# 11. Diophantine Equations

(786) For which positive integer(s) $x$ is it true that

$$(*) \qquad x^3 + (x+1)^3 + (x+2)^3 = (x+3)^3 \ ?$$

(787) Show that the Diophantine equation

$$x^3 + 5 = 117y^3$$

has no solutions.

(788) One day, as the English mathematician Godfrey Harold Hardy (1877–1947) was visiting Srinivasa Ramanujan (1885–1920) at the hospital, the patient commented to his visitor that the number on the license plate of the taxi that had brought him, namely 1729, was a very special number: it is the smallest positive integer which can be written as the sum of two cubes in two different ways, namely

$$1729 = 1^3 + 12^3 = 9^3 + 10^3.$$

Using the identity

$$(*) \quad (3a^2 + 5ab - 5b^2)^3 + (4a^2 - 4ab + 6b^2)^3 = (-5a^2 + 5ab + 3b^2)^3 + (6a^2 - 4ab + 4b^2)^3$$

due to Ramanujan, show that there exist infinitely many positive integers which can be written as a sum of two cubes in two different ways. Does this identity allow one to find the "double" representation of 1729?

(789) Let $a, b, c \in \mathbb{Z}$. Show that $ax + by = b + c$ is solvable in integers $x$ and $y$ if and only if $ax + by = c$ is also solvable.

(790) Let $a, b, c \in \mathbb{Z}$. Show that $ax + by = c$ is solvable in integers $x$ and $y$ if and only if $(a, b) = (a, b, c)$.

(791) Let $a$ and $b$ be positive integers such that $(a, b) = 1$. Show that $ax+by = n$ has positive integer solutions if $n > ab$, while it has no positive integer solution if $n = ab$.

(792) Find the positive integer solution(s) of the system of equations

$$\begin{aligned} x + y + z &= 100, \\ 2x + 5y + \frac{z}{10} &= 100. \end{aligned}$$

(793) The triangle whose sides have lengths 5, 12 and 13 respectively has the property that its perimeter is equal to its area. There exist exactly five such triangles with integer sides. Which are they?

(794) Identify all the integer solutions, if any, to the equation

$$x^2 + 3y = 5.$$

(795) Show that if the positive integers $x, y, z$ are the respective lengths of the sides of a rectangular triangle, then at least one of these three numbers is a multiple of 5.

(796) Let $a, b$ and $c$ be three real nonnegative numbers. Show that the system of equations

$$\begin{aligned} ax + by + cxy &= a + b + c, \\ by + cz + ayz &= a + b + c, \\ cz + ax + bzx &= a + b + c \end{aligned}$$

has one and only one solution in nonnegative integers $x$, $y$, $z$. What is this solution? Why is it so?

(797) Let $a$ and $b$ be positive integers such that $(a, b) = 1$. Show that $ax + by = ab - a - b$ has no solutions in integers $x \geq 0$ and $y \geq 0$.

(798) Let $a$ and $b$ be positive integers such that $(a, b) = 1$. Show that the number of nonnegative solutions of $ax + by = n$ is equal to

$$\left[\frac{n}{ab}\right] \quad \text{or} \quad \left[\frac{n}{ab}\right] + 1.$$

(799) At the fruit counter in a store, apples are sold 5 cents each and oranges are sold 7 cents each. Say Peter purchases four apples and twelve oranges. Peter notices that Paul also bought apples and oranges and that he pays the same total amount as you did, but with a different number of apples and oranges. Knowing that Paul purchased at least three oranges, does Peter have enough information to determine the exact number of apples and oranges purchased by Paul?

(800) Determine the set of solutions of the Diophantine equation $3x + 7y = 11$ located in the third quadrant of the cartesian plane.

(801) Determine the set of solutions of the Diophantine equation $5x + 7y = 11$ located above the line $y = x$.

(802) Assume that the set $E$ of solutions of the Diophantine equation

$$(*) \qquad ax + by = 11$$

is given by

$$E = \{(x, y) : x = 5 - 4t \text{ and } y = 1 - 3t, \text{ where } t \in \mathbb{Z}\}.$$

Determine the values of $a$ and $b$.

(803) Find the primitive solutions of $x^2 + 3y^2 = z^2$, that is those solutions $x$, $y$, $z$ which have no common factor other than 1.

