\alpha xi}} + \frac{1}{1 - r_1 e^{\alpha xi}} \sum_{k=1}^{\infty} (a_{k+1} - r_1 a_k) e^{(\alpha k + \beta)xi}$$

and hence

$$\begin{aligned} f_1(x) &= a_1 e^{(\alpha+\beta)xi} + e^{\alpha xi} \frac{r_1 a_1 e^{(\alpha+\beta)xi}}{1 - r_1 e^{\alpha xi}} + \frac{e^{\alpha xi}}{1 - r_1 e^{\alpha xi}} \sum_{k=1}^{\infty} (a_{k+1} - r_1 a_k) e^{(\alpha k + \beta)xi} \\ &= \frac{a_1 e^{(\alpha+\beta)xi}}{1 - r_1 e^{\alpha xi}} + \frac{e^{\alpha xi}}{1 - r_1 e^{\alpha xi}} \sum_{k=1}^{\infty} L_{r_1}(a_k) e^{(\alpha k + \beta)xi}. \end{aligned} \quad (6)$$

Applying a similar technique to $\sum_{k=1}^{\infty} L_{r_1}(a_k) e^{(\alpha k + \beta)xi}$, we get

$$\sum_{k=1}^{\infty} L_{r_1}(a_k) e^{(\alpha k + \beta)xi} = L_{r_1}(a_1) e^{(\alpha + \beta)xi} + e^{\alpha xi} \Phi_2(x)$$

(where now $\Phi_2(x) = \sum_{k=1}^{\infty} L_{r_1}(a_{k+1}) e^{(\alpha k + \beta)xi}$), where from using (3), we get

$$\begin{aligned} (1 - r_2 e^{\alpha xi}) \Phi_2(x) &= \sum_{k=1}^{\infty} (L_{r_1}(a_{k+1}) - r_2 L_{r_1}(a_k)) e^{(\alpha k + \beta)xi} + r_2 L_{r_1}(a_1) e^{(\alpha + \beta)xi} \\ &= \sum_{k=1}^{\infty} L_{r_1 r_2}(a_k) e^{(\alpha k + \beta)xi} + r_2 L_{r_1}(a_1) e^{(\alpha + \beta)xi}. \end{aligned}$$

Hence

$$\begin{aligned} \sum_{k=1}^{\infty} L_{r_1}(a_k) e^{(\alpha k + \beta)xi} &= L_{r_1}(a_1) e^{(\alpha + \beta)xi} + e^{\alpha xi} \frac{r_2 e^{(\alpha + \beta)xi} L_{r_1}(a_1)}{1 - r_2 e^{\alpha xi}} \\ &\quad + \frac{e^{\alpha xi}}{1 - r_2 e^{\alpha xi}} \sum_{k=1}^{\infty} L_{r_1 r_2}(a_k) e^{(\alpha k + \beta)xi}. \end{aligned}$$

Thus, using (6), we obtain

$$\begin{aligned} f_1(x) &= \frac{a_1 e^{(\alpha+\beta)xi}}{1 - r_1 e^{\alpha xi}} + \frac{L_{r_1}(a_1) e^{(2\alpha+\beta)xi}}{(1 - r_1 e^{\alpha xi})(1 - r_2 e^{\alpha xi})} \\ &\quad + \frac{e^{2\alpha xi}}{(1 - r_1 e^{\alpha xi})(1 - r_2 e^{\alpha xi})} \sum_{k=1}^{\infty} L_{r_1 r_2}(a_k) e^{(\alpha k + \beta)xi}. \end{aligned} \quad (7)$$

Repeating this process p times we find that

$$f_1(x) = \frac{a_1 e^{(\alpha+\beta)xi}}{1 - r_1 e^{\alpha xi}} + \sum_{k=1}^{p-1} L_{r_1 \dots r_k}(a_1) \frac{e^{(\alpha(k+1)+\beta)xi}}{(1 - r_1 e^{\alpha xi}) \dots (1 - r_{k+1} e^{\alpha xi})} + R_p^{(1)}(x), \quad (8)$$

where

$$R_p^{(1)}(x) = \frac{1}{(1 - r_1 e^{\alpha xi}) \dots (1 - r_p e^{\alpha xi})} \sum_{n=1}^{\infty} L_{r_1 \dots r_p}(a_n) e^{(\alpha(n+p)+\beta)xi}. \quad (9)$$

