Chapter Four

USING THE DERIVATIVE

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4.1 USING FIRST AND SECOND DERIVATIVES

What Derivatives Tell Us About a Function and its Graph

As we saw in Chapter 2, the connection between derivatives of a function and the function itself is given by the following:

- If $f' > 0$ on an interval, then f is increasing on that interval. Thus, f is monotonic over that interval.
- If $f' < 0$ on an interval, then f is decreasing on that interval. Thus, f is monotonic over that interval.
- If $f'' > 0$ on an interval, then the graph of f is concave up on that interval.
- If $f'' < 0$ on an interval, then the graph of f is concave down on that interval.

We can do more with these principles now than we could in Chapter 2 because we now have formulas for the derivatives of the elementary functions.

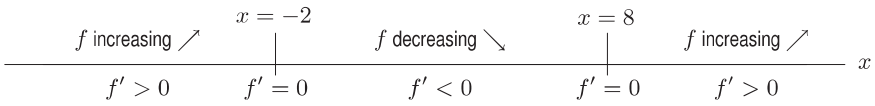
When we graph a function on a computer or calculator, we often see only part of the picture, and we may miss some significant features. Information given by the first and second derivatives can help identify regions with interesting behavior.

Example 1 Consider the function $f(x) = x^3 - 9x^2 - 48x + 52$.

Solution Since f is a cubic polynomial, we expect a graph that is roughly S-shaped. We can use the derivative to determine where the function is increasing and where it is decreasing. The derivative of f is

$$f'(x) = 3x^2 - 18x - 48.$$

To find where $f' > 0$ or $f' < 0$, we first find where $f' = 0$, that is, where $3x^2 - 18x - 48 = 0$. Factoring, we get $3(x+2)(x-8) = 0$, so $x = -2$ or $x = 8$. Since $f' = 0$ *only* at $x = -2$ and $x = 8$, and since f' is continuous, f' cannot change sign on any of the three intervals $x < -2$, or $-2 < x < 8$, or $8 < x$. How can we tell the sign of f' on each of these intervals? One way is to pick a point and substitute into f'. For example, since $f'(-3) = 33 > 0$, we know f' is positive for $x < -2$, so f is increasing for $x < -2$. Similarly, since $f'(0) = -48$ and $f'(10) = 72$, we know that f decreases between $x = -2$ and $x = 8$ and increases for $x > 8$. Summarizing:



We find that $f(-2) = 104$ and $f(8) = -396$. Hence on the interval $-2 < x < 8$ the function decreases from a high of 104 to a low of -396. One more point on the graph is easy to get: the y intercept, $f(0) = 52$. With just these three points we can get a graph. On a calculator, a window of $-10 \le x \le 20$ and $-400 \le y \le 400$ gives the graph in Figure 4.1.

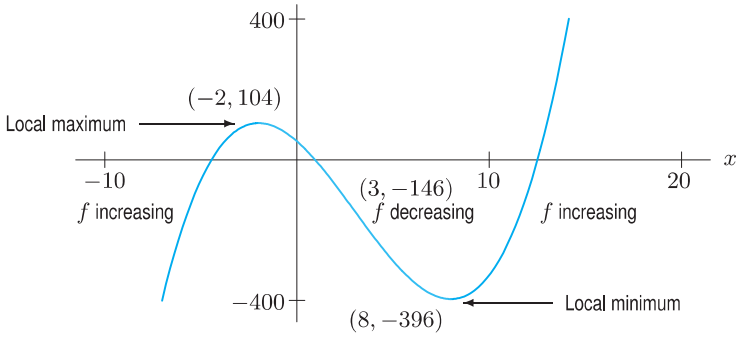
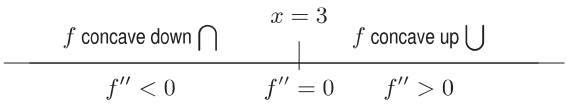


Figure 4.1: Useful graph of $f(x) = x^3 - 9x^2 - 48x + 52$

In Figure 4.1, we see that part of the graph is concave up and part is concave down. We can use the second derivative to analyze concavity. We have

$$f''(x) = 6x - 18.$$

Thus, $f''(x) < 0$ when $x < 3$ and $f''(x) > 0$ when $x > 3$, so the graph of f is concave down for $x < 3$ and concave up for $x > 3$. At $x = 3$, we have $f''(x) = 0$. Summarizing:



Local Maxima and Minima

We are often interested in points such as those marked local maximum and local minimum in Figure 4.1. We have the following definition:

Suppose p is a point in the domain of f:

- f has a **local minimum** at p if $f(p)$ is less than or equal to the values of f for points near p.
- f has a **local maximum** at p if $f(p)$ is greater than or equal to the values of f for points near p.

We use the adjective "local" because we are describing only what happens near p. Local maxima and minima are sometimes called *local extrema.*

How Do We Detect a Local Maximum or Minimum?

In the preceding example, the points $x = -2$ and $x = 8$, where $f'(x) = 0$, played a key role in leading us to local maxima and minima. We give a name to such points:

For any function f, a point p in the domain of f where $f'(p) = 0$ or $f'(p)$ is undefined is called a **critical point** of the function. In addition, the point $(p, f(p))$ on the graph of f is also called a critical point. A **critical value** of f is the value, $f(p)$, at a critical point, p.

Notice that "critical point of f" can refer either to points in the domain of f or to points on the graph of f. You will know which meaning is intended from the context.

Geometrically, at a critical point where $f'(p) = 0$, the line tangent to the graph of f at p is horizontal. At a critical point where $f'(p)$ is undefined, there is no horizontal tangent to the graph—there's either a vertical tangent or no tangent at all. (For example, $x = 0$ is a critical point for the absolute value function $f(x) = |x|$.) However, most of the functions we work with are differentiable everywhere, and therefore most of our critical points are of the $f'(p) = 0$ variety.

The critical points divide the domain of f into intervals on which the sign of the derivative remains the same, either positive or negative. Therefore, if f is defined on the interval between two successive critical points, its graph cannot change direction on that interval; it is either increasing or decreasing. The following result, which is proved on page 181, tells us that all local maxima and minima which are not at endpoints occur at critical points.

Theorem 4.1: Local Extrema and Critical Points

Suppose f is defined on an interval and has a local maximum or minimum at the point $x = a$, which is not an endpoint of the interval. If f is differentiable at $x = a$, then $f'(a) = 0$. Thus, a is a critical point.

Warning! Not every critical point is a local maximum or local minimum. Consider $f(x) = x^3$, which has a critical point at $x = 0$. (See Figure 4.2.) The derivative, $f'(x) = 3x^2$, is positive on both sides of $x = 0$, so f increases on both sides of $x = 0$, and there is neither a local maximum nor a local minimum at $x = 0$.

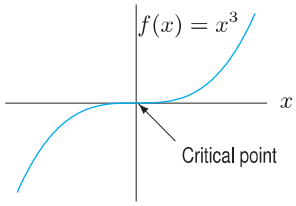


Figure 4.2: Critical point which is not a local maximum or minimum

Testing For Local Maxima and Minima at a Critical Point

If f' has different signs on either side of a critical point p, with $f'(p) = 0$, then the graph changes direction at p and looks like one of those in Figure 4.3. So we have the following criterion:

The First-Derivative Test for Local Maxima and Minima

Suppose p is a critical point of a continuous function f.

- If f' changes from negative to positive at p, then f has a local minimum at p.
- If f' changes from positive to negative at p, then f has a local maximum at p.

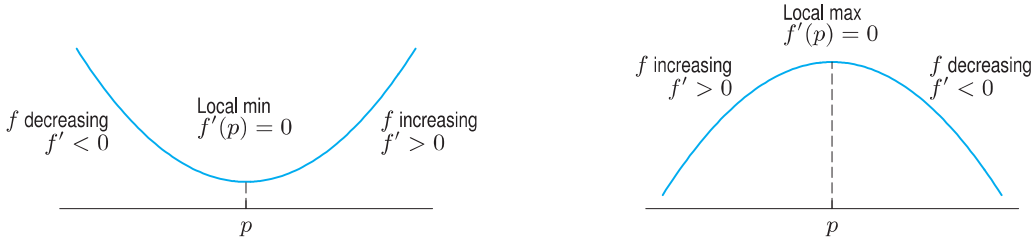


Figure 4.3: Changes in direction at a critical point, p: Local maxima and minima

Example 2 Use a graph of the function $f(x) = \dfrac{1}{x(x-1)}$ to observe its local maxima and minima. Confirm your observation analytically.

Solution The graph in Figure 4.4 suggests that this function has no local minima but that there is a local maximum at about $x = \frac{1}{2}$. Confirming this analytically means using the formula for the derivative to show that what we expect is true. Since $f(x) = (x^2 - x)^{-1}$, we have

$$f'(x) = -1(x^2 - x)^{-2}(2x - 1) = -\frac{2x - 1}{(x^2 - x)^2}.$$

So $f'(x) = 0$ where $2x - 1 = 0$. Thus, the only critical point in the domain of f is $x = \frac{1}{2}$.

Furthermore, $f'(x) > 0$ where $0 < x < 1/2$, and $f'(x) < 0$ where $1/2 < x < 1$. Thus, f increases for $0 < x < 1/2$ and decreases for $1/2 < x < 1$. According to the first derivative test, the critical point $x = 1/2$ is a local maximum.

For $-\infty < x < 0$ or $1 < x < \infty$, there are no critical points and no local maxima or minima. Although $1/(x(x-1)) \to 0$ both as $x \to \infty$ and as $x \to -\infty$, we don't say 0 is a local minimum because $1/\,(x(x-1))$ never actually *equals* 0.

Notice that although $f' > 0$ everywhere that it is defined for $x < 1/2$, the function f is not increasing throughout this interval. The problem is that f and f' are not defined at $x = 0$, so we cannot conclude that f is increasing when $x < 1/2$.

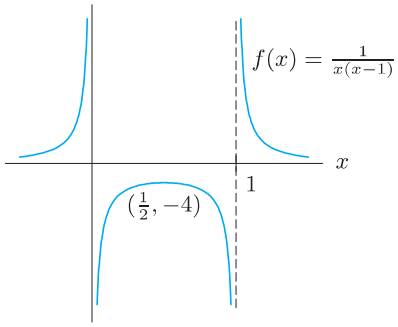


Figure 4.4: Find local maxima and minima

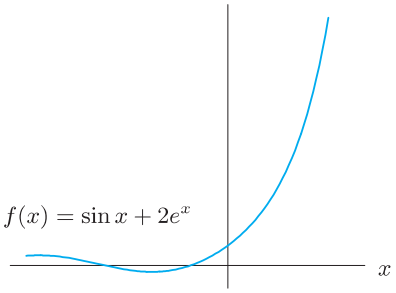


Figure 4.5: Explain the absence of local maxima and minima for $x \geq 0$

Example 3 The graph of $f(x) = \sin x + 2e^x$ is in Figure 4.5. Using the derivative, explain why there are no local maxima or minima for $x \geq 0$.

Solution Local maxima and minima can occur only at critical points. Now $f'(x) = \cos x + 2e^x$, which is defined for all x. We know $\cos x$ is always between -1 and 1, and $2e^x \geq 2$ for $x \geq 0$, so $f'(x)$ cannot be 0 for any $x \geq 0$. Therefore there are no local maxima or minima there.

The Second-Derivative Test for Local Maxima and Minima

Knowing the concavity of a function can also be useful in testing if a critical point is a local maximum or a local minimum. Suppose p is a critical point of f, with $f'(p) = 0$, so that the graph of f has a horizontal tangent line at p. If the graph is concave up at p, then f has a local minimum at p. Likewise, if the graph is concave down, f has a local maximum. (See Figure 4.6.) This suggests:

The Second-Derivative Test for Local Maxima and Minima

- If $f'(p) = 0$ and $f''(p) > 0$ then f has a local minimum at p.
- If $f'(p) = 0$ and $f''(p) < 0$ then f has a local maximum at p.
- If $f'(p) = 0$ and $f''(p) = 0$ then the test tells us nothing.

Figure 4.6: Local maxima and minima and concavity

Example 4 Classify as local maxima or local minima the critical points of $f(x) = x^3 - 9x^2 - 48x + 52$.

Solution As we saw in Example 1 on page 176,

$$f'(x) = 3x^2 - 18x - 48$$

and the critical points of f are $x = -2$ and $x = 8$. We have

$$f''(x) = 6x - 18.$$

Thus $f''(8) = 30 > 0$, so f has a local minimum at $x = 8$. Since $f''(-2) = -30 < 0$, f has a local maximum at $x = -2$.

Warning! The second-derivative test does not tell us anything if both $f'(p) = 0$ and $f''(p) = 0$. For example, if $f(x) = x^3$ and $g(x) = x^4$, both $f'(0) = g'(0) = 0$ and $f''(0) = g''(0) = 0$. The point $x = 0$ is a minimum for g but is neither a maximum nor a minimum for f. However, the first-derivative test is still useful. For example, g' changes sign from negative to positive at $x = 0$, so we know g has a local minimum there.

Concavity and Inflection Points

Investigating points where the slope changes sign led us to critical points. Now we look at points where the concavity changes.

A point at which the graph of a function changes concavity is called an **inflection point** of f. Suppose f' is continuous.

- A point p where f'' is zero or undefined is a possible inflection point.
- To test whether p is an inflection point, check whether f'' changes sign at p.

The words "inflection point of f" can refer either to a point in the domain of f or to a point on the graph of f. The context of the problem will tell you which is meant.

Example 5 Find the critical and inflection points for $g(x) = xe^{-x}$ and sketch the graph of g for $x \geq 0$.

Solution Taking derivatives and simplifying, we have

$$g'(x) = (1 - x)e^{-x} \quad \text{and} \quad g''(x) = (x - 2)e^{-x}.$$

So $x = 1$ is a critical point, and $g' > 0$ for $x < 1$ and $g' < 0$ for $x > 1$. Hence g increases to a local maximum at $x = 1$ and then decreases. Also, $g(x) \to 0$ as $x \to \infty$. There is an inflection point at $x = 2$ since $g'' < 0$ for $x < 2$ and $g'' > 0$ for $x > 2$. The graph is sketched in Figure 4.7.

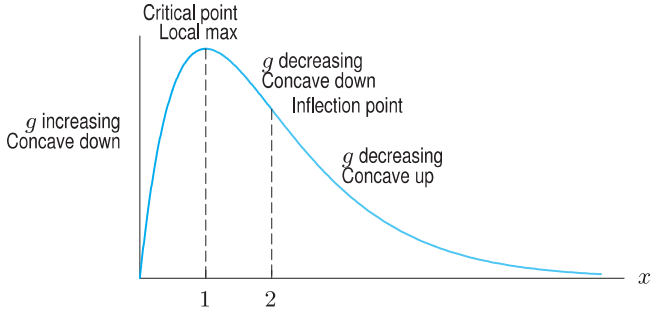


Figure 4.7: Graph of $g(x) = xe^{-x}$

Warning! Not every point x where $f''(x) = 0$ (or f'' is undefined) is an inflection point (just as not every point where $f' = 0$ is a local maximum or minimum). For instance $f(x) = x^4$ has $f''(x) = 12x^2$ so $f''(0) = 0$, but $f'' > 0$ when $x > 0$ and when $x < 0$, so there is *no* change in concavity at $x = 0$. See Figure 4.8.

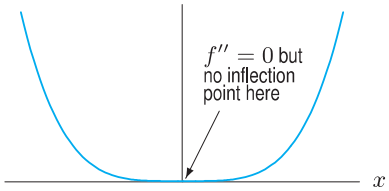


Figure 4.8: Graph of $f(x) = x^4$

Inflection Points and Local Maxima and Minima of the Derivative

Inflection points can also be interpreted in terms of first derivatives. Applying the First Derivative Test for local maxima and minima to f', we obtain the following result:

> Suppose a function f has a continuous derivative. If f'' changes sign at p, then f has an inflection point at p, and f' has a local minimum or a local maximum at p.

Figure 4.9 shows two inflection points. Notice that the curve crosses the tangent line at these points and that the slope of the curve is a maximum or a minimum there.

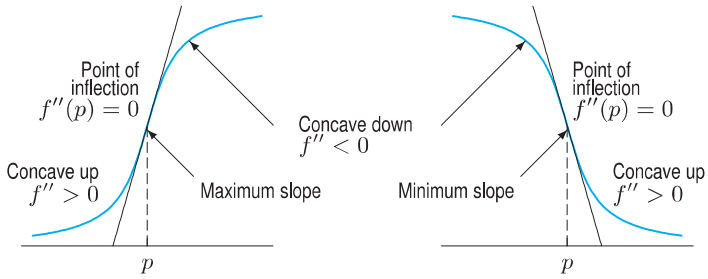


Figure 4.9: Change in concavity at p: Points of inflection

Example 6 Water is being poured into the vase in Figure 4.10 at a constant rate, measured in volume per unit time. Graph $y = f(t)$, the depth of the water against time, t. Explain the concavity and indicate the inflection points.

Solution At first the water level, y, rises slowly because the base of the vase is wide, and it takes a lot of water to make the depth increase. However, as the vase narrows, the rate at which the water is rising increases. Thus, y is increasing at an increasing rate and the graph is concave up. The rate of increase in the water level is at a maximum when the water reaches the middle of the vase, where the diameter is smallest; this is an inflection point. After that, the rate at which y increases decreases again, so the graph is concave down. (See Figure 4.11.)

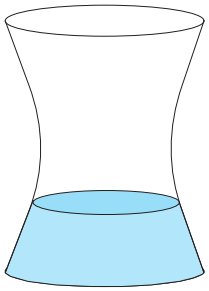


Figure 4.10: A vase

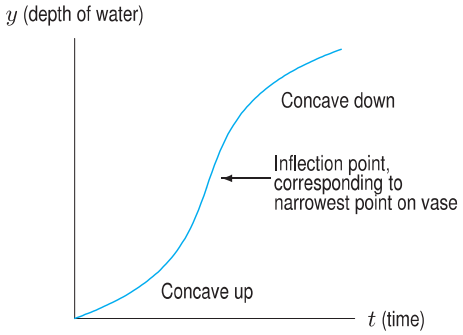


Figure 4.11: Graph of depth of water in the vase, y, against time, t

Showing Local Extrema Are At Critical Points

We now prove Theorem 4.1, which says that inside an interval, local maxima and minima can only occur at critical points. Suppose that f has a local maximum at $x = a$. Assuming that $f'(a)$ is defined, the definition of the derivative gives

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}.$$

Since this is a two-sided limit, we have

$$f'(a) = \lim_{h\to 0^-} \frac{f(a+h)-f(a)}{h} = \lim_{h\to 0^+} \frac{f(a+h)-f(a)}{h}.$$

By the definition of local maximum, $f(a+h) \le f(a)$ for all sufficiently small h. Thus $f(a+h) - f(a) \le 0$ for sufficiently small h. The denominator, h, is positive when we take the limit from the right and negative when we take the limit from the left. Thus

$$\lim_{h\to 0^-} \frac{f(a+h)-f(a)}{h} \ge 0 \quad \text{and} \quad \lim_{h\to 0^+} \frac{f(a+h)-f(a)}{h} \le 0.$$

Since both these limits are equal to $f'(a)$, we have $f'(a) \ge 0$ and $f'(a) \le 0$, so we must have $f'(a) = 0$. The proof for a local minimum at $x = a$ is similar.

Exercises and Problems for Section 4.1

Exercises

In Exercises 1–6, find all critical points and then use the first-derivative test to determine local maxima and minima. Check your answer by graphing.

1. $f(x) = 2x^3+3x^2-36x$

2. $f(x) = 3x^4-4x^3+6$

3. $f(x) = (x^2-4)^7$

4. $f(x) = (x^3-8)^4$

5. $f(x) = \dfrac{x}{x^2+1}$

6. $f(x) = 2x^2e^{5x}+1$

In Exercises 7–8, classify the critical points of the functions as local maxima or local minima or neither.

7. $g(x) = xe^{-x}$

8. $h(x) = x + 1/x$

9. Graph a function which has exactly one critical point, at $x = 2$, and exactly one inflection point, at $x = 4$.

10. Graph a function with exactly two critical points, one of which is a local minimum and the other is neither a local maximum nor a local minimum.

11. **(a)** Use a graph to estimate the x-values of any critical points and inflection points of $f(x) = e^{-x^2}$.
(b) Use derivatives to find the x-values of any critical points and inflection points exactly.

