

5. Congruences

- (266) For which positive integers n is the number $3^n + 1$ a multiple of 10?
- (267) Find the smallest positive residue modulo 7 of $1! + 2! + \cdots + 50!$.
- (268) What is the remainder of the division of $\sum_{i=1}^{111} i!$ by 12?
- (269) Show that for each positive integer n , $10 \cdot 32^n + 1$ is a composite number.
- (270) Is it true that 36 divides $n^6 + n^2 + 4$ for infinitely many positive integers n ? Explain.
- (271) In a letter sent to Christian Huygens (1629–1695) in 1659, Fermat wrote that using his method of infinite descent, he was successful in showing that no integer of the form $3k - 1$ can be written as $x^2 + 3y^2$ (with x and y integers). Is it possible to prove this result in a very simple manner? Explain.
- (272) Let m and n be positive integers such that $p^m \parallel n$ for a certain prime number p . Show that

$$\frac{n!}{p^m} \equiv (-1)^m \prod_{k=0}^{\left\lfloor \frac{\log n}{\log p} \right\rfloor} \left(\left\lfloor \frac{n}{p^k} \right\rfloor - p \left\lfloor \frac{n}{p^{k+1}} \right\rfloor \right)! \pmod{p}.$$

- (273) Let n be a positive integer. Show that the last digit of n^{13} is the same as the last digit of n .
- (274) Find the smallest positive integer n such that $\sqrt[3]{n/7}$ and $\sqrt[11]{n/11}$ are both integers.
- (275) Show that there exists an arbitrarily long sequence of consecutive integers, each divisible by a perfect square.
- (276) Let a and b be integers and let m and n be positive integers. Show that the system of congruences

$$\begin{aligned} x &\equiv a \pmod{m}, \\ x &\equiv b \pmod{n} \end{aligned}$$

has solutions if and only if $(m, n) \mid (a - b)$.

- (277) Let p be a prime number. Show that if k is an integer, $1 \leq k < p$, then $\binom{p}{k} \equiv 0 \pmod{p}$.
- (278) (a) Let x_1, x_2, \dots, x_n be integers. Show that $(x_1 + x_2 + \cdots + x_n)^p \equiv x_1^p + x_2^p + \cdots + x_n^p \pmod{p}$.
- (b) Show that if a and b are integers such that $a^p \equiv b^p \pmod{p}$, then $a^p \equiv b^p \pmod{p^2}$.
- (279) Let p be an odd prime number and let k be an integer such that $1 \leq k < p$. Show that

$$\binom{p-1}{k} \equiv (-1)^k \pmod{p}.$$

- (280) Let p be a prime number and let r be an integer such that $1 \leq r < p$. If $(-1)^r r! \equiv 1 \pmod{p}$, show that

$$(p - r - 1)! \equiv -1 \pmod{p}.$$

Use this result to show that $259! \equiv -1 \pmod{269}$ and $463! \equiv -1 \pmod{479}$.

- (281) Let $\alpha \geq 3$ and $\beta \geq 6$ be two integers. Show that the equation $2^\beta - 1 = 3p^\alpha$ has no solutions for p prime.
- (282) Let p be a prime number and let $n = 2p + 1$. Show that if n is not a multiple of 3 and if $2^{n-1} \equiv 1 \pmod{n}$, then n is prime.
- (283) Let p be a prime number and k a positive integer. Show that

$$(*) \quad a \equiv b \pmod{p^k} \implies a^p \equiv b^p \pmod{p^{k+1}}.$$

Then, prove that if $p > 2$, $p \nmid a$ and $p^k \parallel a - b$, then $p^{k+1} \parallel a^p - b^p$.

- (284) If p is a prime number, can the equation $p^\delta + 1 = 2^\nu$ have solutions with integers $\delta \geq 2$ and $\nu \geq 2$?
- (285) Show that the equation $1 + n + n^2 = m^2$, where m and n are positive integers, is impossible.
- (286) Show that the only solution of the equation $1 + p + p^2 + p^3 + p^4 = q^2$, where p and q are primes, is $\{p, q\} = \{3, 11\}$.
- (287) Let x_1, x_2, x_3, x_4 and x_5 be integers such that

$$x_1^3 + x_2^3 + x_3^3 + x_4^3 + x_5^3 = 0.$$

Show that necessarily one of the x_i 's is a multiple of 7.