Notice that (6) and (7) are the $p = 1$ and $p = 2$ cases of (8), respectively. Since $f_2(x) = f_1(-x)$, we have

$$f_2(x) = \frac{a_1 e^{-(\alpha+\beta)xi}}{1 - r_1 e^{-\alpha xi}} + \sum_{k=1}^{p-1} L_{r_1 \dots r_k}(a_1) \frac{e^{-(\alpha(k+1)+\beta)xi}}{(1 - r_1 e^{-\alpha xi}) \dots (1 - r_{k+1} e^{-\alpha xi})} + R_p^{(2)}(x), \quad (8')$$

where

$$R_p^{(2)}(x) = \frac{1}{(1 - r_1 e^{-\alpha x i}) \dots (1 - r_p e^{-\alpha x i})} \sum_{n=1}^{\infty} L_{r_1 \dots r_p}(a_n) e^{-(\alpha(n+p)+\beta)x i}. \quad (9')$$

Multiplying the equations (8) and (8') by $\frac{1}{2}$ and then summing the two together, using (4) for the sequence $\{\cos(\alpha n + \beta)x\}_{n=1}^{\infty}$, we obtain

$$\begin{aligned} \sum_{n=1}^{\infty} a_n \cos(\alpha n + \beta)x &= \frac{a_1 C_{r_1}(0)}{1 - 2r_1 \cos \alpha x + r_1^2} \\ &+ \sum_{k=1}^{p-1} \frac{L_{r_1 \dots r_k}(a_1) C_{r_1 \dots r_{k+1}}(0)}{(1 - 2r_1 \cos \alpha x + r_1^2) \dots (1 - 2r_{k+1} \cos \alpha x + r_{k+1}^2)} + R_p(x), \end{aligned} \quad (8'')$$

where

$$\begin{aligned} R_p(x) &= \frac{1}{2} \left(R_p^{(1)}(x) + R_p^{(2)}(x) \right) \\ &= \left(\prod_{j=1}^p (1 - 2r_j \cos \alpha x + r_j^2) \right)^{-1} \sum_{n=1}^{\infty} L_{r_1 \dots r_p}(a_n) L_{r_1 \dots r_p} \cos(\alpha n + \beta)x. \end{aligned} \quad (9'')$$

Equalities (8'') and (9'') complete the proof. \square

In completely analogous way we obtain the similar transform for sine series

$$\begin{aligned} \sum_{n=1}^{\infty} a_n \sin(\alpha n + \beta)x &= \frac{a_1 S_{r_1}(0)}{1 - 2r_1 \cos \alpha x + r_1^2} \\ &+ \sum_{k=1}^{p-1} \frac{L_{r_1 \dots r_k}(a_1) S_{r_1 \dots r_{k+1}}(0)}{(1 - 2r_1 \cos \alpha x + r_1^2) \dots (1 - 2r_{k+1} \cos \alpha x + r_{k+1}^2)} \\ &+ \left(\prod_{j=1}^p (1 - 2r_j \cos \alpha x + r_j^2) \right)^{-1} \sum_{n=1}^{\infty} L_{r_1 \dots r_p}(a_n) L_{r_1 \dots r_p} \sin(\alpha n + \beta)x, \end{aligned} \quad (10)$$

where $S_{r_1 \dots r_p}(n) = L_{r_1 \dots r_p} \sin(\alpha n + \beta)x$ ($n = 0, 1, 2, \dots$).

Remark 1. If $L_{r_1 \dots r_p}(a_n) = 0$ for some $p \geq 1$ and for n sufficiently large, then (5) and (10) transform trigonometric series into finite sums.

Remark 2. In particular, for $r_1 = r_2 = \dots = r_p = r$ we obtain the transform (2).

3. ACCELERATING CONVERGENCE OF TRIGONOMETRIC SERIES

The following theorem gives a transform of convergent trigonometric series (1) into an equivalent convergent series.

Theorem 2. Let (1) be a convergent series on x , $\pi/2 \leq |\alpha|x \leq 3\pi/2$ and $\{r_n\}_{n=1}^{\infty}$ a sequence of positive real numbers such that for some real $\lambda > 1$

$$r_n = O(n^{-\lambda}) \quad \text{or} \quad 1/r_n = O(n^{-\lambda}). \quad (11)$$