12. Find all critical points of $f(x) = 10.2x^2e^{-0.4x}$.

13. Find the inflection points of $f(x) = x^4+x^3-3x^2+2$.

For the graphs of f' in Exercises 14–17, decide:
(a) Over what intervals is f increasing? Decreasing?
(b) Does f have local maxima or minima? If so, which, and where?

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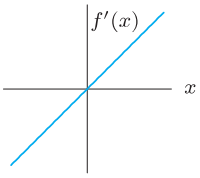

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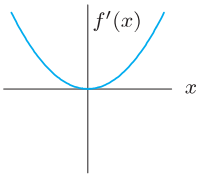

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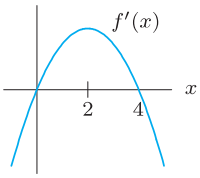

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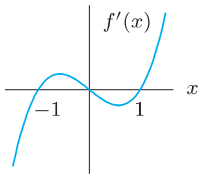

18. Indicate all critical points on the graph of f in Figure 4.12 and determine which correspond to local maxima of f, which to local minima, and which to neither.

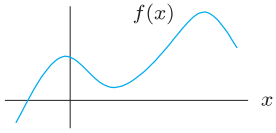

Figure 4.12

19. Indicate on the graph of the derivative function f' in Figure 4.13 the x-values that are critical points of the function f itself. At which critical points does f have local maxima, local minima, or neither?

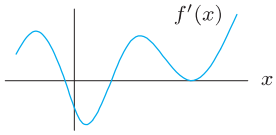

Figure 4.13

20. Indicate on the graph of the derivative f' in Figure 4.14 the x-values that are inflection points of the function f.

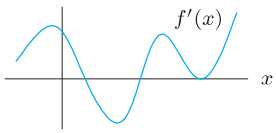

Figure 4.14

21. Indicate on the graph of the second derivative f'' in Figure 4.15 the x-values that are inflection points of the function f.

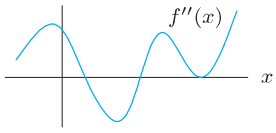

Figure 4.15

Problems

22. **(a)** Show that if a is a positive constant, then $x = 0$ is the only critical point of $f(x) = x + a\sqrt{x}$.
(b) Use derivatives to show that f is increasing and its graph is concave down for all $x > 0$.

23. **(a)** If b is a positive constant and $x > 0$, find all critical points of $f(x) = x - b \ln x$.
(b) Use the second derivative test to determine whether the function has a local maximum or local minimum at each critical point.

24. **(a)** If a is a nonzero constant, find all critical points of

$$f(x) = \frac{a}{x^2} + x.$$

(b) Use the second derivative test to show that if a is positive then the graph has a local minimum, and if a is negative then the graph has a local maximum.

25. If U and V are positive constants, find all critical points of

$$F(t) = Ue^t + Ve^{-t}.$$

26. Give examples of graphs of functions with no, one, and infinitely many critical points.

27. Graph $f(x) = x + \sin x$, and determine where f is increasing most rapidly and least rapidly.

For Problems 28–31, sketch a possible graph of $y = f(x)$, using the given information about the derivatives $y' = f'(x)$ and $y'' = f''(x)$. Assume that the function is defined and continuous for all real x.

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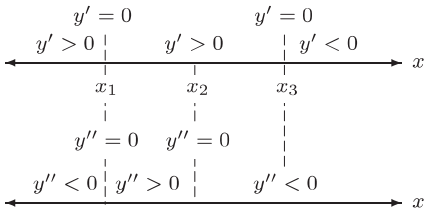

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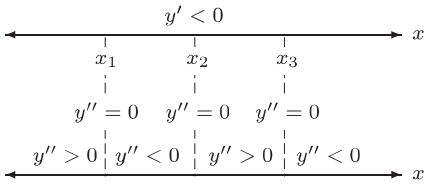

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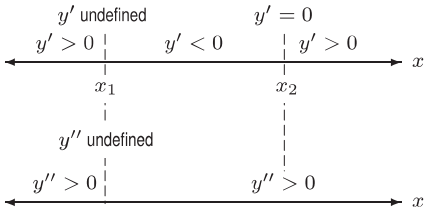

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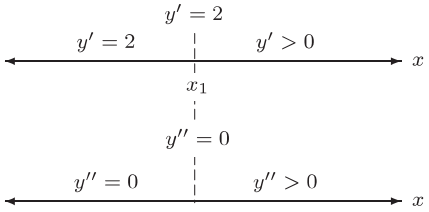


32. Suppose f has a continuous derivative whose values are given in the following table.

(a) Estimate the x-coordinates of critical points of f for $0 \le x \le 10$.
(b) For each critical point, indicate if it is a local maximum of f, local minimum, or neither.

x	0	1	2	3	4	5	6	7	8	9	10
$f'(x)$	5	2	1	-2	-5	-3	-1	2	3	1	-1

33. **(a)** The following table gives values of the differentiable function $y = f(x)$. Estimate the x-values of critical points of $f(x)$ on the interval $0 < x < 10$. Classify each critical point as a local maximum, local minimum, or neither.
(b) Now assume that the table gives values of the continuous function $y = f'(x)$ (instead of $f(x)$). Estimate and classify critical points of the function $f(x)$.

x	0	1	2	3	4	5	6	7	8	9	10
y	1	2	1	-2	-5	-3	-1	2	3	1	-1

34. **(a)** Water is flowing at a constant rate (i.e., constant volume per unit time) into a cylindrical container standing vertically. Sketch a graph showing the depth of water against time.
(b) Water is flowing at a constant rate into a cone-shaped container standing on its point. Sketch a graph showing the depth of the water against time.

35. If water is flowing at a constant rate (i.e., constant volume per unit time) into the vase in Figure 4.16, sketch a graph of the depth of the water against time. Mark on the graph the time at which the water reaches the corner of the vase.

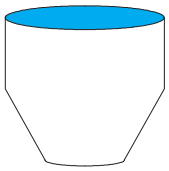


Figure 4.16

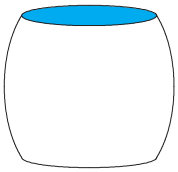


Figure 4.17

36. If water is flowing at a constant rate (i.e., constant volume per unit time) into the Grecian urn in Figure 4.17, sketch a graph of the depth of the water against time. Mark on the graph the time at which the water reaches the widest point of the urn.

37. Find and classify the critical points of $f(x) = x^3(1-x)^4$ as local maxima and minima.

38. If $m, n \geq 2$ are integers, find and classify the critical points of $f(x) = x^m(1-x)^n$.

39. **(a)** Find and simplify the second derivative of

$$P = \frac{1}{1+10e^{-t}}.$$

(b) Sketch P and d^2P/dt^2 on separate axes. Label any asymptotes. On both graphs, label t_0, the t-coordinate of the inflection point on the graph of P.

40. The rabbit population on a small Pacific island is approximated by

$$P = \frac{2000}{1+e^{5.3-0.4t}}$$

with t measured in years since 1774, when Captain James Cook left 10 rabbits on the island.

(a) Using a calculator, graph P. Does the population level off?

(b) Estimate when the rabbit population grew most rapidly. How large was the population at that time?

(c) What natural causes could lead to the shape of the graph of P?

41. Find values of a and b so that the function $f(x) = x^2 + ax + b$ has a local minimum at the point $(6, -5)$.

42. Find the value of a so that the function $f(x) = xe^{ax}$ has a critical point at $x = 3$.

43. Find constants a and b in the function $f(x) = axe^{bx}$ such that $f(\frac{1}{3}) = 1$ and the function has a local maximum at $x = \frac{1}{3}$.

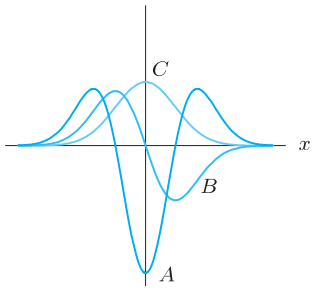
44. You might think the graph of $f(x) = x^2 + \cos x$ should look like a parabola with some waves on it. Sketch the actual graph of $f(x)$ using a calculator or computer. Explain what you see using $f''(x)$.

45. Graph f given that:

- $f'(x) = 0$ at $x = 2$, $f'(x) < 0$ for $x < 2$, $f'(x) > 0$ for $x > 2$,
- $f''(x) = 0$ at $x = 4$, $f''(x) > 0$ for $x < 4$, $f''(x) < 0$ for $x > 4$.

Problems 46–47 show graphs of the three functions f, f', f''. Identify which is which.

46.



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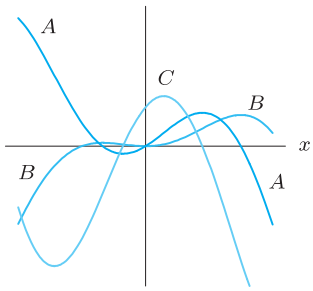

Problems 48–49 show graphs of f, f', f''. Each of these three functions is either odd or even. Decide which functions are odd and which are even. Use this information to identify which graph corresponds to f, which to f', and which to f''.

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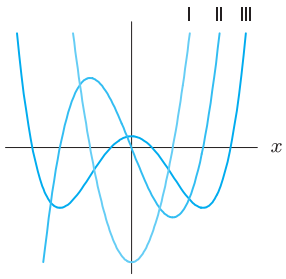

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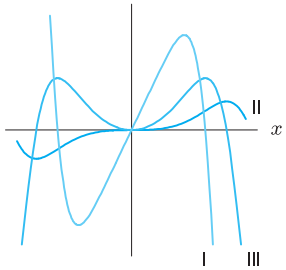


50. Figure 4.18 is a graph of the function

$$h(t) = \frac{t^2 - 2t}{(t-2)(t^2+1)}.$$

The point marked A is an inflection point; point B is a local maximum; and point C is a hole in the graph. Find the coordinates of the point

(a) A **(b)** B **(c)** C

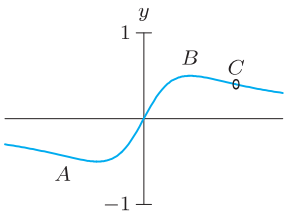


Figure 4.18

51. Assume that the polynomial f has exactly two local maxima and one local minimum, and that these are the only critical points of f.

(a) Sketch a possible graph of f.

(b) What is the largest number of zeros f could have?
(c) What is the least number of zeros f could have?
(d) What is the least number of inflection points f could have?
(e) What is the smallest degree f could have?
(f) Find a possible formula for $f(x)$.

52. Let f be a function with $f(x) > 0$ for all x. Set $g = 1/f$.

(a) If f is increasing in an interval around x_0, what about g?
(b) If f has a local maximum at x_1, what about g?
(c) If f is concave down at x_2, what about g?

53. What happens to concavity when functions are added?

(a) If $f(x)$ and $g(x)$ are concave up for all x, is $f(x) + g(x)$ concave up for all x?
(b) If $f(x)$ is concave up for all x and $g(x)$ is concave down for all x, what can you say about the concavity of $f(x) + g(x)$? For example, what happens if $f(x)$ and $g(x)$ are both polynomials of degree 2?
(c) If $f(x)$ is concave up for all x and $g(x)$ is concave down for all x, is it possible for $f(x) + g(x)$ to change concavity infinitely often?

4.2 OPTIMIZATION

The largest and smallest values of a quantity often have practical importance. For example, automobile engineers want to construct a car that uses the least amount of fuel, scientists want to calculate which wavelength carries the maximum radiation at a given temperature, and urban planners want to design traffic patterns to minimize delays. Such problems belong to the field of mathematics called *optimization*. The next three sections show how the derivative provides an efficient way of solving many optimization problems.

Global Maxima and Minima

The single greatest (or least) value of a function f over a specified domain is called the *global maximum* (or *minimum*) of f. Recall that the local maxima and minima tell us where a function is locally largest or smallest. Now we are interested in where the function is absolutely largest or smallest in a given domain. We say

- f has a **global minimum** at p if $f(p)$ is less than or equal to all values of f.
- f has a **global maximum** at p if $f(p)$ is greater than or equal to all values of f.

Global maxima and minima are sometimes called *extrema* or *optimal values*.

Existence of Global Extrema

The following theorem describes when global extrema are guaranteed to exist:

Theorem 4.2: The Extreme Value Theorem

If f is continuous on the closed interval $a \le x \le b$, then f has a global maximum and a global minimum on that interval.

For a proof of Theorem 4.2, see www.wiley.com/college/hugheshallett.

How Do We Find Global Maxima and Minima?

If f is a continuous function defined on a closed interval $a \le x \le b$ (that is, an interval containing its endpoints), then Theorem 4.2 guarantees that global maxima and minima exist. Figure 4.19 illustrates that the global maximum or minimum of f occurs either at a critical point or at an endpoint of the interval, $x = a$ or $x = b$. These points are the candidates for global extrema.

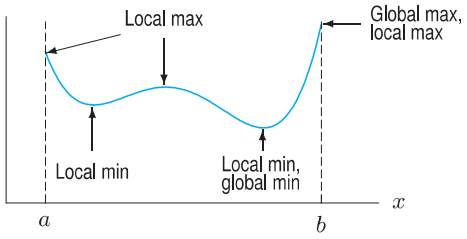


Figure 4.19: Global maximum and minimum on a closed interval $a \le x \le b$

Global Maxima and Minima on a Closed Interval: Test the Candidates

For a continuous function f on a closed interval $a \le x \le b$:

- Find the critical points of f in the interval.
- Evaluate the function at the critical points and at the endpoints, a and b.

.

If the function is defined on an open interval $a < x < b$ (that is, an interval not including its endpoints) or on all real numbers, there may or may not be a global maximum or minimum. For example, there is no global maximum in Figure 4.20 because the function has no actual largest value. The global minimum in Figure 4.20 coincides with the local minimum. There is a global minimum but no global maximum in Figure 4.21.

Global Maxima and Minima on an Open Interval or on All Real Numbers

For a continuous function, f, find the value of f at all the critical points and sketch a graph. Look at values of f when x approaches the endpoints of the interval, or approaches $\pm\infty$, as appropriate. If there is only one critical point, look at the sign of f' on either side of the critical point.

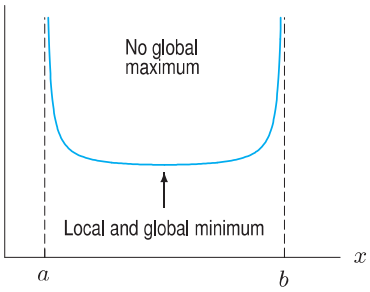


Figure 4.20: Global minimum on $a < x < b$

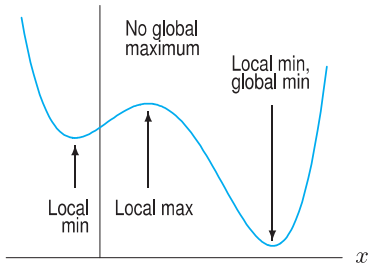


Figure 4.21: Global minimum when the domain is all real numbers

Example 1 Find the global maxima and minima of $f(x) = x^3 - 9x^2 - 48x + 52$ on the following intervals:

(a) $-5 \le x \le 12$ (b) $-5 \le x \le 14$ (c) $-5 \le x < \infty$.

Solution (a) We have previously obtained the critical points $x = -2$ and $x = 8$ using

$$f'(x) = 3x^2 - 18x - 48 = 3(x + 2)(x - 8).$$

We evaluate f at the critical points and the endpoints of the interval:

$$\begin{aligned} f(-5) &= (-5)^3 - 9(-5)^2 - 48(-5) + 52 = -58 \\ f(-2) &= 104 \\ f(8) &= -396 \\ f(12) &= -92. \end{aligned}$$

Comparing these function values, we see that the global maximum on $[-5, 12]$ is 104 and occurs at $x = -2$, and the global minimum on $[-5, 12]$ is -396 and occurs at $x = 8$.

(b) For the interval $[-5, 14]$, we compare

$$f(-5) = -58, \quad f(-2) = 104, \quad f(8) = -396, \quad f(14) = 360.$$

The global maximum is now 360 and occurs at $x = 14$, and the global minimum is still -396 and occurs at $x = 8$. Since the function is increasing for $x > 8$, changing the right-hand end of the interval from $x = 12$ to $x = 14$ alters the global maximum but not the global minimum. See Figure 4.22.

(c) Figure 4.22 shows that for $-5 \le x < \infty$ there is no global maximum, because we can make $f(x)$ as large as we please by choosing x large enough. The global minimum remains -396 at $x = 8$.

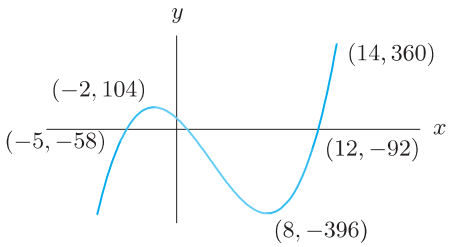


Figure 4.22: Graph of $f(x) = x^3 - 9x^2 - 48x + 52$

Example 2 (a) For a positive constant b, the surge function $f(t) = te^{-bt}$ gives the quantity of a drug in the body for time $t \ge 0$. Find the global maxima and minima of $f(t)$ for $t \ge 0$.

(b) Find the value of b making $t = 10$ the global maximum.

Solution (a) Differentiating and factoring gives

$$f'(t) = 1 \cdot e^{-bt} - bte^{-bt} = (1 - bt)e^{-bt},$$

Thus there is a critical point at $t = 1/b$.

The sign of f' is determined by the sign of $(1 - bt)$, so f' is positive to the left of $t = 1/b$ and negative to the right of $t = 1/b$. Since f increases to the left of $t = 1/b$ and decreases to the right of $t = 1/b$, the global maximum occurs at $t = 1/b$. In addition, $f(0) = 0$ and $f(t) \ge 0$ for all $t \ge 0$, so the global minimum occurs at $t = 0$. Thus

The global maximum value is $f\left(\dfrac{1}{b}\right) = \dfrac{1}{b}e^{-b(1/b)} = \dfrac{e^{-1}}{b}$.

The global minimum value is $f(0) = 0$.

(b) Since $x = 10$ gives the global maximum, we have $1/b = 10$, so $b = 0.1$. See Figure 4.23.

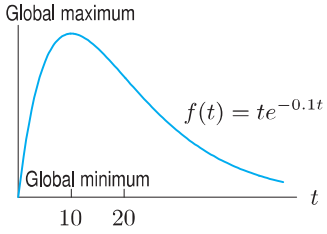


Figure 4.23: Graph of $f(t) = te^{-bt}$ for $b = 0.1$

Example 3 When you cough, your windpipe contracts. The speed, $v(r)$, with which you expel air depends on the radius, r, of your windpipe. If a is the normal (rest) radius of your windpipe, then for $0 \le r \le a$, the speed is given by:

$$v(r) = k(a - r)r^2 \quad \text{where } k \text{ is a positive constant.}$$

What value of r maximizes the speed? For what value is the speed minimized?

Solution For $0 \le r \le a$, the speed at which air is expelled is given by

$$v(r) = k(a - r)r^2 = kar^2 - kr^3.$$

Thus, the derivative is defined for all r and given by

$$v'(r) = 2kar - 3kr^2 = 2kr\left(a - \frac{3}{2}r\right).$$

The derivative is zero if $r = \frac{2}{3}a$ or $r = 0$. These are the critical points of v. To decide if the critical points give global maxima or minima, we evaluate v at the critical point:

$$\begin{aligned} v\left(\frac{2}{3}a\right) &= k\left(a - \frac{2}{3}a\right)\left(\frac{2}{3}a\right)^2 \\ &= k\left(\frac{a}{3}\right)\frac{4a^2}{9} \\ &= \frac{4ka^3}{27}, \end{aligned}$$

and we evaluate v at the endpoints:

$$v(0) = v(a) = 0.$$

Thus, v has a global maximum at $r = \frac{2}{3}a$. The global minimum of $v = 0$ occurs at both endpoints $r = 0$ and $r = a$.

Example 4 When an arrow is shot into the air, its range, R, is defined as the horizontal distance from the archer to the point where the arrow hits the ground. If the ground is horizontal and we neglect air resistance, it can be shown that

$$R = \frac{{v_0}^2 \sin(2\theta)}{g},$$

where v_0 is the initial velocity of the arrow, g is the (constant) acceleration due to gravity, and θ is the angle above horizontal, so $0 \le \theta \le \pi/2$. (See Figure 4.24.) What angle θ maximizes R?