- (288) Show that $2^p + 3^p$ is not a power (> 1) of an integer if p is prime.
- (289) Show that for each positive integer n ,

$$1^n + 2^n + 3^n + 4^n + 5^n + 6^n$$

is divisible by 7 if and only if n is not divisible by 6.

- (290) Is it true that if n is a positive odd integer whose last digit in decimal representation is different from 5, then the last two digits of the decimal representation of n^{400} are 0 and 1? Explain.
- (291) What are the possible values of the last digit of 4^m for each $m \in \mathbb{N}$?
- (292) Show that the difference of two consecutive cubes is never divisible by 3, nor by 5.
- (293) Is it true that $27 \mid (2^{5n+1} + 5^{n+2})$ for each integer $n \geq 0$? Explain.
- (294) Show that for each positive integer k , the number $(13^2)^{2k+1} + (98^2)^{2k+1}$ is divisible by 337.
- (295) Find the last two digits of the decimal representation of $19^{19^{19}}$.
- (296) If a and b are positive integers such that $(ab, 70) = 1$, show that $a^{12} - b^{12} \equiv 0 \pmod{280}$.
- (297) Show that for each integer $n \geq 2$, $n^{13} - n$ is divisible by 2730.
- (298) Find the smallest positive integer which divided by 12, by 17, by 45 or by 70 gives in each case a remainder of 4.
- (299) If n is an arbitrary positive integer, is the number

$$3n^{13} + 4n^{11} + n^7 + 3n^5 + 3n$$

divisible by 7?

- (300) Let p be a prime number; show that $\binom{2p}{p} \equiv 2 \pmod{p}$.
- (301) Show that a 3-digit positive integer whose decimal representation is of the form " abc " (for three digits a , b and c) is divisible by 7 if and only if $2a + 3b + c$ is divisible by 7.

(302) Show that a 6-digit positive integer whose decimal representation is of the form “ $abcabc$ ” (for three digits a , b and c) is necessarily divisible by 13.

(303) Show that $561|2^{561} - 2$ and that $561|3^{561} - 3$.

(304) Given a positive integer n , show that

$$\frac{12}{35}n^{13} + \frac{23}{35}n$$

is an integer.

(305) Does there exist a rational number r such that for each positive integer n relatively prime with 481,

$$\frac{50}{481}n^{36} + r$$

is a positive integer?

(306) Let p be an odd prime number, $p \neq 5$. Show that p divides infinitely many integers amongst $1, 11, 111, 1\,111, \dots$.

(307) According to Fermat’s Little Theorem, if n is an odd prime number and if a is a positive integer such that $(a, n) = 1$, then $a^{n-1} \equiv 1 \pmod{n}$. Show that the reverse of this result is false.

(308) Let $p > 3$ be a prime number. Show that $ab^p - ba^p \equiv 0 \pmod{6p}$ for any integers a and b .

(309) If n is a positive integer, is it true that

$$1 + 2 + 3 + \dots + (n-1) \equiv 0 \pmod{n}?$$

Explain.

(310) For which positive integers n do we have

$$1^2 + 2^2 + 3^2 + \dots + (n-1)^2 \equiv 0 \pmod{n}?$$

(311) Is it true that if n is a positive integer divisible by 4, then

$$1^3 + 2^3 + 3^3 + \dots + (n-1)^3 \equiv 0 \pmod{n}?$$

(312) Prove that for each positive integer n , we have

$$5^n \equiv 1 + 4n \pmod{16} \quad \text{and} \quad 5^n \equiv 1 + 4n + 8n(n-1) \pmod{64}.$$

(313) Show that for each positive integer $k \geq 3$,

$$5^{2^{k-3}} \not\equiv 1 \pmod{2^k} \quad \text{while} \quad 5^{2^{k-2}} \equiv 1 \pmod{2^k}.$$

More generally, show that for $k > 2$ and a given odd integer a , we have

$$a^{2^{k-2}} \equiv 1 \pmod{2^k}.$$

(314) Show that

$$\frac{n^5}{5} + \frac{n^3}{3} + \frac{7n}{15}$$

is an integer for all $n \in \mathbb{N}$. More generally, show that if p and q are prime numbers, then

$$\frac{n^p}{p} + \frac{n^q}{q} + \frac{(pq - p - q)n}{pq}$$

is an integer for all $n \in \mathbb{N}$.

(315) Find the solution of the congruence $x^{24} + 7x \equiv 2 \pmod{13}$.

