The Second Derivative. Increasing and Decreasing Functions

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Aims and Objectives

- Studying by means of the second derivative of the rate of change of the rate of change of a quantity
- Using the derivative to determine where a function is increasing or decreasing
- Finding by means of the first derivative a relative maximum or a relative minimum of a function

Contents

The Second Derivative

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The second derivative of a function is the derivative of its derivative.

If $y = f(x)$, the second derivative is denoted by

$$f''(x) \quad \text{of} \quad \frac{d^2y}{dx^2}.$$

The second derivative gives the rate of change of the rate of change of the original function.

Application: The Change of the Rate of Production

Example

An efficiency study of the morning shift at a certain factory indicates that an average worker who arrives on the job at 8:00 will have produced

$$Q(t) = -t^3 + 11t^2 + 16t$$

radio-transistors t hours later.

1. Compute the worker's rate of production at 12:00?
2. At what rate is the worker's rate of production changing with respect to time at 12:00?
3. Use calculus to estimate the change in the worker's rate of production between 12:00 and 12:10.
4. Compute the actual change in the worker's rate of production between 12:00 and 12:10.

Application: The Change of the Rate... (Continued)

Solution...

1. The worker's rate of production is the first derivative

$$Q'(t) = -3t^2 + 22t + 16$$

ot the output function $Q(t)$.
At 12:00, $t = 4$ and the rate of production is

$$Q'(4) = -3 \cdot 4^2 + 22 \cdot 4 + 16 = 56$$

units per hour.

$\square$

Application: The Change of the Rate... (Continued)

...Solution...

2. Tha rate of change of the rate of production is the second derivative

$$Q''(t) = -6t + 22$$

of the output function.
At 12:00 the rate is

$$Q''(4) = -6 \cdot 4 + 22 = -2$$

units per hour per hour.

□

Application: The Change of the Rate. . . (Continued)

. . . Solution. . .

3. Note that 10 minutes is $\frac{1}{6}$ hour; i.e. $\Delta t = \frac{1}{6}$ hour. Apply the approximation by increments formula to the function $Q'(t)$:

$$\Delta Q' \approx Q''(t)\Delta t,$$

to get, after substituting $t = 4$ and $\Delta t = \frac{1}{6}$:

$$\Delta Q' \approx Q''(4)\Delta t = -2 \cdot \frac{1}{6} = -\frac{1}{3} \approx -0.33$$

units per hour.

□

Application: The Change of the Rate... (Continued)

...Solution.

4. The actual change in the worker's rate of production between 12:00 and 12:10 is

$$Q'\left(4+\frac{1}{6}\right)-Q'(4)=Q'\left(\frac{25}{6}\right)-Q'(4)$$
$$=-3\left(\frac{25}{6}\right)^2+22\left(\frac{25}{6}\right)+16-(-3\cdot 4^2+22\cdot 4+16)$$
$$\approx 55.58-56=-0.42$$

units per hour.

□

Increasing and Decreasing Functions

Increasing and Decreasing Functions

- A function $f(x)$ is *increasing* on an interval $a < x < b$ if $f(x_1) < f(x_2)$ whenever $x_1 < x_2$ for x_1, x_2 in the interval.
    - In words, if $y = f(x)$ increases as x increases.
- The function is *decreasing* on $a < x < b$ if $f(x_1) > f(x_2)$ whenever $x_1 < x_2$ for x_1, x_2 in the interval.
    - In words, if $y = f(x)$ decreases as x increases.

Increasing and Decreasing Functions. (Continued)

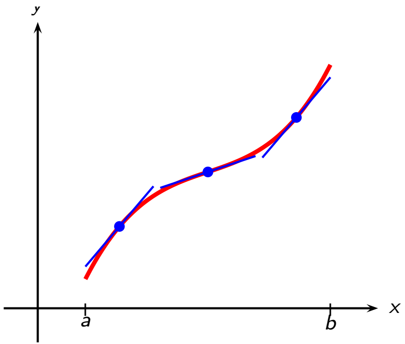


Figure: $f'(x) > 0$ on $a < x < b$, so $f(x)$ is increasing.

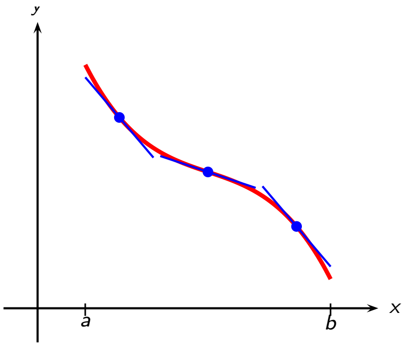


Figure: $f'(x) < 0$ on $a < x < b$, so $f(x)$ is decreasing.

Increasing and Decreasing Functions. (Continued)

Derivative Criteria for Increasing and Decreasing Functions

- A function $f(x)$ is increasing on an interval where $f'(x) > 0$.
- A function $f(x)$ is decreasing on an interval where $f'(x) < 0$.

Example: Determining Intervals of Increase and Decrease

Example

Find the intervals of increase and decrease for

$$f(x) = -\frac{1}{3}x^3 + \frac{1}{2}x^2 + 2x - 1.$$

Solution...