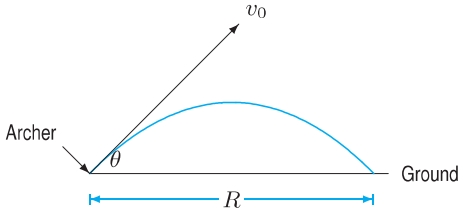


Figure 4.24: Arrow's path

Solution We can find the maximum of this function without using calculus. The maximum value of R occurs when $\sin(2\theta) = 1$, so $\theta = \arcsin(1)/2 = \pi/4$, giving $R = v_0^2/g$.

Let's see how we can do the same problem with calculus. We want to find the global maximum of R for $0 \le \theta \le \pi/2$. First we look for critical points:

$$\frac{dR}{d\theta} = 2\frac{v_0^2 \cos(2\theta)}{g}.$$

Setting $dR/d\theta$ equal to 0, we get

$$0 = \cos(2\theta), \quad \text{or} \quad 2\theta = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2}, \ldots$$

so $\pi/4$ is the only critical point in the interval $0 \le \theta \le \pi/2$. The range at $\theta = \pi/4$ is $R = {v_0}^2/g$.

Now we must check the value of R at the endpoints $\theta = 0$ and $\theta = \pi/2$. Since $R = 0$ at each endpoint (the arrow is shot horizontally or vertically), the critical point $\theta = \pi/4$ gives both a local and a global maximum on $0 \le \theta \le \pi/2$. Therefore, the arrow goes farthest if shot at an angle of $\pi/4$, or $45°$.

A Graphical Example: Minimizing Gas Consumption

Next we look at an example in which a function is given graphically and the optimum values are read from a graph. You already know how to estimate the optimum values of $f(x)$ from a graph of $f(x)$—read off the highest and lowest values. In this example, we see how to estimate the optimum value of the quantity $f(x)/x$ from a graph of $f(x)$ against x.

The question we investigate is how to set driving speeds to maximize fuel efficiency.[1] We assume that gas consumption, g (in gallons/hour), as a function of velocity, v (in mph) is as shown in Figure 4.25. We want to minimize the gas consumption per *mile*, not the gas consumption per hour. Let $G = g/v$ represent the gas consumption per mile. (The units of G are gallons/mile.)

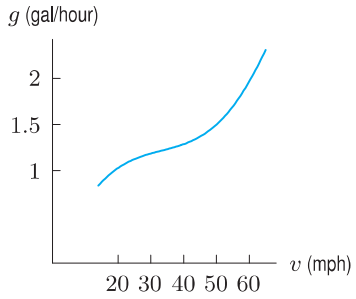


Figure 4.25: Gas consumption versus velocity

Example 5 Using Figure 4.25, estimate the velocity which minimizes G.

Solution We want to find the minimum value of $G = g/v$ when g and v are related by the graph in Figure 4.25. We could use Figure 4.25 to sketch a graph of G against v and estimate a critical point. But there is an easier way.

Figure 4.26 shows that g/v is the slope of the line from the origin to the point P. Where on the curve should P be to make the slope a minimum? From the possible positions of the line shown in

[1] Adapted from Peter D. Taylor, *Calculus: The Analysis of Functions* (Toronto: Wall & Emerson, 1992).

Figure 4.26, we see that the slope of the line is both a local and global minimum when the line is tangent to the curve. From Figure 4.27, we can see that the velocity at this point is about 50 mph. Thus to minimize gas consumption per mile, we should drive about 50 mph.

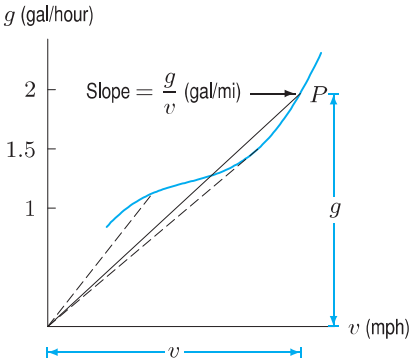


Figure 4.26: Graphical representation of gas consumption per mile, $G = g/v$

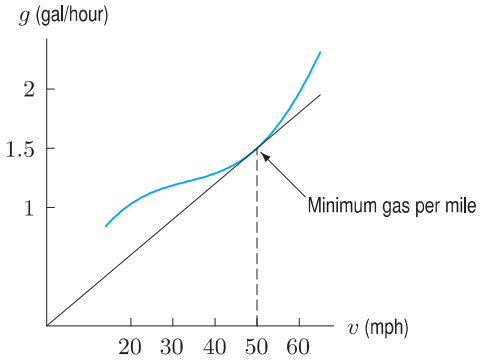


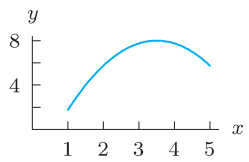
Figure 4.27: Velocity for maximum fuel efficiency

Exercises and Problems for Section 4.2

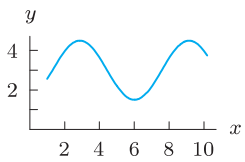
Exercises

For Exercises 1–2, indicate all critical points on the given graphs. Determine which correspond to local minima, local maxima, global minima, global maxima, or none of these. (Note that the graphs are on closed intervals.)

1.



2.



3. Find the value of x that maximizes $y = 12 + 18x - 5x^2$ and the corresponding value of y , by

(a) Estimating the values from a graph of y.
(b) Finding the values using calculus.

4. For $x > 0$, find the x-value and the corresponding y-value that maximizes $y = 25 + 6x^2 - x^3$, by

(a) Estimating the values from a graph of y.
(b) Finding the values using calculus.

In Exercises 5–12, find the global maximum and minimum for the function on the closed interval.

5. $f(x) = x^3 - 3x^2 + 20, \quad -1 \le x \le 3$

6. $f(x) = x^4 - 8x^2, \quad -3 \le x \le 1$

7. $f(x) = xe^{-x^2/2}, \quad -2 \le x \le 2$

8. $f(x) = 3x^{1/3} - x, \quad -1 \le x \le 8$

9. $f(x) = e^{-x} \sin x, \quad 0 \le x \le 2\pi$

10. $f(x) = x - 2\ln(x+1), \quad 0 \le x \le 2$

11. $f(x) = e^x + \cos x, \quad 0 \le x \le \pi$

12. $f(x) = \dfrac{x+1}{x^2+3}, \quad -1 \le x \le 2$

In Exercises 13–15, find the value(s) of x for which:

(a) $f(x)$ has a local maximum or local minimum. Indicate which ones are maxima and which are minima.
(b) $f(x)$ has a global maximum or global minimum.

13. $f(x) = x^{10} - 10x$, and $0 \le x \le 2$

14. $f(x) = x - \ln x$, and $0.1 \le x \le 2$

15. $f(x) = \sin^2 x - \cos x$, and $0 \le x \le \pi$

Problems

In Problems 16–21, find the exact global maximum and minimum values of the function. The domain is all real numbers unless otherwise specified.

16. $g(x) = 4x - x^2 - 5$

17. $f(x) = x + 1/x$ for $x > 0$

18. $g(t) = te^{-t}$ for $t > 0$

19. $f(x) = x - \ln x$ for $x > 0$

20. $f(t) = \dfrac{t}{1+t^2}$

21. $f(t) = (\sin^2 t + 2)\cos t$

22. Find the value(s) of x that give critical points of $y = ax^2 + bx + c$, where a, b, c are constants. Under what conditions on a, b, c is the critical value a maximum? A minimum?

23. What value of w minimizes S if $S - 5pw = 3qw^2 - 6pq$ and p and q are positive constants?

24. Let $y = at^2e^{-bt}$ with a and b positive constants. For $t \geq 0$, what value of t maximizes y? Sketch the curve if $a = 1$ and $b = 1$.

25. A grapefruit is tossed straight up with an initial velocity of 50 ft/sec. The grapefruit is 5 feet above the ground when it is released. Its height at time t is given by

$$y = -16t^2 + 50t + 5.$$

How high does it go before returning to the ground?

26. For some positive constant C, a patient's temperature change, T, due to a dose, D, of a drug is given by

$$T = \left(\frac{C}{2} - \frac{D}{3}\right) D^2.$$

(a) What dosage maximizes the temperature change?
(b) The sensitivity of the body to the drug is defined as dT/dD. What dosage maximizes sensitivity?

27. A warehouse selling cement has to decide how often and in what quantities to reorder. It is cheaper, on average, to place large orders, because this reduces the ordering cost per unit. On the other hand, larger orders mean higher storage costs. The warehouse always reorders cement in the same quantity, q. The total weekly cost, C, of ordering and storage is given by

$$C = \frac{a}{q} + bq, \quad \text{where } a, b \text{ are positive constants.}$$

(a) Which of the terms, a/q and bq, represents the ordering cost and which represents the storage cost?
(b) What value of q gives the minimum total cost?

28. A chemical reaction converts substance A to substance Y; the presence of Y catalyzes the reaction. At the start of the reaction, the quantity of A present is a grams. At time t seconds later, the quantity of Y present is y grams. The rate of the reaction, in grams/sec, is given by

$$\text{Rate} = ky(a - y), \quad k \text{ is a positive constant.}$$

(a) For what values of y is the rate nonnegative? Graph the rate against y.
(b) For what values of y is the rate a maximum?

29. For positive constants A and B , the force between two atoms in a molecule is given by

$$f(r) = -\frac{A}{r^2} + \frac{B}{r^3},$$

where $r > 0$ is the distance between the atoms. What value of r minimizes the force between the atoms?

30. When an electric current passes through two resistors with resistance r_1 and r_2, connected in parallel, the combined resistance, R, can be calculated from the equation

$$\frac{1}{R} = \frac{1}{r_1} + \frac{1}{r_2},$$

where R, r_1, and r_2 are positive. Assume that r_2 is constant.

(a) Show that R is an increasing function of r_1.
(b) Where on the interval $a \leq r_1 \leq b$ does R take its maximum value?

31. As an epidemic spreads through a population, the number of infected people, I, is expressed as a function of the number of susceptible people, S, by

$$I = k \ln\left(\frac{S}{S_0}\right) - S + S_0 + I_0, \quad \text{for } k, S_0, I_0 > 0.$$

(a) Find the maximum number of infected people.
(b) The constant k is a characteristic of the particular disease; the constants S_0 and I_0 are the values of S and I when the disease starts. Which of the following affects the maximum possible value of I? Explain.

- The particular disease, but not how it starts.
- How the disease starts, but not the particular disease.
- Both the particular disease and how it starts.

32. The distance, s, traveled by a cyclist, who starts at 1 pm, is given in Figure 4.28. Time, t, is in hours since noon.

(a) Explain why the quantity, s/t, is represented by the slope of a line from the origin to the point (t, s) on the graph.
(b) Estimate the time at which the quantity s/t is a maximum.
(c) What is the relationship between the quantity s/t and the instantaneous speed of the cyclist at the time you found in part (b)?

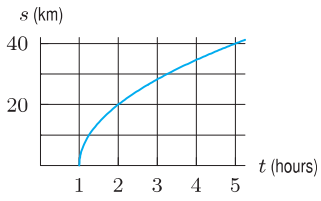


Figure 4.28

33. A line goes through the origin and a point on the curve $y = x^2e^{-3x}$, for $x \geq 0$. Find the maximum slope of such a line. At what x-value does it occur?

34. Two points on the curve $y = \dfrac{x^3}{1 + x^4}$ have opposite x-values, x and $-x$. Find the points making the slope of the line joining them greatest.

35. When birds lay eggs, they do so in clutches of several at a time. When the eggs hatch, each clutch gives rise to a brood of baby birds. We want to determine the clutch size which maximizes the number of birds surviving to adulthood per brood. If the clutch is small, there are few baby birds in the brood; if the clutch is large, there are so many baby birds to feed that most die of starvation. The number of surviving birds per brood as a function of clutch size is shown by the benefit curve in Figure 4.29.[2]

(a) Estimate the clutch size which maximizes the number of survivors per brood.
(b) Suppose also that there is a biological cost to having a larger clutch: the female survival rate is reduced by large clutches. This cost is represented by the dotted line in Figure 4.29. If we take cost into account by assuming that the optimal clutch size in fact maximizes the vertical distance between the curves, what is the new optimal clutch size?

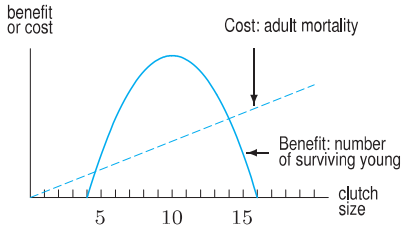


Figure 4.29

36. Let $f(v)$ be the amount of energy consumed by a flying bird, measured in joules per second (a joule is a unit of energy), as a function of its speed v (in meters/sec). Let $a(v)$ be the amount of energy consumed by the same bird, measured in joules per meter.

(a) Suggest a reason (in terms of the way birds fly) for the shape of the graph of $f(v)$ in Figure 4.30.
(b) What is the relationship between $f(v)$ and $a(v)$?
(c) Where is $a(v)$ a minimum?
(d) Should the bird try to minimize $f(v)$ or $a(v)$ when it is flying? Why?

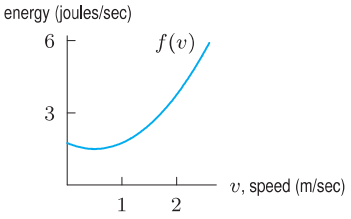


Figure 4.30

37. The forward motion of an aircraft in level flight is reduced by two kinds of forces, known as *induced drag* and *parasite drag*. Induced drag is a consequence of the downward deflection of air as the wings produce lift. Parasite drag results from friction between the air and the entire surface of the aircraft. Induced drag is inversely proportional to the square of speed and parasite drag is directly proportional to the square of speed. The sum of induced drag and parasite drag is called total drag. The graph in Figure 4.31 shows a certain aircraft's induced drag and parasite drag functions.

(a) Sketch the total drag as a function of airspeed.
(b) Estimate two different airspeeds which each result in a total drag of 1000 pounds. Does the total drag function have an inverse? What about the induced and parasite drag functions?
(c) Fuel consumption (in gallons per hour) is roughly proportional to total drag. Suppose you are low on fuel and the control tower has instructed you to enter a circular holding pattern of indefinite duration to await the passage of a storm at your landing field. At what airspeed should you fly the holding pattern? Why?

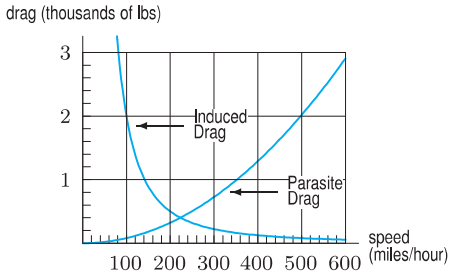


Figure 4.31

38. Let $f(v)$ be the fuel consumption, in gallons per hour, of a certain aircraft as a function of its airspeed, v, in miles per hour. A graph of $f(v)$ is given in Figure 4.32.

(a) Let $g(v)$ be the fuel consumption of the same aircraft, but measured in gallons per mile instead of gallons per hour. What is the relationship between $f(v)$ and $g(v)$?
(b) For what value of v is $f(v)$ minimized?
(c) For what value of v is $g(v)$ minimized?
(d) Should a pilot try to minimize $f(v)$ or $g(v)$?

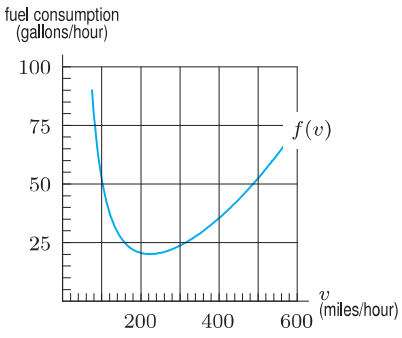


Figure 4.32

[2]Data from C. M. Perrins and D. Lack reported by J. R. Krebs and N. B. Davies in *An Introduction to Behavioural Ecology* (Oxford: Blackwell, 1987).

39. The function $y = t(x)$ is positive and continuous with a global maximum at the point $(3, 3)$. Graph $t(x)$ if $t'(x)$ and $t''(x)$ have the same sign for $x < 3$, but opposite signs for $x > 3$.

40. Figure 4.33 gives the derivative of $g(x)$ on $-2 \le x \le 2$.

(a) Write a few sentences describing the behavior of $g(x)$ on this interval.
(b) Does the graph of $g(x)$ have any inflection points? If so, give the approximate x-coordinates of their locations. Explain your reasoning.
(c) What are the global maxima and minima of g on $[-2, 2]$?
(d) If $g(-2) = 5$, what do you know about $g(0)$ and $g(2)$? Explain.

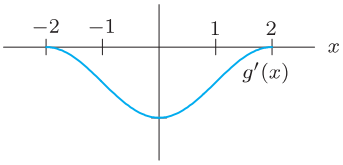


Figure 4.33

41. Figure 4.34 shows the second derivative of $h(x)$ for $-2 \le x \le 1$. If $h'(-1) = 0$ and $h(-1) = 2$,

(a) Explain why $h'(x)$ is never negative on this interval.
(b) Explain why $h(x)$ has a global maximum at $x = 1$.
(c) Sketch a possible graph of $h(x)$ for $-2 \le x \le 1$.

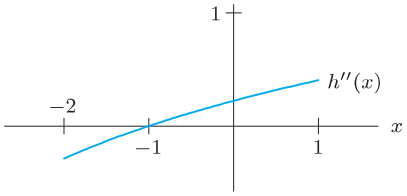


Figure 4.34

42. Show that if $f''(x)$ is continuous and $f(x)$ has exactly two critical points, then $f'(x)$ has a local maximum or local minimum between the two critical points.

43. In this problem we prove a special case of the Mean Value Theorem where $f(a) = f(b) = 0$. This special case is called Rolle's Theorem: If f is continuous on $[a, b]$ and differentiable on (a, b), and if $f(a) = f(b) = 0$, then there is a number c, with $a < c < b$, such that

$$f'(c) = 0.$$

By the Extreme Value Theorem, f has a global maximum and a global minimum on $[a, b]$.

(a) Prove Rolle's theorem in the case that both the global maximum and the global minimum are at endpoints of $[a, b]$. [Hint: $f(x)$ must be a very simple function in this case.]
(b) Prove Rolle's theorem in the case that either the global maximum or the global minimum is not at an endpoint. [Hint: Think about local maxima and minima.]

44. Use Rolle's Theorem to prove the Mean Value Theorem. Suppose that $f(x)$ is continuous on $[a, b]$ and differentiable on (a, b). Let $g(x)$ be the difference between $f(x)$ and the y-value on the secant line joining $(a, f(a))$ to $(b, f(b))$, so

$$g(x) = f(x) - f(a) - \frac{f(b) - f(a)}{b - a}(x - a).$$

(a) Show $g(x)$ on a sketch of $f(x)$.
(b) Use Rolle's Theorem (Problem 43) to show that there must be a point c in (a, b) such that $g'(c) = 0$.
(c) Show that if c is the point in part (b), then

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

4.3 FAMILIES OF FUNCTIONS

We saw in Chapter 1 that knowledge of one function can provide knowledge of the graphs of many others. The shape of the graph of $y = x^2$ also tells us, indirectly, about the graphs of $y = x^2 + 2$, $y = (x + 2)^2$, $y = 2x^2$, and countless other functions. We say that all functions of the form $y = a(x + b)^2 + c$ form a *family of functions*; their graphs are like that of $y = x^2$, except for shifts and stretches determined by the values of a, b, and c. The constants a, b, c are called *parameters*. Different values of the parameters give different members of the family.

One reason for studying families of functions is their use in mathematical modeling. Confronted with the problem of modeling some phenomenon, a crucial first step involves recognizing families of functions which might fit the available data.

Motion Under Gravity: $y = -4.9t^2 + v_0t + y_0$

The position of an object moving vertically under the influence of gravity can be described by a function in the two-parameter family

$$y = -4.9t^2 + v_0t + y_0$$

where t is time in seconds and y is the distance in meters above the ground. Why do we need the parameters v_0 and y_0 to describe all such motions? Notice that at time $t = 0$ we have $y = y_0$. Thus the parameter y_0 gives the height above ground of the object at time $t = 0$. Since $dy/dt = -9.8t + v_0$, the parameter v_0 gives the velocity of the object at time $t = 0$. From this equation we see that $dy/dt = 0$ when $t = v_0/9.8$. This is the time when the object reaches its maximum height.