(316) Because of Wilson’s Theorem, the numbers $2, 3, 4, \dots, 15$ can be arranged in seven pairs $\{x, y\}$ such that $xy \equiv 1 \pmod{17}$. Find these seven pairs.

- (317) Let $m = m_1 m_2 \cdots m_r$, where the m_i 's are integers > 1 and pairwise coprime. Show that

$$m_1^{\phi(m)/\phi(m_1)} + m_2^{\phi(m)/\phi(m_2)} + \cdots + m_r^{\phi(m)/\phi(m_r)} \equiv r - 1 \pmod{m}.$$

- (318) Let p be a prime number and k an integer, $0 < k < p$. Show that

$$(k-1)!(p-k)! \equiv (-1)^k \pmod{p}.$$

- (319) If p and q are distinct prime numbers, is it true that we always have

$$p^{q-1} + q^{p-1} \equiv 1 \pmod{pq}?$$

More generally, if m and n are positive integers such that $(m, n) = 1$, is it true that

$$n^{\phi(m)} + m^{\phi(n)} \equiv 1 \pmod{mn}?$$

- (320) Show that for each positive integer n ,

$$3^{2n+2} \equiv 8n + 9 \pmod{64}.$$

- (321) Let $p \geq 5$ be a prime number. Find the value of $(p!, (p-2)! - 1)$.

- (322) Show that

$$5^{6614} - 12^{857} \equiv 1 \pmod{7}.$$

- (323) *Divisibility tests.* Let N be a positive integer whose decimal representation is $N = a_n 10^n + \cdots + a_2 10^2 + a_1 10 + a_0$, where $0 < a_n \leq 9$ and for $k = 0, \dots, n-1$, $0 \leq a_k \leq 9$. Show that

(a) N is divisible by 3 $\iff a_n + a_{n-1} + \cdots + a_1 + a_0 \equiv 0 \pmod{3}$.

(b) N is divisible by 4 $\iff 10a_1 + a_0 \equiv 0 \pmod{4}$.

(c) N is divisible by 6 $\iff 4(a_n + \cdots + a_1 + a_0) \equiv 3a_0 \pmod{6}$.

(d) N is divisible by 7 $\iff (100a_2 + 10a_1 + a_0) - (100a_5 + 10a_4 + a_3) + (100a_8 + 10a_7 + a_6) - \cdots \equiv 0 \pmod{7}$.

(e) N is divisible by 8 $\iff 100a_2 + 10a_1 + a_0 \equiv 0 \pmod{8}$.

(f) N is divisible by 9 $\iff a_n + a_{n-1} + \cdots + a_0 \equiv 0 \pmod{9}$.

(g) N is divisible by 11 $\iff a_n - a_{n-1} + \cdots + (-1)^n a_0 \equiv 0 \pmod{11}$.

- (324) Assume that 168 divides the integer whose decimal representation is "770ab45c". Find the digits a , b and c .

- (325) Let a be an integer ≥ 2 and let $m \in \mathbb{N}$. If $(a, m) = (a-1, m) = 1$, show that

$$1 + a + a^2 + \cdots + a^{\phi(m)-1} \equiv 0 \pmod{m}.$$

- (326) Let p be a prime number. Show that for each $a \in \mathbb{N}$, we have

$$a^{(p-1)!+1} \equiv a \pmod{p}.$$

- (327) Show that if p is a prime number, then $1^{p-1} + 2^{p-1} + \cdots + (p-1)^{p-1} \equiv -1 \pmod{p}$.

- (328) Show that if p is an odd prime number, then $1^p + 2^p + \cdots + (p-1)^p \equiv 0 \pmod{p}$.

- (329) Let p be an odd prime number. Show that

$$\sum_{k=1}^{p-1} (k-1)!(p-k)!k^{p-1} \equiv 0 \pmod{p}.$$

- (330) Letting p be a prime number of the form $4n + 1$, show that $((2n)!)^2 \equiv -1 \pmod{p}$. More generally, if p is a prime number and if $m + n = p - 1$, $m \geq 0$, $n \geq 0$, show that

$$m!n! \equiv (-1)^{m+1} \pmod{p}.$$

(A similar result was obtained in Problem 318.) Use this last formula to prove that

$$\left\{ \left(\frac{p-1}{2} \right)! \right\}^2 \equiv (-1)^{(p+1)/2} \pmod{p}.$$

- (331) Show that an integer $n > 2$ is prime if and only if n divides the number $2(n-3)! + 1$.
 (332) Show that if p is a prime number and a an arbitrary integer, then p divides the expression $a^p + a(p-1)!$.
 (333) Show that if $\pi = 3.141592\dots$ stands for Archimede's constant and $\pi(x)$ stands for the number of prime numbers $p \leq x$, then

$$\pi(x) = \sum_{2 \leq n \leq x} \left[\cos^2 \left(\pi \frac{(n-1)! + 1}{n} \right) \right],$$

where $[y]$ stands for the largest integer smaller or equal to y .