The derivative of $f(x)$ is

$$f'(x) = -x^2 + x + 2 = -(x+1)(x-2),$$

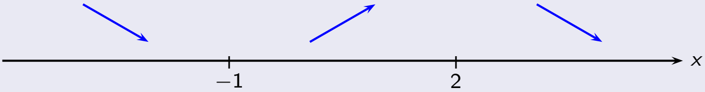
which is continuous and has zeros at $x = -1$ and $x = 2$. So, the sign of $f'(x)$ must stay the same
on each of the intervals $x < -1$, $-1 < x < 2$ and $x > 2$. □

Example: Determining Intervals. . . (Continued)

. . . Solution.

Interval	**Test Number c**	**Sign of $f'(c)$**	**Conclusion**
$x < -1$	-2	$f'(-2) < 0$	$f(x) \searrow$
$-1 < x < 2$	0	$f'(0) > 0$	$f(x) \nearrow$
$x > 2$	3	$f'(3) < 0$	$f(x) \searrow$

Table: Intervals of increase and decrease for the function $f(x) = -\frac{1}{3}x^3 + \frac{1}{2}x^2 + 2x - 1$.



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Example: Determining Intervals. . . (Continued)

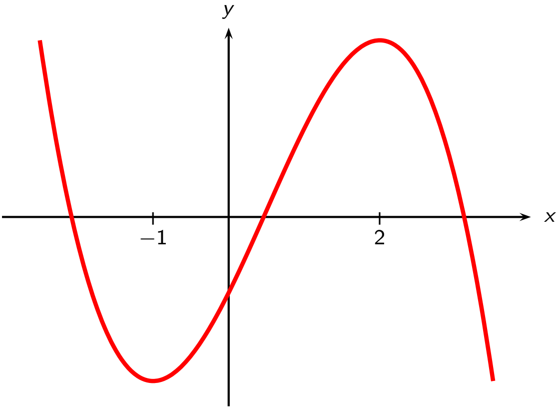


Figure: The graph of $f(x) = -\frac{1}{3}x^3 + \frac{1}{2}x^2 + 2x - 1$.

Example: Determining Intervals. . . (Continued)

Example

Find the intervals of increase and decrease for

$$f(x) = \frac{x^2}{x-1}.$$

Solution. . .

The function is defined for $x \neq 1$, and its derivative is

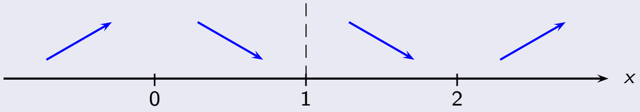
$$f'(x) = \frac{(2x)(x-1) - x^2 \cdot 1}{(x-1)^2} = \frac{x(x-2)}{(x-1)^2},$$

which is continuous except at $x = 1$. □

Example: Determining Intervals. . . (Continued)

Solution. . .

Choosing test numbers on the intervals $x < 0$, $0 < x < 1$, $1 < x < 2$ and $x > 2$ (e.g., -1, $\frac{1}{2}$, $\frac{3}{2}$ and 3) we obtain the arrow diagram shown.



□

Example: Determining Intervals. . . (Continued)

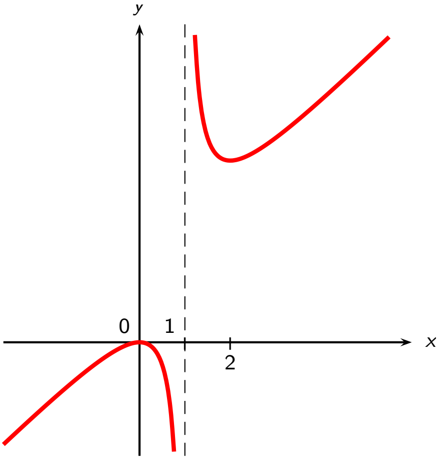


Figure: The graph of $f(x) = \frac{x^2}{x-1}$.

Relative Extrema

Relative Extrema

- A function $f(x)$ is said to have a *relative maximum* at $x = c$ if $f(c) \geq f(x)$ for all x in an interval $a \leq x \leq b$ containing c.
- Similarly, $f(x)$ has a *relative minimum* at $x = c$ if $f(c) \leq f(x)$ for all x in an interval $a \leq x \leq b$ containing c.
- Collectively, the relative maxima and minima of $f(x)$ are called its *relative extrema*.

Relative Extrema. (Continued)

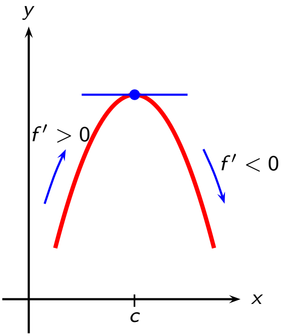


Figure: Relative maximum

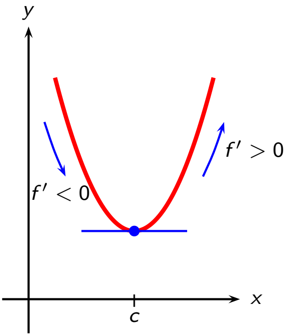


Figure: Relative minimum

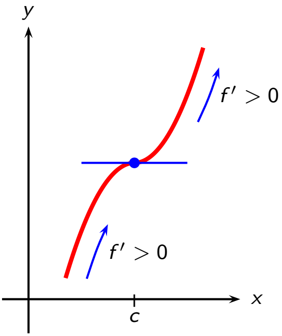


Figure: Not a relative extremum

Critical Points

Critical Points

- A number c in the domain of a differentiable function $f(x)$ is called a *critical number* if $f'(c) = 0$.
- The corresponding point $(c, f(c))$ on the graph of $f(x)$ is called a *critical point* for $f(x)$.

