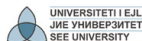


Integration by Parts

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Aims and Objectives

- Introducing a technique for integrating certain products $f(x)g(x)$.

Contents

- 1 Integration by Parts and Why It Works
- 2 How and When to Use Integration by Parts

Integration by Parts

Integration by Parts

If G is an antiderivative of g , then

$$\int f(x)g(x) dx = f(x)G(x) - \int f'(x)G(x) dx.$$

Why Integration by Parts Works

- Use the product rule for differentiation:

$$\frac{d}{dx}[f(x)G(x)] = f'(x)G(x) + f(x)G'(x) = f'(x)G(x) + f(x)g(x).$$

- Express in terms of integrals:

$$f(x)G(x) = \int f'(x)G(x) dx + \int f(x)g(x) dx.$$

- Herefrom

$$\int f(x)g(x) dx = f(x)G(x) - \int f'(x)G(x) dx,$$

which is precisely the formula for integration by parts.

When to Use Integration by Parts

- Use integration by parts to integrate products $f(x)g(x)$, in which:
 - one of the factors, say $g(x)$, can be easily integrated
 - and the other, $f(x)$, becomes simpler when differentiated.

How to Use Integration by Parts

How to Use Integration by Parts

- 1 Select one of the factors to be integrated and the other to be differentiated.
- 2 Integrate the designated factor and multiply it by the other factor.
- 3 Differentiate the designated factor, multiply it by the integrated factor from step 2, and subtract the integral of this product from the result of step 2.
- 4 Complete the procedure by finding the new integral that was formed in step 3.

Examples

Example

Find $\int x e^{2x} dx$.

Solution...

The process of differentiation simplifies x :

$$g(x) = e^{2x} \quad \text{and} \quad f(x) = x.$$

Then

$$G(x) = \frac{1}{2} e^{2x} \quad \text{and} \quad f'(x) = 1.$$



Examples. (Continued)

... Solution.

So

$$\begin{aligned}\int x e^{2x} dx &= x \left(\frac{1}{2} e^{2x} \right) - \int 1 \cdot \left(\frac{1}{2} e^{2x} \right) dx \\ &= \frac{1}{2} x e^{2x} - \frac{1}{2} \int e^{2x} dx \\ &= \frac{1}{2} x e^{2x} - \frac{1}{4} e^{2x} + C = \frac{1}{2} \left(x - \frac{1}{2} \right) e^{2x} + C.\end{aligned}$$



Examples. (Continued)

Example

Find $\int x\sqrt{x+5} dx$.

Solution...

$$g(x) = \sqrt{x+5} \quad \text{and} \quad f(x) = x.$$

Then

$$G(x) = \int \sqrt{x+5} dx = \int (x+5)^{\frac{1}{2}} dx = \frac{2}{3}(x+5)^{\frac{3}{2}} \quad \text{and} \quad f'(x) = 1.$$



Examples. (Continued)

... Solution.

So

$$\begin{aligned}\int x\sqrt{x+5} \, dx &= \frac{2}{3}x(x+5)^{\frac{3}{2}} - \frac{2}{3} \int (x+5)^{\frac{3}{2}} \, dx \\ &= \frac{2}{3}x(x+5)^{\frac{3}{2}} - \frac{2}{3} \cdot \frac{2}{5}(x+5)^{\frac{5}{2}} + C.\end{aligned}$$



Examples. (Continued)

Example

Find $\int \ln x \, dx$.

Solution...

Write $\ln x$ as the product $1 \cdot \ln x$, where 1 is easy to integrate and $\ln x$ is simplified by differentiation:

$$g(x) = 1 \quad \text{and} \quad f(x) = \ln x.$$

Then

$$G(x) = x \quad \text{and} \quad f'(x) = \frac{1}{x}.$$



Examples. (Continued)

... Solution.

So

$$\begin{aligned}\int \ln x \, dx &= \int 1 \cdot \ln x \, dx = x \ln x - \int x \frac{1}{x} dx \\ &= x \ln x - \int dx = x \ln x - x + C = x(\ln x - 1) + C.\end{aligned}$$



Examples. (Continued)

Example

Find the particular solution of the differential equation

$$\frac{dy}{dx} = xe^{x-y}$$

that satisfies the condition $y = \ln 2$ when $x = 0$.

Solution...

Separate the variables:

$$\begin{aligned}\frac{dy}{dx} &= \frac{xe^x}{e^y} \\ e^y dy &= xe^x dx\end{aligned}$$



Examples. (Continued)

... Solution...

$$\int e^y dy = \int x e^x dx$$
$$e^y = \int x e^x dx$$

Integrate by parts:

$$g(x) = e^x \quad \text{and} \quad f(x) = x.$$

Then

$$G(x) = e^x \quad \text{and} \quad f'(x) = 1.$$



Examples. (Continued)

... Solution.

So

$$\begin{aligned}\int x e^x dx &= x e^x - \int 1 \cdot e^x dx \\ &= x e^x - e^x + C = (x - 1) e^x + C.\end{aligned}$$

The general solution of the differential equation:

$$e^y = (x - 1) e^x + C$$



Examples. (Continued)

... Solution.

Determine C :

$$y(0) = \ln 2$$

$$e^{\ln 2} = (0 - 1)e^0 + C$$

$$2 = -1 \cdot 1 + C$$

$$C = 3$$

The particular solution:

$$e^y = (x - 1)e^x + 3.$$



For Further Reading

- <http://fberisha.netfirms.com>
- **Homework:** Exercises from teaching materials
- L. D. Hofmann, G. L. Bradley, *Calculus – for business, economics and life sciences*, pp. 410–418.

Summary

- Integration by parts: If G is an antiderivative of g , then

$$\int f(x)g(x) dx = f(x)G(x) - \int f'(x)G(x) dx.$$