The Bell-Shaped Curve: $y = e^{-(x-a)^2/b}$

This family is related to the *normal density* function, used in probability and statistics.[3] We assume that $b > 0$. See Section 8.8 for applications of the normal distribution.

First we let $b = 1$; see Figure 4.35. The role of the parameter a is to shift the graph of $y = e^{-x^2}$ to the right or left. Notice that the value of y is always positive. Since $y \to 0$ as $x \to \pm\infty$, the x-axis is a horizontal asymptote. Thus $y = e^{-(x-a)^2}$ is the family of horizontal shifts of the bell-shaped curve.

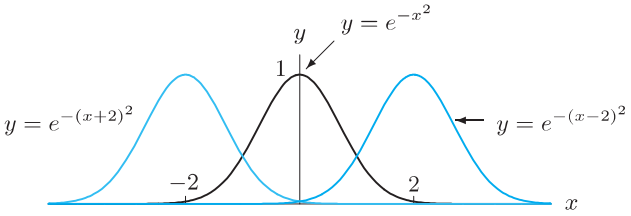


Figure 4.35: The family $y = e^{-(x-a)^2}$

We now consider the role of the parameter b by studying the family with $a = 0$:

$$y = e^{-x^2/b}.$$

To investigate the critical points and points of inflection, we calculate

$$\frac{dy}{dx} = -\frac{2x}{b}e^{-x^2/b}$$

and, using the product rule, we get

$$\frac{d^2y}{dx^2} = -\frac{2}{b}e^{-x^2/b} - \frac{2x}{b}\left(-\frac{2x}{b}e^{-x^2/b}\right) = \frac{2}{b}\left(\frac{2x^2}{b} - 1\right)e^{-x^2/b}.$$

Critical points occur where $dy/dx = 0$, that is, where

$$\frac{dy}{dx} = -\frac{2x}{b}\,e^{-x^2/b} = 0.$$

Since $e^{-x^2/b}$ is never zero, the only critical point is $x = 0$. At that point, $y = 1$ and $d^2y/dx^2 < 0$. Hence, by the second derivative test, there is a local maximum at $x = 0$, and this is also a global maximum.

Inflection points occur where the second derivative changes sign; thus, we start by finding values of x for which $d^2y/dx^2 = 0$. Since $e^{-x^2/b}$ is never zero, $d^2y/dx^2 = 0$ when

$$\frac{2x^2}{b} - 1 = 0.$$

Solving for x gives

$$x = \pm\sqrt{\frac{b}{2}}.$$

[3] Probabilists divide our function by a constant, $\sqrt{\pi b}$, to get the normal density.

Looking at the expression for d^2y/dx^2, we see that d^2y/dx^2 is negative for $x = 0$, and positive as $x \to \pm\infty$. Therefore the concavity changes at $x = -\sqrt{b/2}$ and at $x = \sqrt{b/2}$, so we have inflection points there.

Returning to the two-parameter family $y = e^{-(x-a)^2/b}$, we conclude that there is a maximum at $x = a$, obtained by horizontally shifting the maximum at $x = 0$ of $y = e^{-x^2/b}$ by a units. There are inflection points at $x = a \pm \sqrt{b/2}$ obtained by shifting the inflection points $x = \pm\sqrt{b/2}$ of $y = e^{-x^2/b}$ by a units. (See Figure 4.36.) At these points $y = e^{-1/2} \approx 0.6$.

With this information we can see the effect of the parameters. The parameter a determines the location of the center of the bell and the parameter b determines how narrow or wide the bell is. (See Figure 4.37.) If b is small, then the inflection points are close to a and the bell is sharply peaked near a; if b is large, the inflection points are farther away from a and the bell is spread out.

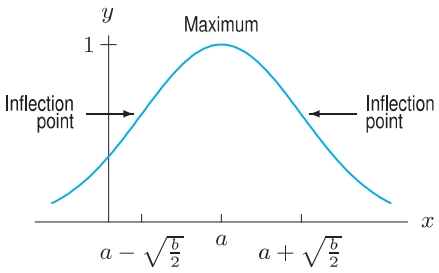


Figure 4.36: Graph of $y = e^{-(x-a)^2/b}$: bell-shaped curve with peak at $x = a$

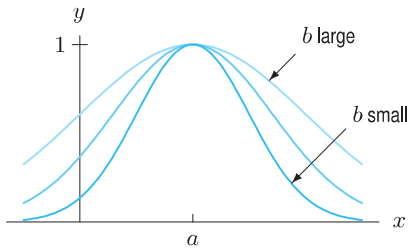


Figure 4.37: Graph of $y = e^{-(x-a)^2/b}$ for fixed a and various b

Exponential Model With a Limit: $y = a(1 - e^{-bx})$

We consider $a, b > 0$. The graph of one member, with $a = 2$ and $b = 1$, is in Figure 4.38. Such a graph represents a quantity which is increasing but leveling off. For example, a body dropped in a thick fluid speeds up initially, but its velocity levels off as it approaches a terminal velocity. Similarly, if a pollutant pouring into a lake builds up toward a saturation level, its concentration may be described in this way. The graph might also represent the temperature of an object in an oven.

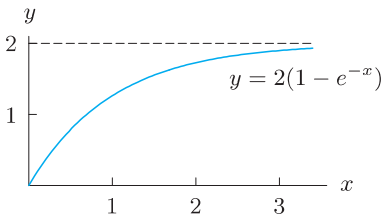


Figure 4.38: One member of the family $y = a(1 - e^{-bx})$, with $a = 2, b = 1$

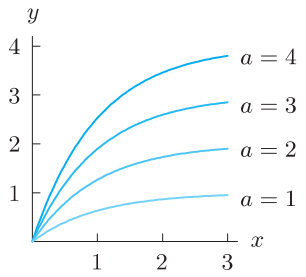


Figure 4.39: Fixing $b = 1$ gives $y = a(1 - e^{-x})$, graphed for various a

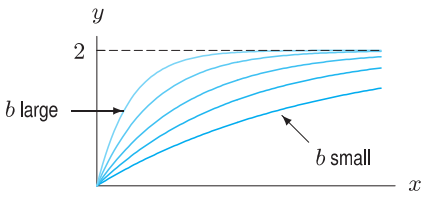


Figure 4.40: Fixing $a = 2$ gives $y = 2(1 - e^{-bx})$, graphed for various b

We examine the effect on the graph of varying a. Fix b at some positive number, say $b = 1$. Substitute different values for a and look at the graphs in Figure 4.39. We see that as x gets larger, y approaches a from below, so a is an upper bound for this function. Analytically this follows from the fact that $e^{-bx} \to 0$ as $x \to \infty$. Physically, the value of a represents the terminal velocity of a falling body or the saturation level of the pollutant in the lake.

Now examine the effect of varying b on the graph. Fix a at some positive number, say $a = 2$. Substitute different values for b and look at the graphs in Figure 4.40. The parameter b determines how sharply the curve rises and how quickly it gets close to the line $y = a$.

Let's confirm this last observation analytically. For $y = a(1-e^{-bx})$, we have $dy/dx = abe^{-bx}$, so the slope of the tangent to the curve at $x = 0$ is ab. For larger b, the curve rises more rapidly at $x = 0$. How long does it take the curve to climb halfway up from $y = 0$ to $y = a$? When $y = a/2$, we have

$$a(1 - e^{-bx}) = \frac{a}{2}, \qquad \text{which leads to} \qquad x = \frac{\ln 2}{b}.$$

If b is large then $(\ln 2)/b$ is small, so in a short distance the curve is already half way up to a. If b is small, then $(\ln 2)/b$ is large and we have to go a long way out to get up to $a/2$. See Figure 4.41.

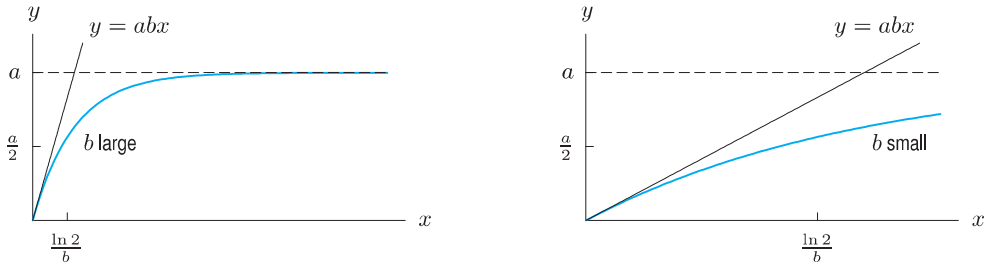


Figure 4.41: Tangent at $x = 0$ to $y = a(1 - e^{-bx})$, with fixed a, and large and small b

The Logistic Model: $y = L/(1 + Ae^{-kt})$

The *logistic* family is often used to model population growth when it is limited by the environment. (See Section 11.7.) We assume that $L, A, k > 0$ and we look at the roles of each of the three parameters in turn.

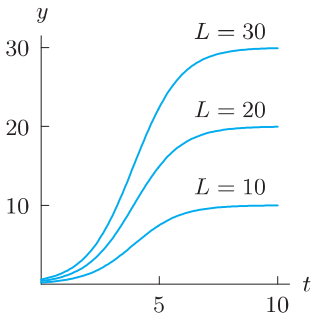


Figure 4.42: Graph of $y = L/(1 + Ae^{-kt})$ varying L with $A = 50$, $k = 1$

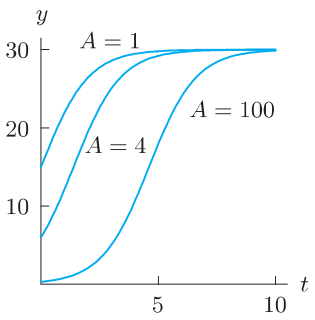


Figure 4.43: Graph of $y = L/(1 + Ae^{-kt})$ varying A with $L = 30$, $k = 1$

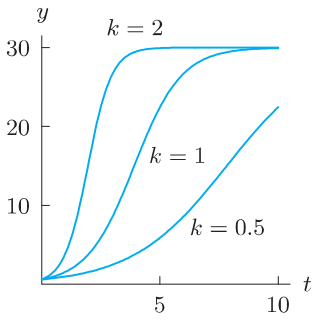


Figure 4.44: Graph of $y = L/(1 + Ae^{-kt})$ varying k, with $L = 30$, $A = 50$

Logistic curves with varying values of L are shown in Figure 4.42. The values of y level off as $t \to \infty$ because $Ae^{-kt} \to 0$ as $t \to \infty$. Thus, as t increases, the values of y approach L. The line $y = L$ is a horizontal asymptote, called the *limiting value* or *carrying capacity*, and representing the maximum sustainable population. The parameter L stretches or shrinks the graph vertically.

In Figure 4.43 we investigate the effect of the parameter A, with k and L fixed. The parameter A alters the point at which the curve cuts the y-axis—larger values of A move the y-intercept closer to the origin. At $t = 0$ we have $y = L/(1 + A)$, confirming that increasing A decreases the value of y at $t = 0$.

Figure 4.44 shows the effect of the parameter k. With L and A fixed, we see that varying k affects the rate at which the function approaches the limiting value L. If k is small, the graph rises slowly; if k is large, the graph rises steeply. At $t = 0$, we have $dy/dt = LAk/(1 + A)^2$, so the initial slope of a logistic curve is proportional to k.

The graphs suggest that none of the curves has a critical point for $t > 0$. Some curves appear to have a point of inflection; others have none. To investigate, we take derivatives:

$$\frac{dy}{dt} = \frac{LAke^{-kt}}{(1 + Ae^{-kt})^2}.$$

Since every factor of dy/dt is positive, the first derivative is always positive. Thus, there are no critical points and the function is always increasing.

Using a computer algebra system or the quotient rule, we find

$$\frac{d^2y}{dt^2} = \frac{LAk^2e^{-kt}(-1 + Ae^{-kt})}{(1 + Ae^{-kt})^3}.$$

Since L, A, k, e^{-kt}, and the denominator are always positive, the sign of d^2y/dt^2 is determined by the sign of $(-1 + Ae^{-kt})$. Points of inflection may occur where $d^2y/dt^2 = 0$. This is where $-1 + Ae^{-kt} = 0$, or

$$Ae^{-kt} = 1.$$

At this value of t,

$$y = \frac{L}{1 + Ae^{-kt}} = \frac{L}{1 + 1} = \frac{L}{2}.$$

In Problem 13, we see that d^2y/dt^2 changes sign at $y = L/2$. Since the concavity changes at $y = L/2$, there is a point of inflection when the population is half the carrying capacity. If the initial population is $L/2$ or above, there is no inflection point. (See the top graph in Figure 4.43.)

To find the value of t at the inflection point, we solve for t in the equation

$$\begin{aligned} Ae^{-kt} &= 1 \\ t &= \frac{\ln(1/A)}{-k} = \frac{\ln A}{k}. \end{aligned}$$

Thus, increasing the value of A moves the inflection point to the right. (See the bottom two graphs in Figure 4.43.)

Exercises and Problems for Section 4.3

Exercises

In Exercises 1–6, investigate the one parameter family of functions. Assume that a is positive.

(a) Graph $f(x)$ using three different values for a.

(b) Using your graph in part (a), describe the critical points of f and how they appear to move as a increases.

(c) Find a formula for the x-coordinates of the critical point(s) of f in terms of a.

1. $f(x) = (x - a)^2$

2. $f(x) = x^3 - ax$

3. $f(x) = ax^3 - x$

4. $f(x) = x - a\sqrt{x}$

5. $f(x) = x^2e^{-ax}$

6. $f(x) = \dfrac{a}{x^2} + x$ for $x > 0$

In Exercises 7–12, investigate the given two parameter family of functions. Assume that a and b are positive.

(a) Graph $f(x)$ using $b = 1$ and three different values for a.

(b) Graph $f(x)$ using $a = 1$ and three different values for b.

(c) In the graphs in parts (a) and (b), how do the critical points of f appear to move as a increases? As b increases?

(d) Find a formula for the x-coordinates of the critical point(s) of f in terms of a and b.

7. $f(x) = (x - a)^2 + b$

8. $f(x) = x^3 - ax^2 + b$

9. $f(x) = ax(x - b)^2$

10. $f(x) = \dfrac{ax}{x^2 + b}$

11. $f(x) = \sqrt{b - (x - a)^2}$

12. $f(x) = \dfrac{a}{x} + bx$ for $x > 0$

Problems

13. **(a)** Derive formulas for the first and second derivatives of the logistic function:

$$y = \frac{L}{1 + Ae^{-kt}} \quad \text{for } L, A, \text{ and } k \text{ positive constants.}$$

(b) Derive a formula for the t value of any inflection point(s).
(c) Use the second derivative to determine the concavity on either side af any inflection points.

14. Find the exact coordinates of the point at which the following curve is steepest:

$$y = \frac{50}{1 + 6e^{-2t}} \qquad \text{for } t \geq 0.$$

15. **(a)** Sketch graphs of $y = xe^{-bx}$ for $b = 1, 2, 3, 4$. Describe the graphical significance of b.
(b) Find the coordinates of the critical point of $y = xe^{-bx}$ and use it to confirm your answer to part (a).

16. Consider the surge function $y = axe^{-bx}$ for $a, b > 0$.

(a) Find the local maxima, local minima, and points of inflection.
(b) How does varying a and b affect the shape of the graph?
(c) On one set of axes, graph this function for several values of a and b.

17. Find a formula for the family of cubic polynomials with an inflection point at the origin. How many parameters are there?

18. If a and b are nonzero constants, find the domain and all critical points of

$$f(x) = \frac{ax^2}{x - b}.$$

19. If A and B are positive constants, find all critical points of

$$f(w) = \frac{A}{w^2} - \frac{B}{w}.$$

20. **(a)** Graph $f(x) = x + a \sin x$ for $a = 0.5$ and $a = 3$.
(b) For what values of a is $f(x)$ increasing for all x?

21. **(a)** Graph $f(x) = x^2 + a \sin x$ for $a = 1$ and $a = 20$.
(b) For what values of a is $f(x)$ concave up for all x?

22. Consider the family of functions $y = f(x) = x - k\sqrt{x}$, with k a positive constant and $x \geq 0$. Show that the graph of $f(x)$ has a local minimum at a point whose x-coordinate is $1/4$ of the way between its x-intercepts.

Find formulas for the functions described in Problems 23–34.

23. A cubic polynomial with a critical point at $x = 2$, an inflection point at $(1, 4)$, and a leading coefficient of 1.

24. A function of the form $y = at + b/t$, with a local minimum $(3, 12)$ and a local maximum at $(-3, -12)$.

25. A function of the form $y = a(1 - e^{-bx})$ with $a, b > 0$ and a horizontal asymptote of $y = 5$.

26. A function of the form $y = be^{-(x-a)^2/2}$ with its maximum at the point $(0, 3)$.

27. A curve of the form $y = e^{-(x-a)^2/b}$ for $b > 0$ with a local maximum at $x = 2$ and points of inflection at $x = 1$ and $x = 3$.

28. A logistic curve with carrying capacity of 12, y-intercept of 4, and point of inflection at $(0.5, 6)$.

29. A function of the form $y = \dfrac{a}{1 + be^{-t}}$ with y-intercept 2 and an inflection point at $t = 1$.

30. A fourth degree polynomial whose graph is symmetric about the y-axis, has a y-intercept of 0, and global maxima at $(1, 2)$ and $(-1, 2)$.

31. A function of the form $y = a \sin(bt^2)$ whose first critical point for positive t occurs at $t = 1$ and whose derivative is 3 when $t = 2$.

32. A function of the form $y = a \cos(bt^2)$ whose first critical point for positive t occurs at $t = 1$ and whose derivative is -2 when $t = 1/\sqrt{2}$.

33. A function of the form $y = ae^{-x} + bx$ with the global minimum at $(1, 2)$.

34. A function of the form $y = bxe^{-ax}$ with a local maximum at $(3, 6)$.

35. Let $f(x) = \sqrt{ax^2 - b}$, where a and b are positive constants.

(a) What is the domain of f?
(b) Evaluate the limit $\lim_{x \to \infty} f'(x)$.
(c) What is the graphical significance of the value of a?

36. Let $f(x) = bxe^{1+bx}$, where b is constant and $b > 0$.

(a) What is the x-coordinate of the critical point of f?
(b) Is the critical point a local maximum or a local minimum?
(c) Show that the y-coordinate of the critical point does not depend on the value of b.

37. The number, N, of people who have heard a rumor spread by mass media by time, t, is given by

$$N(t) = a(1 - e^{-kt}).$$

There are 200,000 people in the population who hear the rumor eventually. If 10% of them heard it the first day, find a and k, assuming t is measured in days.

38. The temperature, T, in $^\circ$ C, of a yam put into a 200°C oven is given as a function of time, t, in minutes, by

$$T = a(1 - e^{-kt}) + b.$$

(a) If the yam starts at 20°C, find a and b.
(b) If the temperature of the yam is initially increasing at 2°C per minute, find k.

39. **(a)** Find all critical points of $f(x) = x^4 + ax^2 + b$.
(b) Under what conditions on a and b does this function have exactly one critical point? What is the one critical point, and is it a local maximum, a local minimum, or neither?
(c) Under what conditions on a and b does this function have exactly three critical points? What are they? Which are local maxima and which are local minima?
(d) Is it ever possible for this function to have two critical points? No critical points? More than three critical points? Give an explanation in each case.

40. If $a > 0$, $b > 0$, show that $f(x) = a(1 - e^{-bx})$ is everywhere increasing and everywhere concave down.

41. A family of functions is given by

$$r(x) = \frac{1}{a + (x - b)^2}.$$

(a) For what values of a and b does the graph of r have a vertical asymptote? Where are the vertical asymptotes in this case?
(b) Find values of a and b so that the function r has a local maximum at the point $(3, 5)$.

42. For any constant a, let $f(x) = ax - x \ln x$ for $x > 0$.
(a) What is the x-intercept of the graph of $f(x)$?
(b) Graph $f(x)$ for $a = -1$ and $a = 1$.
(c) For what values of a does $f(x)$ have a critical point for $x > 0$? Find the coordinates of the critical point and decide if it is a local maximum, a local minimum, or neither.