(804) Show that the only nonzero integer solutions $(x, y, z)$ to the system of equations

$$x + y + z = x^3 + y^3 + z^3 = 3$$

are $(1, 1, 1)$, $(-5, 4, 4)$, $(4, -5, 4)$ and $(4, 4, -5)$.

(805) Show that the equation

$$x^2 = y^3 + z^5$$

has infinitely many solutions in positive integers $x$, $y$ $z$.

(806) Find the four different ways of writing 136 as a sum of two positive integers, one of which is divisible by 5 and the other by 7.

(807) Any solution in positive integers $x, y, z$ of $x^2 + y^2 = z^2$ is called a Pythagorean triple, since in such a case there exists a rectangular triangle whose sides have $x, y, z$ for their respective lengths. Find all Pythagorean triples whose terms form an arithmetic progression.

(808) Find the dimensions of the Pythagorean triangle whose hypotenuse is of length 281.

(809) Show that 60 divides the product of the lengths of the sides of a Pythagorean triangle.

(810) Find every Pythagorean triangle whose area is equal to three times its perimeter.

(811) Find every Pythagorean triangle whose perimeter is equal to twice its area.

(812) Show that $\{x, y, z\} = \{3, 4, 5\}$ is the only solution of $x^2 + y^2 = z^2$ with consecutive integers $x$, $y$, $z$.

(813) Show that $n^2 + (n+1)^2 = 2m^2$ is impossible for $n, m \in \mathbb{N}$.

(814) Show that the equation $x^2 + y^2 = 4z + 7$ has no integer solution.

(815) Find all integer solutions of $x^2 + y^2 = z^4$ such that $(x, y, z) = 1$.

(816) Find all integer points on the line $x + y = 1$ which are located inside the circle centered at the origin and of radius 3.

(817) Find all primitive solutions of the Diophantine equation

$$x^2 + 3136 = z^2.$$

(818) Find all integer solutions of the equation

$$x^2 + y^2 = xy.$$

(819) Find the solutions of the Diophantine equation

$$(*) \qquad x^2 + 2y^2 = 4z^2.$$

(820) Find a triangle such that each of its sides is of integer length and for which an interior angle is equal to twice another interior angle.

(821) Find all positive integer solutions to the equation $x^2 + y^2 = 10$. Do the same for $x^2 + y^2 = 47$.

(822) Find all positive rational solutions of $x^2 + y^2 = 1$.

(823) Find all primitive Pythagorean triangles such that the length of one of their sides is equal to 24.

(824) Show that the radius of any circle inscribed in a Pythagorean triangle is an integer.

(825) Show that the equation $x^2 + y^2 + z^2 = 2239$ has no solutions in positive integers $x, y, z$.

(826) Show that

$$t^2 = x^2 + y^2 + z^2$$

has no nontrivial integer solution with $t$ even and with $x$, $y$, $z$ having no common factor.

(827) Find all primitive solutions of the Diophantine equation $x^2 + 2y^2 = z^2$.

(828) Find all positive integer solutions to the system of equations

$$\begin{cases} a^3 - b^3 - c^3 = 3abc, \\ a^2 = 2(b + c). \end{cases}$$

(829) Find all integer solutions of $y^2 + y = x^4 + x^3 + x^2 + x$.

(830) Find the smallest prime number which can be written in each of the following forms: $x^2 + y^2, x^2 + 2y^2, \ldots, x^2 + 10y^2$.

(831) Determine the set of quadruples $(x, y, x, w)$ verifying $x^3 + y^3 + z^3 = w^3$ and such that $x, y, z$ and $w$ are positive integers in arithmetical progression.

(832) Consider the sequence $8, 26, 56, 98, 152, \ldots$, that is the sequence $\{x_n\}$ defined by $x_1 = 8$ and $x_{n+1} = x_n + 6(2n+1)$, $n \geq 1$, and show that for $n > 1$, $x_n$ cannot be the cube of an integer.

(833) Show that $x^n + 1 = y^{n+1}$ has no solutions in positive integers $x, y, n$ $(n \geq 2)$ with $(x, n+1) = 1$.

(834) Show that neither of the equations

$$3^a + 1 = 5^b + 7^c \quad \text{and} \quad 5^a + 1 = 3^b + 7^c$$

has a solution in integers $a, b, c$ other than $a = b = c = 0$.

(835) Find all integer triples $(x, y, z)$ such that $4^x + 4^y + 4^z$ is a perfect square.