The First Derivative Test for Relative Extrema

The First Derivative Test for Relative Extrema

Let $f(x)$ have a critical point at $x = c$ (i.e., $f'(c) = 0$).
Then, the critical point $(c, f(c))$ is:

- a *relative maximum* if $f'(x) > 0$ to the left of c and $f'(x) < 0$ to the right of c;
- a *relative minimum* if $f'(x) < 0$ to the left of c and $f'(x) > 0$ to the right of c;
- *not a relative extremum* if $f'(x)$ has the same sign on both sides of c.

Example of the Test for Relative Extrema

Example

Find the critical numbers for the function
$f(x) = \frac{3}{4}x^4 - 4x^3 + 6x^2 - 3$
and classify each critical point as a relative maximum, relative minimum, or neither.

Solution...

The derivative of $f(x)$ is continuous everywhere:

$$f'(x) = 3x^3 - 12x^2 + 12x = 3x(x-2)^2.$$

The critical points:

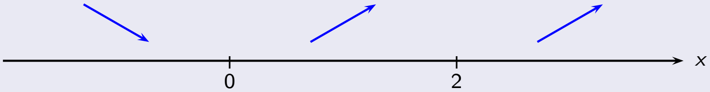
$$f'(x) = 0$$
$$x = 0 \quad \text{or} \quad x = 2$$

□

Example of the Test for Relative Extrema. (Continued)

. . . Solution.

The arrow diagram:



The function has a relative minimum at $x = 0$
and no extremum at $x = 2$. □

Example of the Test for Relative Extrema. (Continued)

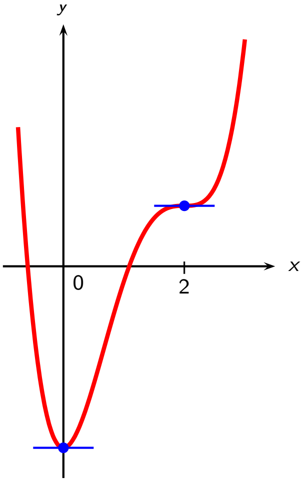


Figure: The graph of $f(x) = \frac{3}{4}x^4 - 4x^3 + 6x^2 - 3$.

Application: Optimization of Revenue

Example

The revenue derived from the sale of x units of a certain commodity is

$$R(x) = \frac{3x - x^2}{x^2 + 3} \quad \text{million euros.}$$

What level of production results in maximum revenue?

Solution...

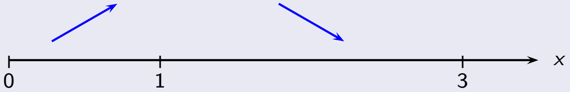
In the application, the revenue function $R(x)$ makes sense only for $x \geq 0$ and $R(x) \geq 0$; i.e. $0 \leq x \leq 3$. □

Application: Optimization of Revenue. (Continued)

. . . Solution.

$$R'(x) = \frac{(3-2x)(x^2+3)-(3x-x^2)(2x)}{(x^2+3)^2} = \frac{-3(x^2+2x-3)}{(x^2+3)^2} = \frac{-3(x-1)(x+3)}{(x^2+3)^2}.$$

The arrow diagram:



The maximum revenue for the level of production $x = 1$. $\square$

Application: Optimization of Revenue. (Continued)

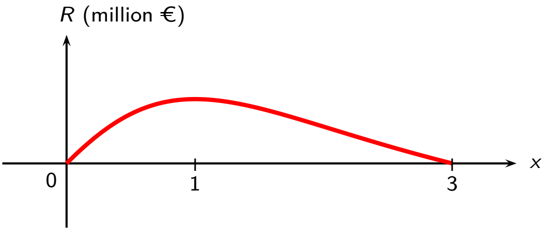


Figure: The graph of $R(x) = \frac{3x-x^2}{x^2+3}$.

For Further Reading

- `http://fberisha.netfirms.com`
- Homework: Exercises from teaching materials
- L. D. Hofmann, G. L. Bradley, *Calculus – for business, economics and life sciences,* pp. 161–210.
- F. M. Berisha, M. Q. Berisha, *Matematikë – për biznes dhe ekonomiks,* pp. 199–213.

Summary

- The rate of change $f''(x)$ of the rate of change $f'(x)$ of a function $f(x)$
- Determining the intervals of increase and decrease of a function $f(x)$ by means of its first derivative $f'(x)$
- Relative extrema
- The first derivative test for relative extrema