- (334) Let $m_1, m_2 \in \mathbb{N}$ be such that $(m_1, m_2) = 1$. If a , r and s are positive integers such that $a^r \equiv 1 \pmod{m_1}$ and $a^s \equiv 1 \pmod{m_2}$. Show that

$$a^{[r,s]} \equiv 1 \pmod{m_1 m_2}.$$

- (335) Let m be a positive integer. Show that for each $a \in \mathbb{N}$,

$$a^m \equiv a^{m-\phi(m)} \pmod{m}.$$

- (336) Let m be a positive odd integer. Show that the sum of the elements of a complete residue system modulo m is congruent to 0 (mod m).
 (337) Let $a, b \in \mathbb{Z}$, $m \in \mathbb{N}$. If E is a complete residue system modulo m and if $(a, m) = 1$, show that

$$E' = \{ax + b \mid x \in E\}$$

is also a complete residue system modulo m .

- (338) Is it possible to construct a reduced residue system modulo 7 made up entirely of multiples of 6? Explain.
 (339) Let $m > 2$ be an integer. Show that the sum of the elements of a reduced residue system modulo m is congruent to 0 (mod m).
 (340) If $\{r_1, r_2, \dots, r_{p-1}\}$ is a reduced residue system modulo a prime number p , show that

$$\prod_{j=1}^{p-1} r_j \equiv -1 \pmod{p}.$$

- (341) Let $a, b \in \mathbb{Z}$, $m \in \mathbb{N}$. Using a counter-example, show that if E is a reduced residue system modulo m and if $(a, m) = 1$, then the set $\{ax + b \mid x \in E\}$ is not necessarily a reduced residue system modulo m .
 (342) Find all integers x, y and z with $2 \leq x \leq y \leq z$ such that

$$xy \equiv 1 \pmod{z}, \quad xz \equiv 1 \pmod{y}, \quad yz \equiv 1 \pmod{x}.$$

- (343) Let n and k be positive integers. Show that there exists a sequence of n consecutive composite integers such that each is divisible by at least k distinct prime numbers. Using this result, find the smallest sequence of four consecutive integers divisible by 3, 5, 7 and 11 respectively.
- (344) Find all positive integers which give the remainder 1, 2 and 3 when divided respectively by 3, 4 and 5.
- (345) Find the smallest integer $a > 2$ such that
- $$2|a, \quad 3|a+1, \quad 4|a+2, \quad 5|a+3, \quad 6|a+4.$$
- (346) Find the cycle and the period of $1/3, 1/3^2, 1/3^3, 1/3^4, 1/7, 1/7^2, 1/7^3$. Let p be an arbitrary prime number for which the period of $1/p$ is m . Using these computations, what should one conjecture regarding the periods of $1/p^2, 1/p^3, \dots, 1/p^n$?
- (347) The decimal expansion of $2/3 = 0.666\dots$ consists in a repetition of $6 = 2 \cdot 3$. The same phenomenon occurs with the decimal expansion of $1/3 = 0.333\dots$. Find all positive rational numbers a/b with $(a, b) = 1$, whose decimal expansion is formed by an infinite repetition of the product of its numerator and of its denominator.
- (348) Show that the period of a fraction m/n with $m < n$, $(m, n) = 1$, $(n, 10) = 1$ is the smallest positive integer h such that $10^h \equiv 1 \pmod{n}$.
- (349) If m/n has the cycle $a_1 a_2 \dots a_h$, show that $m | a_1 a_2 \dots a_h$.
- (350) If $m/n = 0.\overline{a_1 a_2 \dots a_r}$, show that

$$\frac{m}{n} = \frac{a_1 a_2 \dots a_r}{10^r - 1},$$

where the numerator is the number made up of the r digits a_1, a_2, \dots, a_r (and not of their product).