43. Let $f(x) = x^2 + \cos(kx)$, for $k > 0$.
(a) Graph f for $k = 0.5, 1, 3, 5$. Find the smallest number k at which you see points of inflection in the graph of f.
(b) Explain why the graph of f has no points of inflection if $k \leq \sqrt{2}$, and infinitely many points of inflection if $k > \sqrt{2}$.
(c) Explain why f has only a finite number of critical points, no matter what the value of k.

44. Let $f(x) = e^x - kx$, for $k > 0$.
(a) Graph f for $k = 1/4, 1/2, 1, 2, 4$. Describe what happens as k changes.
(b) Show that f has a local minimum at $x = \ln k$.
(c) Find the value of k for which the local minimum is the largest.

45. Let $y = Ae^{-Bx^2}$ for positive A, B. Analyze the effect of varying A and B on the shape of the curve. Illustrate your answer with sketches.

46. Let $g(x) = x - ke^x$, where k is any constant. For what value(s) of k does the function g have a critical point?

47. Let $h(x) = e^{-x} + kx$, where k is any constant. For what value(s) of k does h have
(a) No critical points?
(b) One critical point?
(c) A horizontal asymptote?

48. Sketch several members of the family $y = e^{-ax} \sin bx$ for $b = 1$, and describe the graphical significance of the parameter a.

49. Sketch several members of the family $e^{-ax} \sin bx$ for $a = 1$, and describe the graphical significance of the parameter b.

50. Consider the family

$$y = \frac{A}{x + B}.$$

(a) If $B = 0$, what is the effect of varying A on the graph?
(b) If $A = 1$, what is the effect of varying B?
(c) On one set of axes, graph the function for several values of A and B.

51. For positive a, b, the potential energy, U, of a particle is

$$U = b\left(\frac{a^2}{x^2} - \frac{a}{x}\right) \quad \text{for } x > 0.$$

(a) Find the intercepts and asymptotes.
(b) Compute the local maxima and minima.
(c) Sketch the graph.

52. The force, F, on a particle with potential energy U is given by

$$F = -\frac{dU}{dx}.$$

Using the expression for U in Problem 51, graph F and U on the same axes, labeling intercepts and local maxima and minima.

53. The Lennard-Jones model predicts the potential energy $V(r)$ of a two-atom molecule as a function of the distance r between the atoms to be

$$V(r) = \frac{A}{r^{12}} - \frac{B}{r^6}, \quad r > 0,$$

where A and B are positive constants.

(a) Evaluate $\lim_{r \to 0^+} V(r)$, and interpret your answer.
(b) Find the critical point of $V(r)$. Is it a local maximum or local minimum?
(c) The inter-atomic force is given by $F(r) = -V'(r)$. At what distance r is the inter-atomic force zero? (This is called the equilibrium size of the molecule.)
(d) Describe how the parameters A and B affect the equilibrium size of the molecule.

54. For positive A, B, the force between two atoms is a function of the distance, r, between them:

$$f(r) = -\frac{A}{r^2} + \frac{B}{r^3} \qquad r > 0.$$

(a) What are the zeros and asymptotes of f?
(b) Find the coordinates of the critical points and inflection points of f.
(c) Graph f.
(d) Illustrating your answers with a sketch, describe the effect on the graph of f of:
(i) Increasing B, holding A fixed
(ii) Increasing A, holding B fixed

55. Consider the family of functions $y = a\cosh(x/a)$ for $a > 0$. Sketch graphs for $a = 1, 2, 3$. Describe in words the effect of increasing a.

56. Let $y = Ae^x + Be^{-x}$ for any constants A, B.

(a) Sketch the function for
(i) $A = 1,\ B = 1$ (ii) $A = 1,\ B = -1$
(iii) $A = 2,\ B = 1$ (iv) $A = 2,\ B = -1$
(v) $A = -2,\ B = -1$ (vi) $A = -2,\ B = 1$
(b) Describe in words the general shape of the graph if A and B have the same sign. What effect does the sign of A have on the graph?
(c) Describe in words the general shape of the graph if A and B have different signs. What effect does the sign of A have on the graph?
(d) For what values of A and B does the function have a local maximum? A local minimum? Justify your answer using derivatives.

57. An organism has size W at time t. For positive constants A, b, and c, the Gompertz growth function gives

$$W = Ae^{-e^{b-ct}}, \quad t \geq 0.$$

(a) Find the intercepts and asymptotes.
(b) Find the critical points and inflection points.
(c) Graph W for various values of A, b, and c.
(d) A certain organism grows fastest when it is about 1/3 of its final size. Would the Gompertz growth function be useful in modeling its growth? Explain.

4.4 OPTIMIZATION, GEOMETRY, AND MODELING

Finding global maxima and minima is often made possible by having a formula for the function to be maximized or minimized. The process of translating a problem into a function with a known formula is called *mathematical modeling*. The examples that follow give the flavor of modeling.

Example 1 What are the dimensions of an aluminum can that holds 40 in^3 of juice and that uses the least material (i.e., aluminum)? Assume that the can is cylindrical, and is capped on both ends.

Solution It is often a good idea to think about a problem in general terms before trying to solve it. Since we're trying to use as little material as possible, why not make the can very small, say, the size of a peanut? We can't, since the can must hold 40 in^3. If we make the can short, to try to use less material in the sides, we'll have to make it fat as well, so that it can hold 40 in^3. Saving aluminum in the sides might actually use more aluminum in the top and bottom in a short, fat can. See Figure 4.45(a).

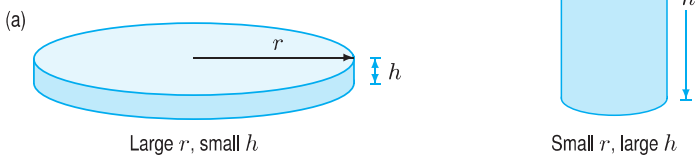


Figure 4.45: Various cylindrical-shaped cans

Table 4.1 *Height, h, and material, M, used in can for various choices of radius, r*

r (in)	h (in)	M (in^2)
0.2	318.31	400.25
1.0	12.73	86.27
2.0	3.18	65.09
3.0	1.41	83.13
4.0	0.80	120.64
10.0	0.13	636.49

If we try to save material by making the top and bottom small, the can has to be tall to accommodate the 40 in^3 of juice. So any savings we get by using a small top and bottom might be outweighed by the height of the sides. See Figure 4.45(b). This is, in fact, true, as Table 4.1 shows.

The table gives the height h and amount of material M used in the can for some choices of the radius, r. You can see that r and h change in opposite directions, and that more material is used at the extremes (very large or very small r) than in the middle. From the table it appears that the optimal radius for the can lies somewhere in $1.0 \leq r \leq 3.0$. If we consider the material used, M, as a function of the radius, r, a graph of this function looks like Figure 4.46. The graph shows that the global minimum we want is at a critical point.

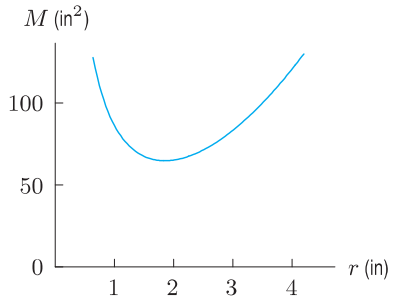


Figure 4.46: Total material used in can, M, as a function of radius, r

Both the table and the graph were obtained from a mathematical model, which in this case is a formula for the material used in making the can. Finding such a formula depends on knowing the geometry of a cylinder, in particular its area and volume. We have

$$M = \text{Material used in the can} = \text{Material in ends} + \text{Material in the side}$$

where

$$\text{Material in ends} = 2 \cdot \text{Area of a circle with radius } r = 2 \cdot \pi r^2,$$
$$\text{Material in the side} = \text{Surface area of cylinder with height } h \text{ and radius } r = 2\pi rh.$$

We have

$$M = 2\pi r^2 + 2\pi rh.$$

However, h is not independent of r: if r grows, h shrinks, and vice-versa. To find the relationship, we use the fact that the volume of the cylinder, $\pi r^2 h$, is equal to the constant 40 in^3:

$$\text{Volume of can } = \pi r^2 h = 40, \quad \text{giving} \quad h = \frac{40}{\pi r^2}.$$

This means

$$\text{Material in the side} = 2\pi rh = 2\pi r \frac{40}{\pi r^2} = \frac{80}{r}.$$

Thus we obtain the formula for the total material, M, used in a can of radius r:

$$M = 2\pi r^2 + \frac{80}{r}.$$

The domain of this function is all $r > 0$ because the radius of the can cannot be negative or zero.

Now we use calculus to find the minimum of M. We look for critical points:

$$\frac{dM}{dr} = 4\pi r - \frac{80}{r^2} = 0 \quad \text{at a critical point,} \quad \text{so} \quad 4\pi r = \frac{80}{r^2}.$$

Therefore,

$$\pi r^3 = 20, \quad \text{so} \quad r = \left(\frac{20}{\pi}\right)^{1/3} \approx 1.85 \text{ inches},$$

which agrees with our graph. We also have

$$h = \frac{40}{\pi r^2} = \frac{40}{\pi(1.85)^2} = 3.7 \text{ inches.}$$

Thus, the material used, $M = 2\pi(1.85)^2 + 80/1.85 = 64.7$ in^2.

To confirm that we have found the global minimum, we look at the formula for dM/dr. For small r, the $-80/r^2$ term dominates and for large r, the $4\pi r$ term dominates, so dM/dr is negative for $r < 1.85$ and positive for $r > 1.85$. Thus, M is decreasing for $r < 1.85$ and increasing for $r > 1.85$, so the global minimum occurs at $r = 1.85$.

Practical Tips for Modeling Optimization Problems

1. Make sure that you know what quantity or function is to be optimized.
2. If possible, make several sketches showing how the elements that vary are related. Label your sketches clearly by assigning variables to quantities which change.
3. Try to obtain a formula for the function to be optimized in terms of the variables that you identified in the previous step. If necessary, eliminate from this formula all but one variable. Identify the domain over which this variable varies.
4. Find the critical points and evaluate the function at these points and the endpoints (if relevant) to find the global maxima and minima.

The next example, another problem in geometry, illustrates this approach.

Example 2 Alaina wants to get to the bus stop as quickly as possible. The bus stop is across a grassy park, 2000 feet west and 600 feet north of her starting position. Alaina can walk west along the edge of the park on the sidewalk at a speed of 6 ft/sec. She can also travel through the grass in the park, but only at a rate of 4 ft/sec. What path will get her to the bus stop the fastest?

Solution

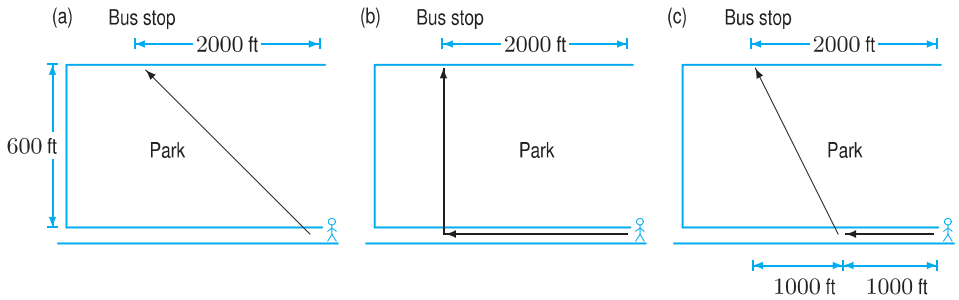


Figure 4.47: Three possible paths to the bus stop

We might first think that she should take a path that is the shortest distance. Unfortunately, the path that follows the shortest distance to the bus stop is entirely in the park, where her speed is slow. (See Figure 4.47(a).) That distance is $\sqrt{2000^2 + 600^2} \approx 2088$ feet, which takes her about 522 seconds to traverse. She could instead walk quickly the entire 2000 feet along the sidewalk, which leaves her just the 600-foot northward journey through the park. (See Figure 4.47(b).) This method would take $2000/6 + 600/4 \approx 483$ seconds total walking time.

But can she do even better? Perhaps another combination of sidewalk and park gives a shorter travel time. For example, what is the travel time if she walks 1000 feet west along the sidewalk and the rest of the way through the park? (See Figure 4.47(c).) The answer is about 458 seconds.

We make a model for this problem. We label the distance that Alaina walks west along the sidewalk x and the distance she walks through the park y, as in Figure 4.48. Then the total time, t, is

$$t = t_{\text{sidewalk}} + t_{\text{park}}.$$

Since

$$\text{Time} = \text{Distance}/\text{Speed},$$

and she can walk 6 ft/sec on the sidewalk and 4 ft/sec in the park, we have

$$t = \frac{x}{6} + \frac{y}{4}.$$

Now, by the Pythagorean Theorem, $y = \sqrt{(2000-x)^2 + 600^2}$. Therefore

$$t = \frac{x}{6} + \frac{\sqrt{(2000-x)^2 + 600^2}}{4} \qquad \text{for } 0 \le x \le 2000.$$

We can find the critical points of this function analytically. (See Exercise 8 on page 207.) Alternatively, we can graph the function on a calculator and estimate the critical point, which is $x \approx 1463$ feet. This gives a minimum total time of about 445 seconds.

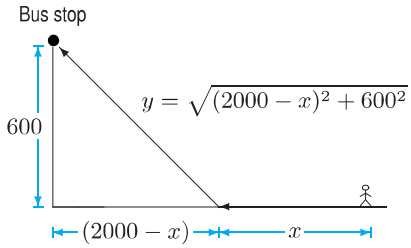


Figure 4.48: Modeling time to bus stop

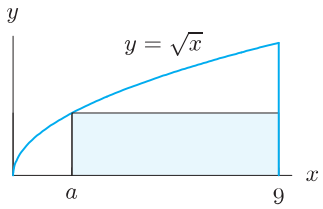


Figure 4.49: Find the rectangle of maximum area with one corner on $y = \sqrt{x}$

Example 3 Figure 4.49 shows the curves $y = \sqrt{x}$, $x = 9$, $y = 0$, and a rectangle with vertical sides at $x = a$ and $x = 9$. Find the dimensions of the rectangle having the maximum possible area.

Solution We want to choose a to maximize the area of the rectangle with corners at $(a, \sqrt{a})$ and $(9, \sqrt{a})$. The area of this rectangle is given by

$$R = \text{Height} \cdot \text{Length} = \sqrt{a}(9-a) = 9a^{1/2} - a^{3/2}.$$

We are restricted to $0 \le a \le 9$. To maximize this area, we set $dR/da = 0$ to find the critical point, and then check that the area at the critical point is greater than the area if $a = 0$ or $a = 9$. Since $R = 0$ if $a = 0$ or $a = 9$, all we need to do is to find where $dR/da = 0$:

$$\frac{dR}{da} = \frac{9}{2}a^{-1/2} - \frac{3}{2}a^{1/2} = 0$$

$$\frac{9}{2\sqrt{a}} = \frac{3\sqrt{a}}{2}$$

$$18 = 6a$$

$$a = 3.$$

Since $a = 3$, the rectangle with the maximum area has length $9 - 3 = 6$ and height $\sqrt{3}$.

Example 4 A closed box has a fixed surface area A and a square base with side x.

(a) Find a formula for the volume, V, of the box as a function of x. What is the domain of V?
(b) Graph V against x.
(c) Find the maximum value of V.

Solution

(a) The height of the box is h, as shown in Figure 4.50. The box has six sides, four with area xh and two, the top and bottom, with area x^2. Thus,

$$4xh + 2x^2 = A.$$

So

$$h = \frac{A - 2x^2}{4x}.$$

Then, the volume, V, is given by

$$V = x^2 h = x^2 \left(\frac{A - 2x^2}{4x} \right) = \frac{x}{4} \left(A - 2x^2 \right) = \frac{A}{4} x - \frac{1}{2} x^3.$$

Since the area of the top and bottom combined must be less than A, we have $2x^2 \leq A$. Thus, the domain of V is $0 \leq x \leq \sqrt{A/2}$.

(b) Figure 4.51 shows the graph for $x \geq 0$. (Note that A is a positive constant.)

(c) To find the maximum, we differentiate, regarding A as a constant:

$$\frac{dV}{dx} = \frac{A}{4} - \frac{3}{2} x^2 = 0$$

so

$$x = \pm \sqrt{\frac{A}{6}}.$$

Since $x \geq 0$ in the domain of V, we use $x = \sqrt{A/6}$. Figure 4.51 indicates that at this value of x, the volume is a maximum.

From the formula, we see that $dV/dx > 0$, so V is increasing, for $x < \sqrt{A/6}$ and that $dV/dx < 0$, so V is decreasing, for $x > \sqrt{A/6}$. Thus, $x = \sqrt{A/5}$ gives the global maximum.

Evaluating V at $x = \sqrt{A/6}$ and simplifying, we get

$$V = \frac{A}{4} \sqrt{\frac{A}{6}} - \frac{1}{2} \left(\sqrt{\frac{A}{6}} \right)^3 = \left(\frac{A}{6} \right)^{3/2}.$$

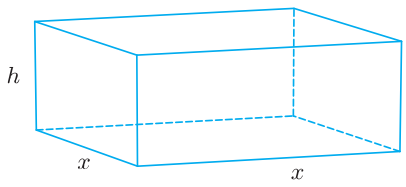


Figure 4.50: Box with base of side x, height h, surface area A, and volume V

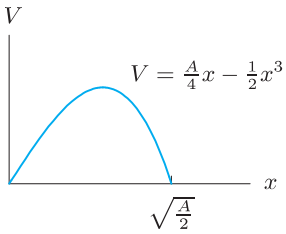


Figure 4.51: Volume, V, against length of side of base, x

Example 5 A light is suspended at a height h above the floor. (See Figure 4.52.) The illumination at the point P is inversely proportional to the square of the distance from the point P to the light and directly proportional to the cosine of the angle θ. How far from the floor should the light be to maximize the illumination at the point P?

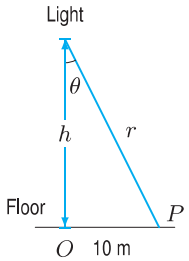


Figure 4.52: How high should the light be?

Solution If the illumination is represented by I, then we know that for some $k \geq 0$,

$$I = \frac{k \cos\theta}{r^2}.$$

Since $r^2 = h^2 + 10^2$ and $\cos\theta = h/r = h/\sqrt{h^2 + 10^2}$, we have, for $h \geq 0$,

$$I = \frac{kh}{(h^2 + 10^2)^{3/2}}.$$

To find the height at which I is maximized, we differentiate using the quotient rule:

$$\begin{aligned}
\frac{dI}{dh} &= \frac{k(h^2 + 10^2)^{3/2} - kh(\frac{3}{2}(h^2 + 10^2)^{1/2}(2h))}{[(h^2 + 10^2)^{3/2}]^2} \\
&= \frac{(h^2 + 10^2)^{1/2}[k(h^2 + 10^2) - 3kh^2]}{(h^2 + 10^2)^3} \\
&= \frac{k(h^2 + 10^2) - 3kh^2}{(h^2 + 10^2)^{5/2}} \\
&= \frac{k(10^2 - 2h^2)}{(h^2 + 10^2)^{5/2}}.
\end{aligned}$$

Setting $dI/dh = 0$ gives

$$\begin{aligned}
10^2 - 2h^2 &= 0 \\
h &= \sqrt{50} \text{ meters.}
\end{aligned}$$

Since $dI/dh > 0$ for $h < \sqrt{50}$ and $dI/dh < 0$ for $h > \sqrt{50}$, there is a local maximum at $h = \sqrt{50}$ meters. There is only one critical point, so the global maximum of I occurs at that point. Thus, the illumination is greatest if the light is suspended at a height of $\sqrt{50} \approx 7$ meters above the floor.

Finding Upper and Lower Bounds

A problem which is closely related to finding maxima and minima is finding the *bounds* of a function. In Example 1 on page187, the value of $f(x)$ on the interval $[-5, 12]$ ranges from -396 to 104. Thus

$$-396 \leq f(x) \leq 104,$$

and we say that -396 is a *lower bound* for f and 104 is an *upper bound* for f on $[-5, 12]$. (See Appendix A for more on bounds.) Of course, we could also say that

$$-400 \leq f(x) \leq 150,$$

so that f is also bounded below by -400 and above by 150 on $[-5, 12]$. However, we consider the -396 and 104 to be the *best possible bounds* because they describe more accurately how the function $f(x)$ behaves on $[-5, 12]$.