Then

$$\begin{aligned} \sum_{n=1}^{\infty} a_n \cos(\alpha n + \beta)x &= \frac{a_1 C_{r_1}(0)}{1 - 2r_1 \cos \alpha x + r_1^2} \\ &+ \sum_{k=1}^{\infty} \frac{L_{r_1 \dots r_k}(a_1) C_{r_1 \dots r_{k+1}}(0)}{(1 - 2r_1 \cos \alpha x + r_1^2) \dots (1 - 2r_{k+1} \cos \alpha x + r_{k+1}^2)}. \end{aligned} \quad (12)$$

Proof. In order to prove (12) we need to show that in (8''), $\lim_{p \rightarrow \infty} R_p(x) = 0$. Let $r_n(x)$ be the remainder of the convergent series $\sum_{n=1}^{\infty} a_n e^{(\alpha n + \beta)x i}$. Then $\lim_{n \rightarrow \infty} r_n(x) = 0$. Thus, using (4) we get

$$\begin{aligned} R_p^{(1)}(x) &= \left(\prod_{j=1}^p (1 - r_j e^{\alpha x i}) \right)^{-1} \sum_{n=1}^{\infty} e^{(\alpha(n+p) + \beta)x i} \sum_{k=0}^p (-1)^k E_k a_{n+p-k} \\ &= \left(\prod_{j=1}^p (1 - r_j e^{\alpha x i}) \right)^{-1} \sum_{k=0}^p (-1)^{p-k} e^{(p-k)x i} E_{p-k} r_k(x). \end{aligned}$$

In the lemma of Introduction we put

$$t_{pk} = \left(\prod_{j=1}^p (1 - r_j e^{\alpha x i}) \right)^{-1} (-1)^{p-k} e^{(p-k)x i} E_{p-k} \quad (0 \leq k \leq p).$$

We have to show that the conditions of the lemma are satisfied. For each nonnegative integer p we have

$$\begin{aligned} |t_{pk}| &= \left| \left(\prod_{j=1}^p (1 - r_j e^{\alpha x i}) \right)^{-1} E_{p-k} \right| \\ &= \left| \sum_{1 \leq i_1 < \dots < i_{p-k} \leq p} \left(\prod_{s=p-k+1}^p (1 - r_{i_s} e^{\alpha x i}) \right)^{-1} \prod_{j=1}^{p-k} \frac{r_{i_j}}{1 - r_{i_j} e^{\alpha x i}} \right|. \end{aligned}$$

Without lost in generality we may assume that $\alpha > 0$. Then $\pi/2 \leq \alpha x \leq 3\pi/2$, and since $r_j \geq 0$ ($j = 1, \dots, p$), we have

$$|1 - r_j e^{\alpha x i}| > 1, \quad |r_j / (1 - r_j e^{\alpha x i})| < 1 \quad (j = 1, \dots, p). \quad (13)$$

Hence

$$|t_{pk}| \leq \sum_{1 \leq i_1 < \dots < i_{p-k} \leq p} \prod_{j=1}^{p-k} \left| \frac{r_{i_j}}{1 - r_{i_j} e^{\alpha x i}} \right| \leq \binom{p}{p-k} M^{p-k} \leq p^k M^{p-k},$$

where $M = \max_{1 \leq j \leq p} |r_j / (1 - r_j e^{\alpha x i})| < 1$. Thus $\lim_{p \rightarrow \infty} t_{pk} = 0$, so the condition 1 of the lemma holds. Notice that

$$\sum_{k=0}^p |t_{pk}| \leq \left| \prod_{j=1}^p (1 - r_j e^{\alpha x i}) \right|^{-1} \sum_{k=0}^p E_{p-k} = \left| \prod_{j=1}^p (1 - r_j e^{\alpha x i}) \right|^{-1} \prod_{k=1}^p (1 + r_k). \quad (14)$$

From (11) we derive that one of the two series $\sum_{j=1}^{\infty} r_j$ or $\sum_{j=1}^{\infty} 1/r_j$ converges, and hence one of the two infinite products $\prod_{j=1}^{\infty} (1 + r_j)$ or $\prod_{j=1}^{\infty} (1 + 1/r_j)$ converges. Put

$$K = \begin{cases} \prod_{j=1}^{\infty} (1 + r_j), & \text{if } \prod_{j=1}^{\infty} (1 + r_j) \text{ converges} \\ \prod_{j=1}^{\infty} (1 + 1/r_j), & \text{if } \prod_{j=1}^{\infty} (1 + 1/r_j) \text{ converges,} \end{cases}$$

using (14) and (13), we conclude that the condition 2 of the lemma is also satisfied. Thus $\lim_{p \rightarrow \infty} R_p^{(1)}(x) = 0$. On the other hand, since

$$|1 - r_j e^{\alpha x i}| = |1 - r_j e^{-\alpha x i}| \quad (j = 1, 2, \dots),$$

the inequalities (13) hold true if we replace x by $-x$. This however, based on (9) and (9'), means that $\lim_{p \rightarrow \infty} R_p^{(2)}(x) = 0$. Hence, using (9''), $\lim_{p \rightarrow \infty} R_p(x) = 0$. Now (12) follows from (8''). \square