(836) Show that there exist solutions in positive integers $a, b, c, x, y$ to the system of equations

$$\begin{cases} a + b + c = x + y, \\ a^3 + b^3 + c^3 = x^3 + y^3. \end{cases}$$

Show, in particular, that there exist infinitely many solutions such that $a, b, c$ are in arithmetic progression.

(837) Solve each of the following Diophantine equations: (here $m$ is a nonnegative integer)

$$x^m(x^2 + y) = y^{m+1},$$
$$x^m(x^2 + y^2) = y^{m+1}.$$

(838) Can the following equations be verified for an appropriate choice of integers $x, a, b, c, d$?

$$(x+1)^2 + a^2 = (x+2)^2 + b^2 = (x+3)^2 + c^2 = (x+4)^2 + d^2.$$

(839) Does the equation

$$x^2 + y^2 + z^2 = xyz - 1$$

have integer solutions?

(840) Find all pairs of real numbers $(x, y)$ which satisfy the two equations:

$$(*) \qquad 2x^3 - x^2 + y^2 = 1,$$
$$(**) \qquad 2y^3 - y^2 + x^2 = 1.$$

(841) Find all positive integer solutions $x, y$ of $x^y = y^{x-y}$.

(842) Find all integers solutions $x, y, z$ to the system of equations

$$\begin{cases} 2x(1 + y + y^2) = 3(1 + y^4), \\ 2y(1 + z + z^2) = 3(1 + z^4), \\ 2z(1 + x + x^2) = 3(1 + x^4). \end{cases}$$

(843) Prove that there exist infinitely many integers $a, b, c, d$ such that $a > b > c > d > 1$ and $a!\,d! = b!\,c!$.

(844) Show that the equation $x^3 + y^3 + z^3 = 4$ has no solutions in integers. What about the equation $x^3 + y^3 + z^3 = 5$?

(845) Does the Diophantine equation $x^4 = 4y^2 + 4y - 80$ have any solutions? If so, what are they? If no, explain why.

(846) Does the Diophantine equation $x^4 + y^4 + z^4 = 363932239$ have any solutions? If so, what are they? If no, explain why.

(847) Let $a$ be an arbitrary integer. Does the Diophantine equation

$$303x + 57y = a^2 + 1$$

have any solution?

(848) Does the Diophantine equation

$$x^4 = 4y^2 + 4y - 15$$

have any solution?

(849) Do integers $x, y, z$ exist such that

$$x^4 + (2y+1)^4 = z^2?$$

(850) Determine the set of positive solutions of the Diophantine equation

$$x^2 = y^4 + 8.$$

(851) Let $p$ be an odd prime number. Assume that $q = p + 8$ is also a prime number. Analyze the set of solutions of the Diophantine equation

$$x^2 = y^4 + pq$$

and give one such solution explicitly.

(852) Does the Diophantine equation

$$x^2 + y^2 + 2x + 4y + 4z + 2 = 0$$

have any solution?

(853) Does the Diophantine equation

$$x^4 + y^4 + z^4 + u^4 = 3xyzu$$

have any nonzero solution?

(854) Does the Diophantine equation

$$x^3 + 2y^3 = 4z^3$$

have any nonzero solution?

(855) Find all integer solutions of $x^2 + y^2 = 8z + 7$.

(856) Show that $x^4 + y^4 = 7z^2$ has no solutions in $\mathbb{N}$. What about the equation $x^4 + y^4 = 5z^2$?

(857) Does the equation $x^4 + x^2 = y^4 + 5$ have any solution in integers $x$ and $y$?

(858) Let $0 < x < y < z$ be integers such that $x^2 + y^2 = z^2$. Show that for each integer $n > 2$, $x^n + y^n = z^n$ is impossible.

(859) Prove that the equation

$$x^3 + 3y^3 = 9z^3$$

has no nontrivial integer solution.

(860) Let $p$ be a prime number. Does the Diophantine equation

$$x^4 + py^4 + p^2z^4 = p^3w^4$$

have any trivial solution?

(861) Show that

$$x^2 + y^2 + z^2 = 2xyz$$

has no nontrivial integer solution.

(862) Determine all rational solutions of the equation

$$x^3 + y^3 = x^2 + y^2.$$

(863) Show that the Diophantine equation

$$\frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_n} + \frac{1}{x_1 x_2 \cdots x_n} = 1$$

has at least one solution for each integer $n \geq 1$.