Example 6 An object on a spring oscillates about its equilibrium position at $y = 0$. Its distance from equilibrium is given as a function of time, t, by

$$y = e^{-t} \cos t.$$

Find the greatest distance the object goes above and below the equilibrium for $t \geq 0$.

Solution We are looking for bounds for the function. What does the graph of the function look like? We can think of it as a cosine curve with a decreasing amplitude of e^{-t}; in other words, it is a cosine curve squashed between the graphs of $y = e^{-t}$ and $y = -e^{-t}$, forming a wave with lower and lower crests and shallower and shallower troughs. (See Figure 4.53.)

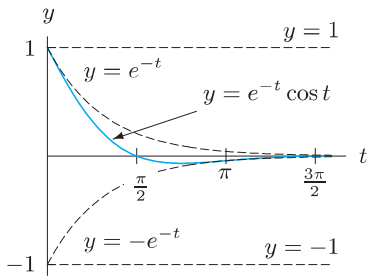


Figure 4.53: $f(t) = e^{-t}\cos t$ for $t \geq 0$

From the graph we can see that for $t \geq 0$, the graph lies between the horizontal lines $y = -1$ and $y = 1$. This means that -1 and 1 are bounds:

$$-1 \leq e^{-t}\cos t \leq 1.$$

The line $y = 1$ is the best possible upper bound because the graph does come up that high (at $t = 0$). However, we can find a better lower bound if we find the global minimum value of f for $t \geq 0$; this minimum occurs in the first trough between $t = \pi/2$ and $t = 3\pi/2$ because later troughs are squashed closer to the t-axis. At the minimum, $dy/dt = 0$. The product rule gives

$$\frac{dy}{dt} = (-e^{-t})\cos t + e^{-t}(-\sin t) = -e^{-t}(\cos t + \sin t) = 0.$$

Since e^{-t} is never 0, we must have

$$\cos t + \sin t = 0, \quad \text{so} \quad \frac{\sin t}{\cos t} = -1.$$

The smallest positive solution of

$$\tan t = -1 \quad \text{is} \quad t = \frac{3\pi}{4}.$$

Thus, the global minimum we see on the graph occurs at $t = 3\pi/4$. The value of y at that minimum is

$$y = e^{-3\pi/4}\cos\left(\frac{3\pi}{4}\right) \approx -0.067.$$

Rounding down so that the inequalities still hold for all $t \geq 0$ gives

$$-0.07 < e^{-t}\cos t \leq 1.$$

Notice how much smaller in magnitude the lower bound is than the upper. This is a reflection of how quickly the factor e^{-t} causes the oscillation to die out.

Exercises and Problems for Section 4.4

Exercises

1. The bending moment M of a beam, supported at one end, at a distance x from the support is given by

$$M = \tfrac{1}{2}wLx - \tfrac{1}{2}wx^2,$$

where L is the length of the beam, and w is the uniform load per unit length. Find the point on the beam where the moment is greatest.

2. The potential energy, U, of a particle moving along the x-axis is given by

$$U = b\left(\frac{a^2}{x^2} - \frac{a}{x}\right),$$

where a and b are positive constants and $x > 0$. What value of x minimizes the potential energy?

3. An electric current, I, in amps, is given by

$$I = \cos(wt) + \sqrt{3}\sin(wt),$$

where $w \neq 0$ is a constant. What are the maximum and minimum values of I?

4. A smokestack deposits soot on the ground with a concentration inversely proportional to the square of the distance from the stack. With two smokestacks 20 miles apart, the concentration of the combined deposits on the line joining them, at a distance x from one stack, is given by

$$S = \frac{k_1}{x^2} + \frac{k_2}{(20-x)^2}$$

where k_1 and k_2 are positive constants which depend on the quantity of smoke each stack is emitting. If $k_1 = 7k_2$, find the point on the line joining the stacks where the concentration of the deposit is a minimum.

5. A wave of wavelength λ traveling in deep water has speed, v, given by

$$v = k\sqrt{\frac{\lambda}{c} + \frac{c}{\lambda}},$$

where c and k are positive constants. As λ varies, does such a wave have a maximum or minimum velocity? If so, what is it? Explain.

6. The efficiency of a screw, E, is given by

$$E = \frac{(\theta - \mu\theta^2)}{\mu + \theta}, \quad \theta > 0,$$

where θ is the angle of pitch of the thread and μ is the coefficient of friction of the material, a (positive) constant. What value of θ maximizes E?

7. A circular ring of wire of radius r_0 lies in a plane perpendicular to the x-axis and is centered at the origin. The ring has a positive electric charge spread uniformly over it. The electric field in the x-direction, E, at the point x on the axis is given by

$$E = \frac{kx}{(x^2 + r_0^2)^{3/2}} \quad \text{for} \quad k > 0.$$

At what point on the x-axis is the field greatest? Least?

8. Find analytically the exact critical point of the function which represents the time, t, to walk to the bus stop in Example 2. Recall that t is given by

$$t = \frac{x}{6} + \frac{\sqrt{(2000-x)^2 + 600^2}}{4}.$$

9. A woman pulls a sled which, together with its load, has a mass of m kg. If her arm makes an angle of θ with her body (assumed vertical) and the coefficient of friction (a positive constant) is μ, the least force, F, she must exert to move the sled is given by

$$F = \frac{mg\mu}{\sin\theta + \mu\cos\theta}.$$

If $\mu = 0.15$, find the maximum and minimum values of F for $0 \le \theta \le \pi/2$. Give answers as multiples of mg.

10. In a chemical reaction, substance A combines with substance B to form substance Y. At the start of the reaction, the quantity of A present is a grams, and the quantity of B present is b grams. At time t seconds after the start of the reaction, the quantity of Y present is y grams. Assume $a < b$ and $y \le a$. For certain types of reactions, the rate of the reaction, in grams/sec, is given by

$$\text{Rate} = k(a-y)(b-y), \quad k \text{ is a positive constant.}$$

(a) For what values of y is the rate nonnegative? Graph the rate against y.
(b) Use your graph to find the value of y at which the rate of the reaction is fastest.

In Exercises 11–16, find the best possible bounds for the function.

11. $x^3 - 4x^2 + 4x$, for $0 \le x \le 4$

12. e^{-x^2}, for $|x| \le 0.3$

13. x^3e^{-x} for $x \ge 0$

14. $x + \sin x$, for $0 \le x \le 2\pi$

15. $\ln(1+x)$, for $x \ge 0$

16. $\ln(1+x^2)$, for $-1 \le x \le 2$

Problems

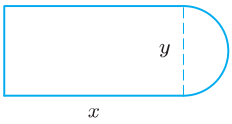
In Problems 17–19 the figures are made of rectangles and semicircles.

(a) Find a formula for the area.

(b) Find a formula for the perimeter.

(c) Find the dimensions x and y that maximize the area given that the perimeter is 100.

17.



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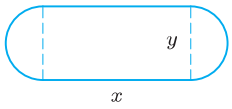

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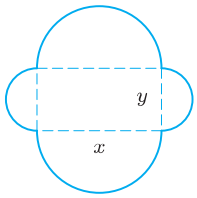


For the solids in Problems 20–23, find the dimensions giving the minimum surface area, given that the volume is 8 cm^3.

20. A closed rectangular box, with a square base x by x cm and height h cm.

21. An open-topped rectangular box, with a square base x by x cm and height h cm.

22. A closed cylinder with radius r cm and height h cm.

23. A cylinder open at one end with radius r cm and height h cm.

In Problems 24–27, find the minimum and maximum values of the expression where x and y are lengths in Figure 4.54 and $0 \le x \le 10$.

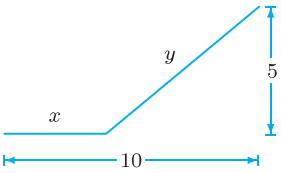


Figure 4.54

24. x **25.** y **26.** $x + 2y$ **27.** $2x + y$

28. The hypotenuse of a right triangle has one end at the origin and one end on the curve $y = x^2e^{-3x}$, with $x \ge 0$. One of the other two sides is on the x-axis, the other side is parallel to the y-axis. Find the maximum area of such a triangle. At what x-value does it occur?

29. A rectangle has one side on the x-axis and two vertices on the curve

$$y = \frac{1}{1+x^2}.$$

Find the vertices of the rectangle with maximum area.

30. A rectangle has one side on the x-axis, one side on the y-axis, one vertex at the origin and one on the curve $y = e^{-2x}$ for $x \ge 0$. Find the

(a) Maximum area **(b)** Minimum perimeter

31. A hemisphere of radius 1 sits on a horizontal plane. A cylinder stands with its axis vertical, the center of its base at the center of the sphere, and its top circular rim touching the hemisphere. Find the radius and height of the cylinder of maximum volume.

32. If you have 100 feet of fencing and want to enclose a rectangular area up against a long, straight wall, what is the largest area you can enclose?

33. A rectangular beam is cut from a cylindrical log of radius 30 cm. The strength of a beam of width w and height h is proportional to wh^2. (See Figure 4.55.) Find the width and height of the beam of maximum strength.

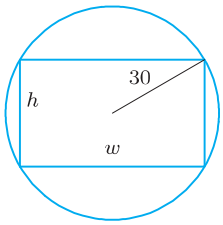


Figure 4.55

34. A rectangular swimming pool is to be built with an area of 1800 square feet. The owner wants 5-foot wide decks along either side and 10-foot wide decks at the two ends. Find the dimensions of the smallest piece of property on which the pool can be built satisfying these conditions.

35. A square-bottomed box with no top has a fixed volume, V. What dimensions minimize the surface area?

36. Which point on the parabola $y = x^2$ is nearest to $(1, 0)$? Find the coordinates to two decimals. [Hint: Minimize the square of the distance—this avoids square roots.]

37. Find the coordinates of the point on the parabola $y = x^2$ which is closest to the point (3,0).

38. The cross-section of a tunnel is a rectangle of height h surmounted by a semicircular roof section of radius r (See Figure 4.56). If the cross-sectional area is A, determine the dimensions of the cross section which minimize the perimeter.

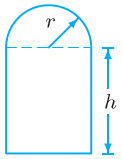


Figure 4.56

39. Four equally massive particles can be made to rotate, equally spaced, around a circle of radius r. This is physically possible provided the radius and period T of the rotation are chosen so that the following *action* function is at its global minimum:

$$A(r) = \frac{r^2}{T} + \frac{T}{r}, \quad r > 0.$$

(a) Find the radius r at which $A(r)$ has a global minimum.
(b) If the period of the rotation is doubled, determine whether the radius of the rotation increases or decreases, and by approximately what percentage.

40. Of all rectangles with given area, A, which has the shortest diagonals?

41. **(a)** A cruise line offers a trip for \$2000 per passenger. If at least 100 passengers sign up, the price is reduced for *all* the passengers by \$10 for every additional passenger (beyond 100) who goes on the trip. The boat can accommodate 250 passengers. What number of passengers maximizes the cruise line's total revenue? What price does each passenger pay then?
(b) The cost to the cruise line for n passengers is $80{,}000 + 400n$. What is the maximum profit that the cruise line can make on one trip? How many passengers must sign up for the maximum to be reached and what price will each pay?

42. The cost of fuel to propel a boat through the water (in dollars per hour) is proportional to the cube of the speed. A certain ferry boat uses \$100 worth of fuel per hour when cruising at 10 miles per hour. Apart from fuel, the cost of running this ferry (labor, maintenance, and so on) is \$675 per hour. At what speed should it travel so as to minimize the cost *per mile* traveled?

43. **(a)** For which positive number x is $x^{1/x}$ largest? Justify your answer.
[Hint: You may want to write $x^{1/x} = e^{\ln(x^{1/x})}$.]
(b) For which positive integer n is $n^{1/n}$ largest? Justify your answer.
(c) Use your answer to parts (a) and (b) to decide which is larger: $3^{1/3}$ or $\pi^{1/\pi}$.

44. The *arithmetic mean* of two numbers a and b is defined as $(a+b)/2$; the *geometric mean* of two positive numbers a and b is defined as $\sqrt{ab}$.

(a) For two positive numbers, which of the two means is larger? Justify your answer.
[Hint: Define $f(x) = (a+x)/2 - \sqrt{ax}$ for fixed a.]
(b) For three positive numbers a, b, c, the arithmetic and geometric mean are $(a+b+c)/3$ and $\sqrt[3]{abc}$, respectively. Which of the two means of three numbers is larger? [Hint: Redefine $f(x)$ for fixed a *and* b.]

45. A bird such as a starling feeds worms to its young. To collect worms, the bird flies to a site where worms are to be found, picks up several in its beak, and flies back to its nest. The *loading curve* in Figure 4.57 shows how the number of worms (the load) a starling collects depends on the time it has been searching for them.[4] The curve is concave down because the bird can pick up worms more efficiently when its beak is empty; when its beak is partly full, the bird becomes much less efficient. The traveling time (from nest to site and back) is represented by the distance PO in Figure 4.57. The bird wants to maximize the rate at which it brings worms to the nest, where

$$\text{Rate worms arrive} = \frac{\text{Load}}{\text{Traveling time + Searching time}}$$

(a) Draw a line in Figure 4.57 whose slope is this rate.
(b) Using the graph, estimate the load which maximizes this rate.
(c) If the traveling time is increased, does the optimal load increase or decrease? Why?

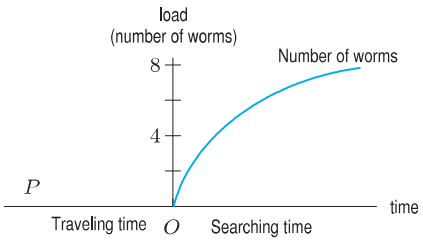


Figure 4.57

46. On the same side of a straight river are two towns, and the townspeople want to build a pumping station, S. See Figure 4.58. The pumping station is to be at the river's edge with pipes extending straight to the two towns. Where should the pumping station be located to minimize the total length of pipe?

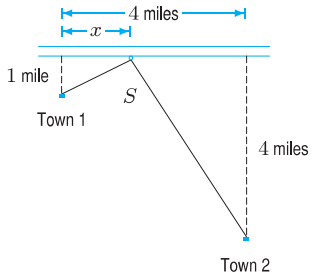


Figure 4.58

[4] Alex Kacelnick (1984). Reported by J. R. Krebs and N. B. Davis, *An Introduction to Behavioural Ecology* (Oxford: Blackwell, 1987).

47. A pigeon is released from a boat (point B in Figure 4.59) floating on a lake. Because of falling air over the cool water, the energy required to fly one meter over the lake is twice the corresponding energy e required for flying over the bank ($e = 3$ joule/meter). To minimize the energy required to fly from B to the loft, L, the pigeon heads to a point P on the bank and then flies along the bank to L. The distance $\overline{AL}$ is 2000 m, and $\overline{AB}$ is 500 m.

(a) Express the energy required to fly from B to L via P as a function of the angle θ (the angle BPA).
(b) What is the optimal angle θ?
(c) Does your answer change if $\overline{AL}$, $\overline{AB}$, and e have different numerical values?

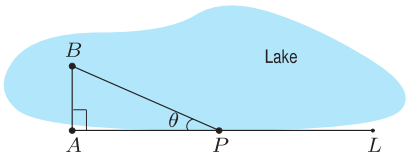


Figure 4.59

48. To get the best view of the Statue of Liberty in Figure 4.60, you should be at the position where θ is a maximum. If the statue stands 92 meters high, including the pedestal, which is 46 meters high, how far from the base should you be? [Hint: Find a formula for θ in terms of your distance from the base. Use this function to maximize θ, noting that $0 \le \theta \le \pi/2$.]

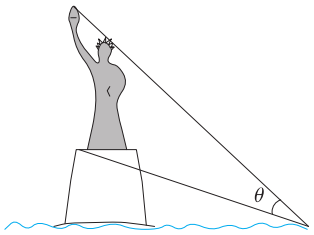


Figure 4.60

49. A light ray starts at the origin and is reflected off a mirror along the line $y = 1$ to the point $(2, 0)$. See Figure 4.61. Fermat's Principle says that light's path minimizes the time of travel.[5] The speed of light is a constant.

(a) Using Fermat's principle, find the optimal position of P.
(b) Using your answer to part (a), derive the Law of Reflection, that $\theta_1 = \theta_2$.

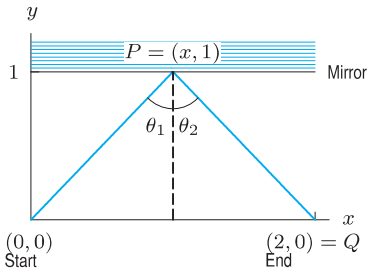


Figure 4.61

50. When a ray of light travels from one medium to another (for example, from air to water), it changes direction. This phenomenon is known as refraction. In Figure 4.62, light is traveling from A to B. The amount of refraction depends on the velocities, v_1 and v_2, of light in the two media and on Fermat's Principle which states that the light's time of travel, T, from A to B, is a minimum.

(a) Find an expression for T in terms of x and the constants a, b, v_1, v_2, and c.
(b) Show that if R is chosen so that the time of travel is minimized, then

$$\frac{\sin\theta_1}{\sin\theta_2} = \frac{v_1}{v_2}.$$

This result is known as Snell's Law and the ratio v_1/v_2 is called the index of refraction of the second medium with respect to the first medium.

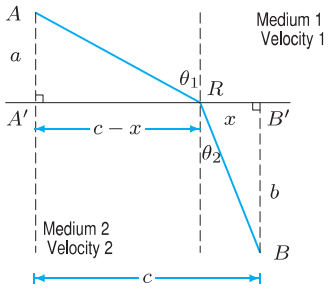


Figure 4.62

51. Show that when the value of x in Figure 4.62 is chosen according to Snell's Law, the time taken by the light ray is a minimum.

[5] See, for example, D. Halliday, R. Resnik, K. Kane, *Physics* Vol 2, 4th ed, (New York: Wiley, 1992), p. 909.

4.5 APPLICATIONS TO MARGINALITY

Management decisions within a particular business usually aim at maximizing profit for the company. In this section we will see how the derivative can be used to maximize profit. Profit depends on both production cost and revenue (or income) from sales. We begin by looking at the cost and revenue functions.

The **cost function**, $C(q)$, gives the total cost of producing a quantity q of some good.

What sort of function do we expect C to be? The more goods that are made, the higher the total cost, so C is an increasing function. In fact, cost functions usually have the general shape shown in Figure 4.63. The intercept on the C-axis represents the *fixed costs*, which are incurred even if nothing is produced. (This includes, for instance, the machinery needed to begin production.) The cost function increases quickly at first and then more slowly because producing larger quantities of a good is usually more efficient than producing smaller quantities—this is called *economy of scale*. At still higher production levels, the cost function starts to increase faster again as resources become scarce, and sharp increases may occur when new factories have to be built. Thus, the graph of $C(q)$ may start out concave down and become concave up later on.

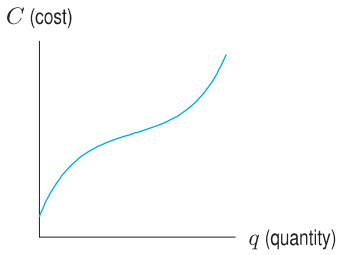


Figure 4.63: Cost as a function of quantity

The **revenue function**, $R(q)$, gives the total revenue received by a firm from selling a quantity q of some good.

Revenue is income obtained from sales. If the price per item is p, and the quantity sold is q, then

$$\text{Revenue} = \text{Price} \times \text{Quantity}, \quad \text{so} \quad R = pq.$$

If the price per item does not depend on the quantity sold, then the graph of $R(q)$ is a straight line through the origin with slope equal to the price p. See Figure 4.64. In practice, for large values of q, the market may become glutted, causing the price to drop, giving $R(q)$ the shape in Figure 4.65.

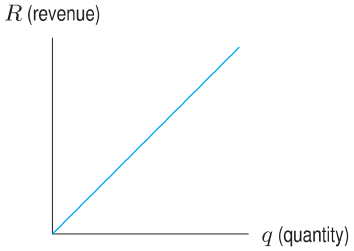


Figure 4.64: Revenue: Constant price

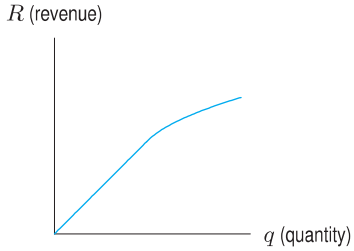


Figure 4.65: Revenue: Decreasing price

The profit is usually written as π. (Economists use π to distinguish it from the price, p; this π has nothing to do with the area of a circle, and merely stands for the Greek equivalent of the letter "p.") The profit resulting from producing and selling q items is defined by

$$\text{Profit} = \text{Revenue} - \text{Cost}, \quad \text{so} \quad \pi(q) = R(q) - C(q).$$

Example 1 If cost, C, and revenue, R, are given by the graph in Figure 4.66, for what production quantities, q, does the firm make a profit? Approximately what production level maximizes profit?

