Differentiation

The Derivative: Slope and Rates

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Aims and Objectives

- Notion of derivative and its interpretations: slope of the graph and rate of change of the function.
- Identifying the relation between derivative and increasing and decreasing functions.
- Applying the derivative to business applications
- Introducing the symbolics for the derivative

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Rate of Change of a Function

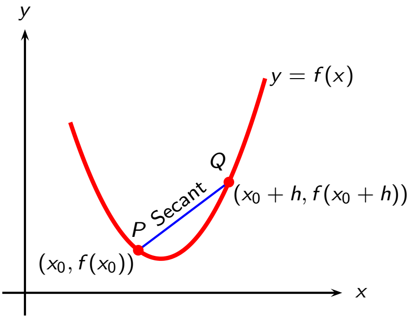

Figure: The graph of $f(x)$ with the secant line through $P(x_0, f(x_0))$ and $Q(x_0 + h, f(x_0 + h))$.

- Average rate of change of a function:

$$v_{\text{ave}} = \frac{f(x_0 + h) - f(x_0)}{h}$$

- Instantaneous rate of change:

$$v = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

Slope of the Tangent Line of a Curve

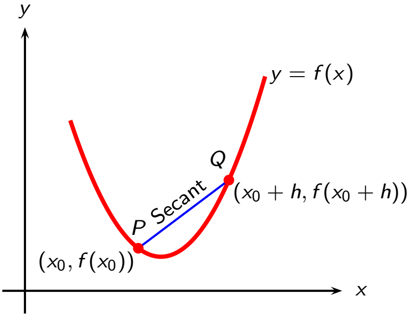

Figure: As $h \to 0$ the secant line tends toward the tangent line through P.

- Slope of the secant line through P, Q:

$$m_{\text{sec}} = \frac{f(x_0 + h) - f(x_0)}{(x_0 + h) - x_0} = \frac{f(x_0 + h) - f(x_0)}{h}$$

- Slope of the tangent line:

$$m_{\tan} = \lim_{h \to 0} m_{\text{sec}} = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

Notion of the Derivative of a Function

The Derivative of a Function

- The *derivative* of a function $f(x)$ with respect to x is the function $f'(x)$ given by

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h},$$

and the process of computing the derivative is called *differentiation.*
- We say that $f(x)$ is *differentiable* at x_0 if $f'(x_0)$ exists.

The Derivative. Slope. Rate of Change

Slope. Rate of Change

- The *slope* of the tangent line to the curve $y = f(x)$ at the point $(x_0, f(x_0))$ is given by $m_{\tan} = f'(x_0)$.
- Instantaneous rate of change of the quantity $f(x)$ with respect to x when $x = x_0$ is equal to $f'(x_0)$.

Example of an Equation of a Tangent Line

Example

Compute the derivative of $f(x) = x^2$,
then use it to find the slope of the curve at the point $x = -1$.
What is the equation of the tangent line at this point?

Example of an Equation of a Tangent Line. (Continued)

Solution. . .

According to the definition of the derivative

$$
\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{(x+h)^2 - x^2}{h} \\
&= \lim_{h\to 0} \frac{(x^2+2xh+h^2)-x^2}{h} = \lim_{h\to 0} \frac{2xh+h^2}{h} \\
&= \lim_{h\to 0} \frac{h(2x+h)}{h} = \lim_{h\to 0}(2x+h) = 2x.
\end{aligned}
$$

The slope of the tangent line to the curve $y = x^2$ at the point $x = -1$:

$$f'(-1) = 2(-1) = -2$$

□

The Graph of $y = x^2$

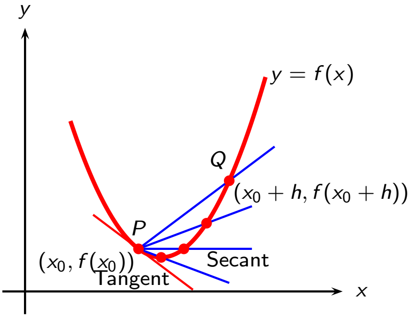

Figure: The tangent line to the curve $y = x^2$ at the point $(-1, 1)$.

Example of an Equation of a Tangent Line. (Continued)

. . . Solution.