Obviously, the result of Theorem 2 can be applied for transform (10) of sine series. The obtained transform, analogous with (12), is

$$\sum_{n=1}^{\infty} a_n \sin(\alpha n + \beta)x = \frac{a_1 S_{r_1}(0)}{1 - 2r_1 \cos \alpha x + r_1^2} + \sum_{k=1}^{\infty} \frac{L_{r_1 \dots r_k}(a_1) S_{r_1 \dots r_{k+1}}(0)}{(1 - 2r_1 \cos \alpha x + r_1^2) \dots (1 - 2r_{k+1} \cos \alpha x + r_{k+1}^2)}. \quad (15)$$

Remark 3. Let $r_j > 0$ ($j = 1, 2, \dots$) and $\pi/2 \leq |\alpha|x \leq 3\pi/2$. Suppose for $p = 1, 2, \dots$ that $L_{r_1 \dots r_p}(a_n) \neq 0$ for n sufficiently large and that $\lim_{n \rightarrow \infty} \frac{L_{r_1 \dots r_p}(a_{n+1})}{L_{r_1 \dots r_p}(a_n)}$ exists. Since, according to (3),

$$\begin{aligned} \frac{L_{r_1}(a_n)}{a_n} &= \frac{a_{n+1}}{a_n} - r_1 \\ \frac{L_{r_1 \dots r_{p+1}}(a_n)}{L_{r_1 \dots r_p}(a_n)} &= \frac{L_{r_1 \dots r_p}(a_{n+1})}{L_{r_1 \dots r_p}(a_n)} - r_{p+1} \quad (p = 1, 2, \dots), \end{aligned} \quad (16)$$

if we require the additional condition that the sequence $\{r_n\}_{n=1}^{\infty}$ is chosen so that

$$r_1 = \lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n}, \quad r_{p+1} = \lim_{n \rightarrow \infty} \frac{L_{r_1 \dots r_p}(a_{n+1})}{L_{r_1 \dots r_p}(a_n)} \quad (p = 1, 2, \dots),$$

then the sequences (16) are null-sequences. Whence

$$\lim_{n \rightarrow \infty} \frac{L_{r_1 \dots r_p}(a_n)}{a_n} = \lim_{n \rightarrow \infty} \frac{L_{r_1}(a_n)}{a_n} \frac{L_{r_1 r_2}(a_n)}{L_{r_1}(a_n)} \dots \frac{L_{r_1 \dots r_p}(a_n)}{L_{r_1 \dots r_{p-1}}(a_n)} = 0 \quad (p = 1, 2, \dots).$$

Therefore, according to (8''), (9''), (9), (9') and (13), we conclude that the series on the right-hand side of (5) and (10) converge faster than the ones on the left-hand side.

If we require that the sequence $\{r_n\}_{n=1}^{\infty}$ from Remark 3 satisfies the condition (11), then the right-hand sides of (12) and (15) converge faster than the left-hand sides.

Example 1. Let $a_n = 1/(a^n + b^n)$ ($0 < a < b$). Then

$$r_1 = \lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = \frac{1}{b}, \quad r_{p+1} = \lim_{n \rightarrow \infty} \frac{L_{r_1 \dots r_p}(a_{n+1})}{L_{r_1 \dots r_p}(a_n)} = \frac{a^p}{b^{p+1}} \quad (p = 1, 2, \dots).$$

Obviously, the sequence $\{r_n\}_{n=1}^{\infty}$ satisfies the condition (11) of Theorem 2. In particular, put $a = 2$, $b = 3$, $\alpha = 1$, $\beta = 0$, $x = 3\pi/4$; then in order to calculate the approximate sum of the number series $\sum_{n=1}^{\infty} \frac{1}{2^n + 3^n} \cos \frac{3n\pi}{4}$ with an error not greater than 10^{-6} we must compute the sum of first 12 terms. Applying the transform (5) of Theorem 1, the same accuracy is obtained by computing the sum of first 7 terms for $p = 1$, 4 terms for $p = 2$, and 2 terms for $p = 3$.

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