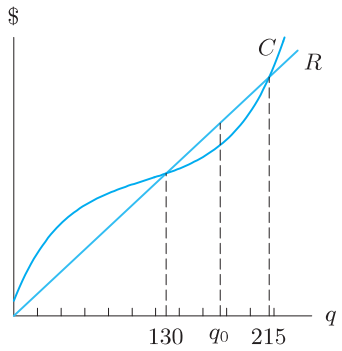


Figure 4.66: Costs and revenues for Example 1

Solution The firm makes a profit whenever revenues are greater than costs, that is, when $R > C$. The graph of R is above the graph of C approximately when $130 < q < 215$. Production between $q = 130$ units and $q = 215$ units generates a profit. The vertical distance between the cost and revenue curves is largest at q_0, so q_0 units gives maximum profit.

Marginal Analysis

Many economic decisions are based on an analysis of the costs and revenues "at the margin." Let's look at this idea through an example.

Suppose we are running an airline and we are trying to decide whether to offer an additional flight. How should we decide? We'll assume that the decision is to be made purely on financial grounds: if the flight will make money for the company, it should be added. Obviously we need to consider the costs and revenues involved. Since the choice is between adding this flight and leaving things the way they are, the crucial question is whether the *additional costs* incurred are greater or smaller than the *additional revenues* generated by the flight. These additional costs and revenues are called the *marginal costs* and *marginal revenues*.

Suppose $C(q)$ is the function giving the total cost of running q flights. If the airline had originally planned to run 100 flights, its costs would be $C(100)$. With the additional flight, its costs would be $C(101)$. Therefore,

$$\text{Additional cost "at the margin"} = C(101) - C(100).$$

Now

$$C(101) - C(100) = \frac{C(101) - C(100)}{101 - 100},$$

and this quantity is the average rate of change of cost between 100 and 101 flights. In Figure 4.67 the average rate of change is the slope of the line joining the $C(100)$ and $C(101)$ points on the graph. If the graph of the cost function is not curving fast near the point, the slope of this line is close to the slope of the tangent line there. Therefore, the average rate of change is close to the instantaneous rate of change. Since these rates of change are not very different, many economists choose to define marginal cost, MC, as the instantaneous rate of change of cost with respect to quantity:

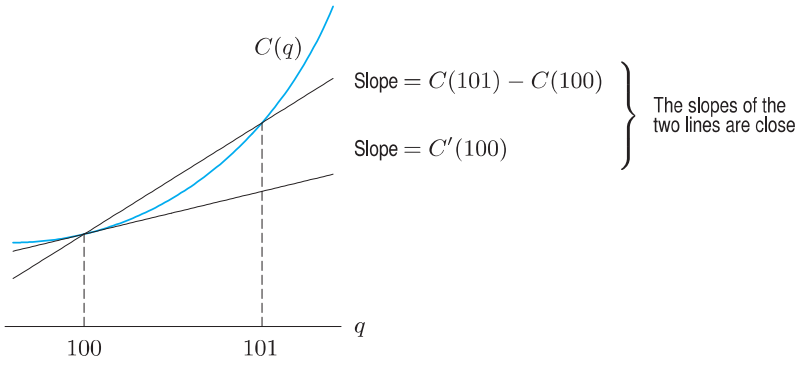


Figure 4.67: Marginal cost: Slope of one of these lines

$$\text{Marginal cost} = MC = C'(q) \qquad \text{so} \qquad \text{Marginal cost} \approx C(q+1) - C(q).$$

Similarly, if the revenue generated by q flights is $R(q)$ and the number of flights increases from 100 to 101, then

$$\text{Additional revenue "at the margin"} = R(101) - R(100).$$

Now $R(101) - R(100)$ is the average rate of change of revenue between 100 and 101 flights. As before, the average rate of change is usually almost equal to the instantaneous rate of change, so economists often define:

$$\text{Marginal revenue} = MR = R'(q) \qquad \text{so} \qquad \text{Marginal revenue} \approx R(q+1) - R(q).$$

We often refer to total cost, C, and total revenue, R, to distinguish them from marginal cost, MC, and marginal revenue, MR. If the words cost and revenue are used alone, they are understood to mean total cost and total revenue.

Example 2 If $C(q)$ and $R(q)$ for the airline are given in Figure 4.68, should the company add the 101^{st} flight?

Solution The marginal revenue is the slope of the revenue curve, and the marginal cost is the slope of the cost curve at the point 100. From Figure 4.68, you can see that the slope at the point A is smaller than the slope at B, so $MC < MR$. This means that the airline will make more in extra revenue than it will spend in extra costs if it runs another flight, so it should go ahead and run the 101^{st} flight.

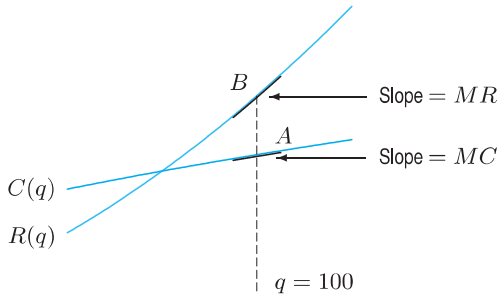


Figure 4.68: Cost and revenue for Example 2

Since MC and MR are derivative functions, they can be estimated from the graphs of total cost and total revenue.

Example 3 If R and C are given by the graphs in Figure 4.69, sketch graphs of $MR = R'(q)$ and $MC = C'(q)$.

Figure 4.69: Total revenue and total cost for Example 3

Solution The revenue graph is a line through the origin, with equation

$$R = pq$$

where p is the price, which is a constant. The slope is p and

$$MR = R'(q) = p.$$

The total cost is increasing, so the marginal cost is always positive (above the q-axis). For small q values, the total cost curve is concave down, so the marginal cost is decreasing. For larger q, say $q > 100$, the total cost curve is concave up and the marginal cost is increasing. Thus the marginal cost has a minimum at about $q = 100$. (See Figure 4.70.)

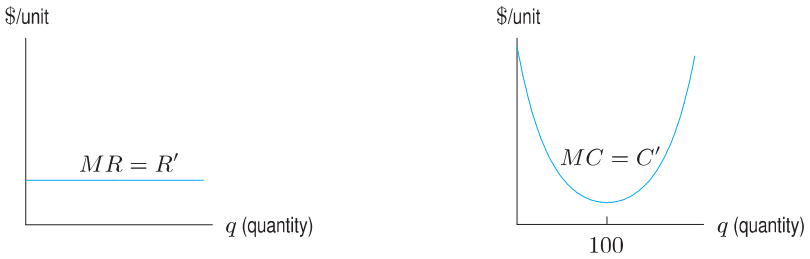


Figure 4.70: Marginal revenue and costs for Example 3

Maximizing Profit

Now let's look at how to maximize profit, given functions for total revenue and total cost.

Example 4 Find the maximum profit if the total revenue and total cost are given, for $0 \le q \le 200$, by the curves R and C in Figure 4.71.

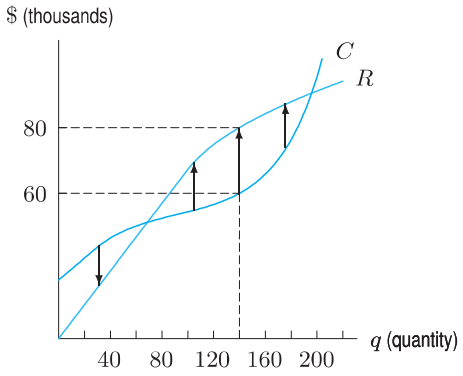


Figure 4.71: Maximum profit at $q = 140$

Solution The profit is represented by the vertical difference between the curves and is marked by the vertical arrows on the graph. When revenue is below cost, the company is taking a loss; when revenue is above cost, the company is making a profit. We can see that the profit is largest at about $q = 140$, so this is the production level we're looking for. To be sure that the local maximum is a global maximum, we need to check the endpoints. At $q = 0$ and $q = 200$, the profit is negative, so the global maximum is indeed at $q = 140$.

To find the actual maximum profit, we estimate the vertical distance between the curves at $q = 140$. This gives a maximum profit of $\$80{,}000 - \$60{,}000 = \$20{,}000$.

Suppose we wanted to find the minimum profit. In this example, we must look at the endpoints, when $q = 0$ or $q = 200$. We see the minimum profit is negative (a loss), and it occurs at $q = 0$.

Maximum Profit Occurs Where $MR = MC$

In Example 4, observe that at $q = 140$ the slopes of the two curves in Figure 4.71 are equal. To the left of $q = 140$, the revenue curve has a larger slope than the cost curve, and the profit increases as q increases. The company will make more money by producing more units, so production should increase toward $q = 140$. To the right of $q = 140$, the slope of the revenue curve is less than the slope of the cost curve, and the profit is decreasing. The company will make more money by producing fewer units so production should decrease toward $q = 140$. At the point where the slopes are equal, the profit has a local maximum; otherwise the profit could be increased by moving toward that point. Since the slopes are equal at $q = 140$, we have $MR = MC$ there.

Now let's look at the general situation. To maximize or minimize profit over an interval, we optimize the profit, π, where

$$\pi(q) = R(q) - C(q).$$

We know that global maxima and minima can only occur at critical points or at endpoints of an interval. To find critical points of π, look for zeros of the derivative:

$$\pi'(q) = R'(q) - C'(q) = 0.$$

So

$$R'(q) = C'(q),$$

that is, the slopes of the revenue and cost curves are equal. This is the same observation that we made in the previous example. In economic language,

> The maximum (or minimum) profit can occur where
>
> Marginal cost = Marginal revenue.

Of course, maximal or minimal profit does not *have* to occur where $MR = MC$; there are also the endpoints to consider.

Example 5 Find the quantity q which maximizes profit if the total revenue, $R(q)$, and total cost, $C(q)$, are given in dollars by

$$R(q) = 5q - 0.003q^2$$
$$C(q) = 300 + 1.1q,$$

where $0 \leq q \leq 800$ units. What production level gives the minimum profit?

Solution We look for production levels that give marginal revenue = marginal cost:

$$MR = R'(q) = 5 - 0.006q$$
$$MC = C'(q) = 1.1.$$

So $5 - 0.006q = 1.1$, giving

$$q = 3.9/0.006 = 650 \text{ units.}$$

Does this value of q represent a local maximum or minimum of π? We can tell by looking at production levels of 649 units and 651 units. When $q = 649$ we have $MR = \$1.106$, which is greater than the (constant) marginal cost of \$1.10. This means that producing one more unit will bring in more revenue than its cost, so profit will increase. When $q = 651$, $MR = \$1.094$, which is *less* than MC, so it is not profitable to produce the 651^{st} unit. We conclude that $q = 650$ is a local maximum for the profit function π. The profit earned by producing and selling this quantity is $\pi(650) = R(650) - C(650) = \$967.50$.

To check for global maxima we need to look at the endpoints. If $q = 0$, the only cost is \$300 (the fixed costs) and there is no revenue, so $\pi(0) = -\$300$. At the upper limit of $q = 800$, we have $\pi(800) = \$900$. Therefore, the maximum profit is at the production level of 650 units, where $MR = MC$. The minimum profit (a loss) occurs when $q = 0$ and there is no production at all.

Exercises and Problems for Section 4.5

Exercises

1. Total cost and revenue are approximated by the functions $C = 5000 + 2.4q$ and $R = 4q$, both in dollars. Identify the fixed cost, marginal cost per item, and the price at which this commodity is sold.

2. A company estimates that the total revenue, R, in dollars, received from the sale of q items is $R = \ln(1 + 1000q^2)$. Calculate and interpret the marginal revenue if $q = 10$.

3. The revenue from selling q items is $R(q) = 500q - q^2$, and the total cost is $C(q) = 150 + 10q$. Write a function that gives the total profit earned, and find the quantity which maximizes the profit.

4. Revenue is given by $R(q) = 450q$ and cost is given by $C(q) = 10{,}000 + 3q^2$. At what quantity is profit maximized? What is the total profit at this production level?

5. **(a)** Fixed costs are \$3 million; variable costs are \$0.4 million per item. Write a formula for total cost as a function of quantity, q.
(b) The item in part (a) is sold for \$0.5 million each. Write a formula for revenue as a function of q.
(c) Write a formula for the profit function for this item.

6. Figure 4.72 shows cost and revenue. For what production levels is the profit function positive? Negative? Estimate the production at which profit is maximized.

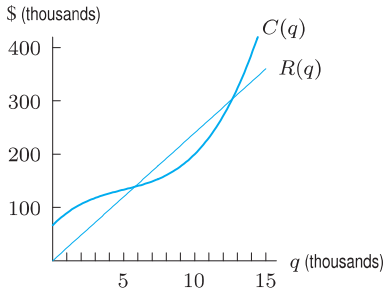


Figure 4.72

7. Figure 4.73 gives cost and revenue. What are fixed costs? What quantity maximizes profit, and what is the maximum profit earned?

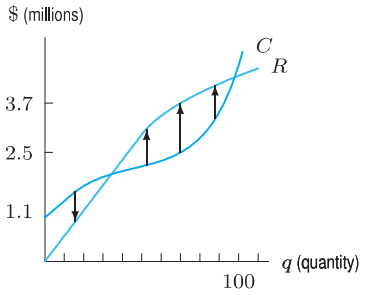


Figure 4.73

8. Table 4.2 shows cost, $C(q)$, and revenue, $R(q)$.

(a) At approximately what production level, q, is profit maximized? Explain your reasoning.
(b) What is the price of the product?
(c) What are the fixed costs?

Table 4.2

q	0	500	1000	1500	2000	2500	3000
$R(q)$	0	1500	3000	4500	6000	7500	9000
$C(q)$	3000	3800	4200	4500	4800	5500	7400

9. Table 4.3 shows marginal cost, MC, and marginal revenue, MR.

(a) Use the marginal cost and marginal revenue at a production of $q = 5000$ to determine whether production should be increased or decreased from 5000.
(b) Estimate the production level that maximizes profit.

Table 4.3

q	5000	6000	7000	8000	9000	10000
MR	60	58	56	55	54	53
MC	48	52	54	55	58	63

Problems

10. If $C'(500) = 75$ and $R'(500) = 100$, should the quantity produced be increased or decreased from $q = 500$ in order to increase profits?

11. When production is 2000, marginal revenue is \$4 per unit and marginal cost is \$3.25 per unit. Do you expect maximum profit to occur at a production level above or below 2000? Explain.

12. Using the cost and revenue graphs in Figure 4.74, sketch the following functions. Label the points q_1 and q_2.

(a) Total profit **(b)** Marginal cost
(c) Marginal revenue

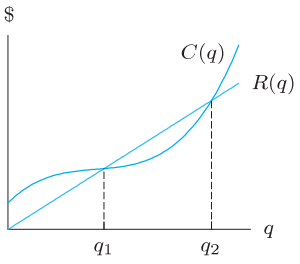


Figure 4.74

13. A manufacturer's cost of producing a product is given in Figure 4.74. The manufacturer can sell the product for a price p each (regardless of the quantity sold), so that the total revenue from selling a quantity q is $R(q) = pq$.

(a) The difference $\pi(q) = R(q) - C(q)$ is the total profit. For which quantity q_0 is the profit a maximum? Mark your answer on a sketch of the graph.
(b) What is the relationship between p and $C'(q_0)$? Explain your result both graphically and analytically. What does this mean in terms of economics? (Note that p is the slope of the line $R(q) = pq$. Note also that $\pi(q)$ has a maximum at $q = q_0$, so $\pi'(q_0) = 0$.)
(c) Graph $C'(q)$ and p (as a horizontal line) on the same axes. Mark q_0 on the q-axis.

14. The marginal revenue and marginal cost for a certain item are graphed in Figure 4.75. Do the following quantities maximize profit for the company? Explain your answer.

(a) $q = a$ **(b)** $q = b$

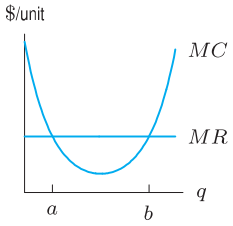


Figure 4.75

15. Let $C(q)$ be the total cost of producing a quantity q of a certain product. See Figure 4.76.

(a) What is the meaning of $C(0)$?
(b) Describe in words how the marginal cost changes as the quantity produced increases.
(c) Explain the concavity of the graph (in terms of economics).
(d) Explain the economic significance (in terms of marginal cost) of the point at which the concavity changes.
(e) Do you expect the graph of $C(q)$ to look like this for all types of products?

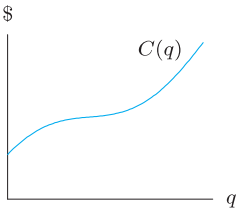


Figure 4.76

16. The total cost $C(q)$ of producing q goods is given by:

$$C(q) = 0.01q^3 - 0.6q^2 + 13q.$$

(a) What is the fixed cost?
(b) What is the maximum profit if each item is sold for \$7? (Assume you sell everything you produce.)
(c) Suppose exactly 34 goods are produced. They all sell when the price is \$7 each, but for each \$1 increase in price, 2 fewer goods are sold. Should the price be raised, and if so by how much?

17. A company manufactures only one product. The quantity, q, of this product produced per month depends on the amount of capital, K, invested (i.e., the number of machines the company owns, the size of its building, and so on) and the amount of labor, L, available each month. We assume that q can be expressed as a *Cobb-Douglas production function*:

$$q = cK^{\alpha}L^{\beta}$$

where c, α, β are positive constants, with $0 < \alpha < 1$ and $0 < \beta < 1$. In this problem we will see how the Russian government could use a Cobb-Douglas function to estimate how many people a newly privatized industry might employ. A company in such an industry has only a small amount of capital available to it and needs to use all of it, so K is fixed. Suppose L is measured in man-hours per month, and that each man-hour costs the company w rubles (a ruble is the unit of Russian currency). Suppose the company has no other costs besides labor, and that each unit of the good can be sold for a fixed price of p rubles. How many man-hours of labor per month should the company use in order to maximize its profit?

18. An agricultural worker in Uganda is planting clover to increase the number of bees making their home in the region. There are 100 bees in the region naturally, and for every acre put under clover, 20 more bees are found in the region.

(a) Draw a graph of the total number, $N(x)$, of bees as a function of x, the number of acres devoted to clover.
(b) Explain, both geometrically and algebraically, the shape of the graph of:
(i) The marginal rate of increase of the number of bees with acres of clover, $N'(x)$.
(ii) The average number of bees per acre of clover, $N(x)/x$.

19. If you invest x dollars in a certain project, your return is $R(x)$. You want to choose x to maximize your return per dollar invested,[6] which is

$$r(x) = \frac{R(x)}{x}.$$

(a) The graph of $R(x)$ is in Figure 4.77, with $R(0) = 0$. Illustrate on the graph that the maximum value of $r(x)$ is reached at a point at which the line from the origin to the point is tangent to the graph of $R(x)$.
(b) Also, the maximum of $r(x)$ occurs at a point at which the slope of the graph of $r(x)$ is zero. On the same axes as part (a), sketch $r(x)$. Illustrate that the maximum of $r(x)$ occurs where its slope is 0.
(c) Show, by taking the derivative of the formula for $r(x)$, that the conditions in part (a) and (b) are equivalent: the x-value at which the line from the origin is tangent to the graph of R is the same as the x-value at which the graph of r has zero slope.

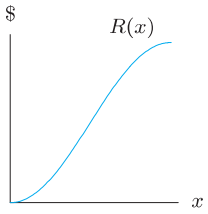


Figure 4.77

Problems 20–21 involve the *average cost* of manufacturing a quantity q of a good, which is defined to be

$$a(q) = \frac{C(q)}{q}.$$

20. Figure 4.78 shows the cost of production, $C(q)$, as a function of quantity produced, q.

(a) For some q_0, sketch a line whose slope is the marginal cost, MC, at that point.
(b) For the same q_0, explain why the average cost $a(q_0)$ can be represented by the slope of the line from that point on the curve to the origin.
(c) Use the method of Example 5 on page 189 to explain why at the value of q which minimizes $a(q)$, the average and marginal costs are equal.