Find the y-coordinate at the point of tangency:

$$y = f(-1) = (-1)^2 = 1.$$

The tangent line passes through the point $(-1, 1)$ with slope -2. Its equation:

$$\begin{aligned} y - 1 &= (-2)[x - (-1)] \\ y &= -2x - 1. \end{aligned}$$

$\square$

Example of a Business Application

Example

A manufacturer estimates that when x units of a commodity are produced and sold the revenue derived will be
$R(x) = 0.5x^2 + 3x - 2$ thousand euros.
At what rate is the revenue changing with respect to the level of production x when 3 units are being produced?
Is the revenue increasing or decreasing at this point?

Example of a Business Application. (Continued)

Solution. . .

For $x \geq 0$, the *difference quotient* of $R(x)$ is

$$\begin{aligned}
&\frac{R(x+h)-R(x)}{h} \\
&= \frac{[0.5(x+h)^2+3(x+h)-2]-[0.5x^2+3x-2]}{h} \\
&= \frac{[0.5(x^2+2xh+h^2)+3x+3h-2]-0.5x^2-3x+2}{h} \\
&= \frac{xh+0.5h^2+3h}{h} = x+0.5h+3.
\end{aligned}$$

□

Example of a Business Application. (Continued)

. . . Solution.

Thus, the derivative of $R(x)$ is

$$R'(x) = \lim_{h \to 0} \frac{R(x+h) - R(x)}{h} = \lim_{h \to 0} (x + 0.5h + 3) = x + 3,$$

and since

$$R'(3) = 3 + 3 = 6,$$

it follows that the revenue is changing at the rate $6,000$ € per unit when 3 units are being produced.
Since $R'(3) = 6 > 0$; i.e. since $R'(3)$ is **positive**,
the tangent line at the point on the graph of the revenue function where $x = 3$ must be sloped upward.
This observation suggests that revenue is increasing when $x = 3$. □

The Graph of the Revenue Function

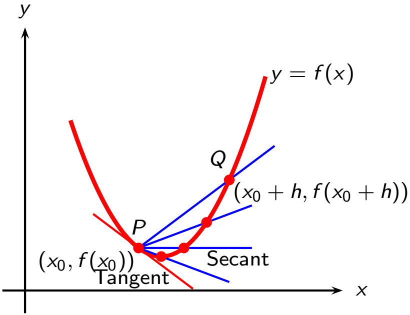

Figure: The graph of $R(x) = 0.5x^2 + 3x - 2$, for $x \geq 0$, with tangent line at the point where $x = 3$.

Derivative Notation

- The derivative $f'(x)$ of $y = f(x)$ is sometimes written as $\frac{dy}{dx}$, and the value of the derivative at $x = c$ (i.e., $f'(c)$):

$$\left.\frac{dy}{dx}\right|_{x=c}.$$

- For example, if $y = x^2$, then

$$\frac{dy}{dx} = 2x.$$

- Sometimes a statement such as

$$\text{"if } y = x^2\text{, then } \frac{dy}{dx} = 2x\text{"}$$

is shortened by simply writing

$$\frac{d}{dx}(x^2) = 2x.$$

Differentiability and Continuity

- If a function is differentiable at a point $P(x_0, f(x_0))$, then its graph has a nonvertical tangent line at P, and all points on the graph "near" P are "close" to the tangent.
- Intuitively, this suggests that a function must be continuous at any point where it is differentiable, since the graph cannot have a "hole" or "gap" at any point where a tangent can be drawn.
- But, the converse is not true; i.e., a continuous function need not be everywhere differentiable.

For Further Reading

- http://fberisha.netfirms.com
- Homework: Exercises from teaching materials
- L. D. Hofmann, G. L. Bradley, *Calculus – for business, economics and life sciences,* pp. 98–109.
- F. M. Berisha, M. Q. Berisha, *Matematikë – për biznes dhe ekonomiks,* pp. 153–161.

Summary

- Secant line; tangent line
- Derivative: $f'(x) = \lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$
- Interpretations:
  - Geometric: [Derivative]=[Slope of the tangent line]
  - Change: [Derivative]=[Rate of change]
- Differentiable functions