(864) Show that the Diophantine equation

$$x^2 + y^2 + z^2 = x^2y^2$$

has no nontrivial solution.

(865) Show that the equation

$$x^8 + x^7 + x^6 + x^5 + x^4 + x^3 + x^2 + x + 1 = 1\,234\,567\,891\,314$$

has no integer solution.

(866) Prove that the Diophantine equation

$$(x + y)^2 + (x + z)^2 = (y + z)^2$$

has no solutions in odd integers $x$, $y$, $z$.

(867) Let $p$ be a fixed prime number. Find all positive integer solutions of $x^2 + py^2 = z^2$.

(868) Show that there exist infinitely many solutions to the Diophantine equation $x^2 + 4y^2 = z^3$.

(869) Find all solutions, for $x$, $y$ integers and $n$ positive integers, to the Diophantine equation $x^n + y^n = xy$.

(870) Show that the equation $n^x + n^y = n^z$ has positive integer solutions only if $n = 2$.

(871) Show that the equation $n^x + n^y + n^w = n^z$ has positive integer solutions only if $n = 2$ or 3.

(872) Show that the *abc* conjecture implies the following result: The equation $x^p + y^q = z^r$ has no solutions in positive integers $p, q, r, x, y, z$ with $z \geq z_0$ and

$$(*) \qquad \frac{1}{p} + \frac{1}{q} + \frac{1}{r} < 1,$$

so that in particular the Fermat equation $x^n + y^n = z^n$ has no nontrivial solution for $n \geq 4$ and $z$ sufficiently large.

(873) Show that if the *abc* conjecture is true, then there can exist only a finite number of triples of consecutive powerful numbers.

(874) Show that if the *abc* conjecture is true, then there exist only a finite number of positive integers $n$ such that $n^3 + 1$ is a powerful number. Moreover, find two numbers $n$ with this property.

(875) Erdős conjectured that the equation $x + y = z$ has only a finite number of solutions in 4-powerful integers $x, y, z$ pairwise coprime. Show that the *abc* conjecture implies this conjecture.

(876) Show that if the *abc* conjecture is true, then there exist only a finite number of 4-powerful numbers which can be written as the sum of two 3-powerful numbers pairwise coprime.

(877) Given an integer $n \geq 2$, let $P(n)$ stand for the largest prime factor of $n$. Prove that it follows from the *abc* conjecture that, for each real number $y > 0$, the set $A_y := \{p \text{ prime} : P(p^2 - 1) \leq y\}$ is a finite set and therefore has a largest element $p = p(y)$.

(878) In 1877, Edouard Lucas (1842–1891) observed that although 2701 is a composite number, we have that $2^{2700} \equiv 1 \pmod{2701}$, thus providing a counter-example to the reverse of Fermat's Little Theorem. More generally, show that one can construct a large family of such counter-examples

by considering the numbers $n = pq$, where $p$ and $q$ are prime numbers such that $p \equiv 1 \pmod 4$ and $q = 2p - 1$.

(879) Show that if the *abc* conjecture is true, then for any $\varepsilon > 0$, there exists a positive constant $M = M(\varepsilon)$ such that for all triples $(x_1, x_2, x_3)$ of positive integers, pairwise coprime and verifying $x_1 + x_2 = x_3$, we have that

$$(*) \qquad \min(x_1, x_2, x_3) \le M\,(\gamma(x_i))^{3+\varepsilon} \qquad (i = 1, 2, 3).$$

(880) In 1979, Enrico Bombieri naively claimed that: "the equation

$$\binom{x}{n} + \binom{y}{n} = \binom{z}{n} \qquad (n \ge 3)$$

had no solutions in positive integers $x, y, z$." Was Bombieri right? If so, prove it; if no, provide a counter-example.

(881) Let $p$ be an odd prime number and let $\alpha_1, \alpha_2, \ldots, \alpha_r$ be positive integers not exceeding $p - 1$. Show that the Diophantine equation

$$n^p = x_1^{\alpha_1} + x_2^{\alpha_2} + \cdots + x_r^{\alpha_r}$$

has solutions in positive integers $n, x_1, x_2, \ldots, x_r$.

(882) Even though, according to Fermat's Last Theorem, for each prime number $p \ge 3$, the equation $x^p + y^p = z^p$ has no solutions in positive integers $x, y, z$, show that the equation $x^{p-1} + y^{p-1} = z^p$ always has solutions (besides the trivial one $x = y = z = 2$).