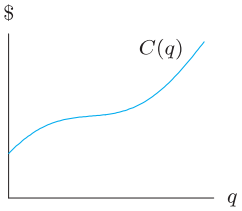


Figure 4.78

21. The average cost per item to produce q items is given by

$$a(q) = 0.01q^2 - 0.6q + 13, \quad \text{for} \quad q > 0.$$

(a) What is the total cost, $C(q)$, of producing q goods?
(b) What is the minimum marginal cost? What is the practical interpretation of this result?
(c) At what production level is the average cost a minimum? What is the lowest average cost?
(d) Compute the marginal cost at $q = 30$. How does this relate to your answer to part (c)? Explain this relationship both analytically and in words.

[6] From Peter D. Taylor, *Calculus: The Analysis of Functions* (Toronto: Wall & Emerson, 1992).

22. The production function $f(x)$ gives the number of units of an item that a manufacturing company can produce from x units of raw material. The company buys the raw material at price w dollars per unit and sells all it produces at a price of p dollars per unit. The quantity of raw material that maximizes profit is denoted by x^*.

(a) Do you expect the derivative $f'(x)$ to be positive or negative? Justify your answer.
(b) Explain why the formula $\pi(x) = pf(x) - wx$ gives the profit $\pi(x)$ that the company earns as a function of the quantity x of raw materials that it uses.
(c) Evaluate $f'(x^*)$.
(d) Assuming it is nonzero, is $f''(x^*)$ positive or negative?
(e) If the supplier of the raw materials is likely to change the price w, then it is appropriate to treat x^* as a function of w. Find a formula for the derivative dx^*/dw and decide whether it is positive or negative.
(f) If the price w goes up, should the manufacturing company buy more or less of the raw material?

In many applications, we want to maximize or minimize some quantity subject to a condition. Such constrained optimization problems are solved using Lagrange multipliers in multivariable calculus; Problems 23–25 show an alternate method.[7]

23. Minimize $x^2 + y^2$ while satisfying $x + y = 4$ using the following steps.

(a) Graph $x + y = 4$. On the same axes, graph $x^2 + y^2 = 1$, $x^2 + y^2 = 4$, $x^2 + y^2 = 9$.
(b) Explain why the minimum value of $x^2 + y^2$ on $x + y = 4$ occurs at the point at which a graph of $x^2 + y^2 =$ Constant is tangent to the line $x + y = 4$.
(c) Using your answer to part (b) and implicit differentiation to find the slope of the circle, find the minimum value of $x^2 + y^2$ such that $x + y = 4$.

24. The quantity Q of an item which can be produced from quantities x and y of two raw materials is given by $Q = 10xy$ at a cost of $C = x + 2y$ thousand dollars. If there is a budget of \$10 thousand for raw materials, find the maximum production using the following steps.

(a) Graph $x+2y = 10$ in the first quadrant. On the same axes, graph $Q = 10xy = 100$, $Q = 10xy = 200$, and $Q = 10xy = 300$.
(b) Explain why the maximum production occurs at a point at which a production curve is tangent to the cost line $C = 10$.
(c) Using your answer to part (b) and implicit differentiation to find the slope of the curve, find the maximum production under this budget.

25. With quantities x and y of two raw materials available, $Q = x^{1/2}y^{1/2}$ thousand items can be produced at a cost of $C = 2x + y$ thousand dollars. Using the following steps, find the minimum cost to produce 1 thousand items.

(a) Graph $x^{1/2}y^{1/2} = 1$. On the same axes, graph $2x + y = 2$, $2x + y = 3$, and $2x + y = 4$.
(b) Explain why the minimum cost occurs at a point at which a cost line is tangent to the production curve $Q = 1$.
(c) Using your answer to part (b) and implicit differentiation to find the slope of the curve, find the minimum cost to meet this production level.

4.6 RATES AND RELATED RATES

Derivatives represent rates of change. In this section, we see how to calculate rates in a variety of situations.

Example 1 A spherical snowball is melting. Its radius decreases at a constant rate of 2 cm per minute from an initial value of 70 cm. How fast is the volume decreasing half an hour later?

Solution The radius, r, starts at 70 cm and decreases at 2 cm/min. At time t minutes since the start,

$$r = 70 - 2t \text{ cm}.$$

The volume of the snowball is given by

$$V = \frac{4}{3}\pi r^3 = \frac{4}{3}\pi(70 - 2t)^3 \text{ cm}^3.$$

The rate at which the volume is changing at time t is

$$\frac{dV}{dt} = \frac{4}{3}\pi \cdot 3(70 - 2t)^2(-2) = -8\pi(70 - 2t)^2 \text{ cm}^3/\text{min}.$$

[7] Kelly Black "Putting Constraints in Optimization for First-Year Calculus Students," p. 310-312 *SIAM Review* Vol. 39 No. 2, June 1997.

The volume is measured in cm^3, and time is in minutes, so after half an hour $t = 30$, and

$$\left.\frac{dV}{dt}\right|_{t=30} = -8\pi(70 - 2 \cdot 30)^2 = -800\pi \text{ cm}^3/\text{min}.$$

Thus, the rate at which the volume is increasing is $-800\pi \approx -2500 \text{ cm}^3/\text{min}$; the rate at which the volume is decreasing is about $2500 \text{ cm}^3/\text{min}$.

Example 2 A skydiver of mass m jumps from a plane at time $t = 0$. Under certain assumptions, the distance, $s(t)$, he has fallen in time t is given by

$$s(t) = \frac{m^2 g}{k^2}\left(\frac{kt}{m} + e^{-kt/m} - 1\right) \quad \text{for some positive constant } k.$$

(a) Find $s'(0)$ and $s''(0)$ and interpret in terms of the skydiver.
(b) Relate the units of $s'(t)$ and $s''(t)$ to the units of t and $s(t)$.

Solution (a) Differentiating using the chain rule gives

$$s'(t) = \frac{m^2 g}{k^2}\left(\frac{k}{m} + e^{-kt/m}\left(-\frac{k}{m}\right)\right) = \frac{mg}{k}\left(1 - e^{-kt/m}\right)$$

$$s''(t) = \frac{mg}{k}(-e^{kt/m})\left(-\frac{k}{m}\right) = ge^{-kt/m}.$$

Since $e^{-k\cdot 0/m} = 1$, evaluating at $t = 0$ gives

$$s'(0) = \frac{mg}{k}(1 - 1) = 0 \quad \text{and} \quad s''(0) = g.$$

The first derivative of distance is velocity, so the fact that $s'(0) = 0$ tells us that the skydiver starts with zero velocity. The second derivative of distance is acceleration, so the fact that $s''(0) = g$ tells us that the skydiver's initial acceleration is g, the acceleration due to gravity.

(b) The units of velocity, $s'(t)$, and acceleration, $s''(t)$, are given by

$$\text{Units of } s'(t) \text{ are } \frac{\text{Units of } s(t)}{\text{Units of } t} = \frac{\text{Units of distance}}{\text{Units of time}}; \quad \text{for example, meters/sec.}$$

$$\text{Units of } s''(t) \text{ are } \frac{\text{Units of } s'(t)}{\text{Units of } t} = \frac{\text{Units of distance}}{(\text{Units of time})^2}; \quad \text{for example, meters/sec}^2.$$

Related Rates

In Example 1, the radius of the snowball decreased at a constant rate. A more realistic scenario is for the radius to decrease at different rates at different times. Then, we may not be able to write a formula for V as a function of t. However, we may still be able to calculate dV/dt, as in the following example.

Example 3 A spherical snowball melts in such a way that the instant at which its radius is 20 cm, its radius is decreasing at 3 cm/min. At what rate is the volume of the ball of snow changing at that instant?

Solution Since the snowball is spherical, we again have that

$$V = \frac{4}{3}\pi r^3.$$

We can no longer write a formula for r in terms of t, but we know that

$$\frac{dr}{dt} = -3 \quad \text{when} \quad r = 20.$$

We want to know dV/dt when $r = 20$. Think of r as an (unknown) function of t and differentiate the expression for V with respect to t using the chain rule:

$$\frac{dV}{dt} = \frac{4}{3}\pi \cdot 3r^2 \frac{dr}{dt} = 4\pi r^2 \frac{dr}{dt}.$$

At the instant at which $r = 20$ and $dr/dt = -3$, we have

$$\frac{dV}{dt} = 4\pi \cdot 20^2 \cdot (-3) = -4800\pi \approx -15{,}080 \text{ cm}^3/\text{min}.$$

Notice that we have sidestepped the problem of not knowing r as a function of t by calculating the derivatives only at the moment we are interested in.

Example 4 Figure 4.79 shows the fuel consumption, g, in miles per gallon (mpg), of a car traveling at v mph. At one moment, the car was going 70 mph and its deceleration was 8000 miles/hour^2. How fast was the fuel consumption changing at that moment? Include units.

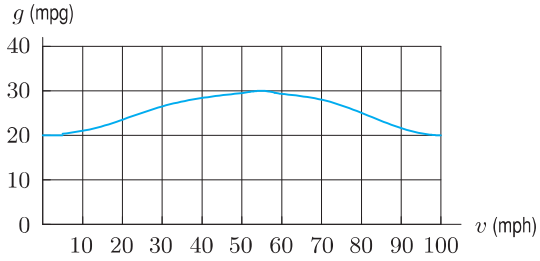


Figure 4.79: Fuel consumption versus velocity

Solution Acceleration is rate of change of velocity, dv/dt, and we are told that the deceleration is 8000 miles/hour^2, so we know $dv/dt = -8000$ when $v = 70$. We want dg/dt. The chain rule gives

$$\frac{dg}{dt} = \frac{dg}{dv} \cdot \frac{dv}{dt}.$$

The value of dg/dv is the slope of the curve in Figure 4.79 at $v = 70$. Since the points $(30, 40)$ and $(100, 20)$ lie approximately on the tangent to the curve at $v = 70$, we can estimate the derivative

$$\frac{dg}{dv} \approx \frac{20 - 40}{100 - 30} = -\frac{2}{7} \text{ mpg/mph}.$$

Thus,

$$\frac{dg}{dt} = \frac{dg}{dv} \cdot \frac{dv}{dt} \approx -\frac{2}{7}(-8000) \approx 2300 \text{ mpg/hr}.$$

In other words, fuel consumption is increasing at about $2300/60 \approx 38$ mpg per minute. Since we approximated dg/dv, we can only get a rough estimate for dg/dt.

A famous problem involves the rate at which a ladder slips down a wall as the foot of the ladder moves.

Example 5
(a) A 3-meter ladder stands against a high wall. The foot of the ladder moves outward at a constant speed of 0.1 meter/sec. When the foot is 1 meter from the wall, how fast is the top of the ladder falling? What about when the foot is 2 meters from the wall?
(b) If the foot of the ladder moves out at a constant speed, how does the speed at which the top falls change as the foot gets farther out?

Solution (a) Let the foot be x meters from the base of the wall and let the top be y meters from the base. See Figure 4.80. Then, since the ladder is 3 meters long, by Pythagoras' Theorem,

$$x^2 + y^2 = 3^2 = 9.$$

Thinking of x and y as functions of t, we differentiate with respect to t using the chain rule:

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0.$$

We are interested in the moment at which $dx/dt = 0.1$ and $x = 1$. We want to know dy/dt, so we solve, giving

$$\frac{dy}{dt} = -\frac{x}{y}\frac{dx}{dt}.$$

When the foot of the ladder is 1 meter from the wall, $x = 1$ and $y = \sqrt{9 - 1^2} = \sqrt{8}$, so

$$\frac{dy}{dt} = -\frac{1}{\sqrt{8}}0.1 = -0.035 \text{ meter/sec.}$$

Thus, the top falls at 0.035 meter/sec.

When the foot is 2 meters from the wall, $x = 2$ and $y = \sqrt{9 - 2^2} = \sqrt{5}$, so

$$\frac{dy}{dt} = -\frac{2}{\sqrt{5}}0.1 = -0.089 \text{ meter/sec.}$$

Thus, the top falls at 0.089 meter/sec. Notice that the top falls faster when the base of the ladder is farther from the wall.

(b) As the foot of the ladder moves out, x increases and y decreases. Looking at the expression

$$\frac{dy}{dt} = -\frac{x}{y}\frac{dx}{dt},$$

we see that if dx/dt is constant, the magnitude of dy/dt increases as the foot gets farther out. Thus, the top falls faster and faster.

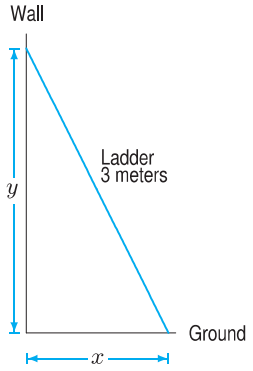


Figure 4.80: Side view of ladder standing against wall (x, y in meters)

Example 6 An airplane, flying at 450 km/hr at a constant altitude of 5 km, is approaching a camera mounted on the ground. Let θ be the angle of elevation above the ground at which the camera is pointed. See Figure 4.81. When $\theta = \pi/3$, how fast does the camera have to rotate in order to keep the plane in view?

Solution Suppose the plane is vertically above the point B. Let x be the distance between B and C in km. The fact that the plane is moving toward C at 450 km/hr means that x is decreasing and $dx/dt = -450$ km/hr. From Figure 4.81, we see that $\tan\theta = 5/x$.

Differentiating $\tan\theta = 5/x$ with respect to t and using the chain rule gives

$$\frac{1}{\cos^2\theta}\frac{d\theta}{dt} = -5x^{-2}\frac{dx}{dt}.$$

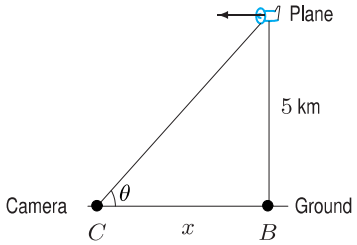


Figure 4.81: Plane approaching a camera at C (side view; x in km)

We want to calculate $d\theta/dt$ when $\theta = \pi/3$. At that moment, $\cos\theta = 1/2$ and $\tan\theta = \sqrt{3}$, so $x = 5/\sqrt{3}$. Substituting gives

$$\frac{1}{(1/2)^2}\frac{d\theta}{dt} = -5\left(\frac{5}{\sqrt{3}}\right)^{-2}\cdot(-450)$$
$$\frac{d\theta}{dt} = 67.5 \text{ radians/hour}.$$

Since there are 60 minutes in an hour, the camera must turn at roughly 1 radian per minute if it is to remain pointed at the plane. With 1 radian $\approx 60^\circ$, this is a rotation of about one degree per second.

Exercises and Problems for Section 4.6

Exercises

1. According to the US Census, the world population P, in billions, was approximately

$$P = 6.7e^{0.011t},$$

where t is in years since January 1, 2007. At what rate was the world's population increasing on that date? Give your answer in millions of people per year. [The world population growth rate has actually decreased since 2007.]

2. Atmospheric pressure decays exponentially as altitude increases. With pressure, P, in inches of mercury and altitude, h, in feet above sea level, we have

$$P = 30e^{-3.23\times10^{-5}h}.$$

(a) At what altitude is the atmospheric pressure 25 inches of mercury?

(b) A glider measures the pressure to be 25 inches of mercury and experiences a pressure increase of 0.1 inches of mercury per minute. At what rate is it changing altitude?

3. The Dubois formula relates a person's surface area, s, in meters^2, to weight, w, in kg, and height,t h, in cm, by

$$s = 0.01w^{0.25}h^{0.75}.$$

(a) What is the surface area of a person who weighs 60 kg and is 150 cm tall?

(b) The person in part (a) stays constant height but increases in weight by 0.5 kg/year. At what rate is his surface area increasing when his weight is 62 kg?

4. With time, t, in minutes, the temperature, H, in degrees Celsius, of a bottle of water put in the refrigerator at $t = 0$ is given by

$$H = 4 + 16e^{-0.02t}.$$

How fast is the water cooling initially? After 10 minutes? Give units.

5. The power, P, dissipated when a 9-volt battery is put across a resistance of R ohms is given by

$$P = \frac{81}{R}.$$

What is the rate of change of power with respect to resistance?

6. With length, l, in meters, the period T, in seconds, of a pendulum is given by

$$T = 2\pi\sqrt{\frac{l}{9.8}}.$$

(a) How fast does the period increase as l increases?
(b) Does this rate of change increase or decrease as l increases?

7. At time t, in hours, a lake is covered with ice of thickness y cm, where $y = 0.2t^{1.5}$.

(a) How fast is the ice forming when $t = 1$? When $t = 2$? Give units.
(b) If ice forms for $0 \le t \le 3$, at what time in this interval is the ice thickest? At what time is the ice forming fastest?

8. A dose, D, of a drug causes a temperature change, T, in a patient. For C a positive constant, T is given by

$$T = \left(\frac{C}{2} - \frac{D}{3}\right) D^3.$$

(a) What is the rate of change of temperature change with respect to dose?
(b) For what doses does the temperature change increase as the dose increases?

9. For positive constants k and g, the velocity, v, of a particle of mass m at time t is given by

$$v = \frac{mg}{k}\left(1 - e^{-kt/m}\right).$$

At what rate is the velocity changing at time 0? At $t = 1$? What do your answers tell you about the motion?

10. The average cost per item, C, in dollars, of manufacturing a quantity q of cell phones is given by

$$C = \frac{a}{q} + b \qquad \text{where } a, b \text{ are positive constants.}$$

(a) Find the rate of change of C as q increases. What are its units?
(b) If production increases at a rate of 100 cell phones per week, how fast is the average cost changing? Is the average cost increasing or decreasing?

11. For positive constants A and B, the force, F, between two atoms in a molecule at a distance r apart is given by

$$F = -\frac{A}{r^2} + \frac{B}{r^3}.$$

(a) How fast does force change as r increases? What type of units does it have?
(b) If at some time t the distance is changing at a rate k, at what rate is the force changing with time? What type of units does this rate of change have?

12. An item costs \$500 at time $t = 0$ and costs \$$P$ in year t. When inflation is $r\%$ per year, the price is given by

$$P = 500e^{rt/100}.$$

(a) If r is a constant, at what rate is the price rising (in dollars per year)
(i) Initially? (ii) After 2 years?
(b) Now suppose that r is increasing by 0.3 per year when $r = 4$ and $t = 2$. At what rate (dollars per year) is the price increasing at that time?

13. A voltage V across a resistance R generates a current

$$I = \frac{V}{R}.$$

If a constant voltage of 9 volts is put across a resistance that is increasing at a rate of 0.2 ohms per second when the resistance is 5 ohms, at what rate is the current changing?

14. The gravitational force, F, on a rocket at a distance, r, from the center of the earth is given by

$$F = \frac{k}{r^2},$$

where $k = 10^{13}$ newton $\cdot$ km^2. When the rocket is 10^4 km from the center of the earth, it is moving away at 0.2 km/sec. How fast is the gravitational force changing at that moment? Give units. (A newton is a unit of force.)

15. The potential, ϕ, of a charge distribution at a point on the positive x-axis is given by

$$\phi = 2\pi\left(\sqrt{x^2+4} - x\right)$$

where x is in centimeters. A particle at $x = 3$ is moving to the left at a rate of 0.2 cm/sec. At what rate is its potential changing?

16. A pyramid has height h and a square base with side x. The volume of a pyramid is $V = \frac{1}{3}x^2h$. If the height remains fixed and the side of the base is decreasing by 0.002 meter/yr, what rate is the volume decreasing when the height is 120 meters and the width is 150 meters?

17. If θ is the angle between a line through the origin and the positive x-axis, the area, in cm^2, of part of a rose petal is

$$A = \frac{9}{16}(4\theta - \sin(4\theta)).$$

If the angle θ is increasing at a rate of 0.2 radians per minute, at what rate is the area changing when $\theta = \pi/4$?

18. A thin uniform rod of length l cm and a small particle lie on a line separated by a distance of a cm. If K is a positive constant and F is measured in newtons, the gravitational force between them is

$$F = \frac{K}{a(a+l)}.$$

(a) If a is increasing at the rate 2 cm/min when $a = 15$ and $l = 5$, how fast is F decreasing?
(b) If l is decreasing at the rate 2 cm/min when $a = 15$ and $l = 5$, how fast is F increasing?

Problems

19. The depth of soot deposited from a smokestack is given by $D = K(r+1)e^{-r}$, where r is the distance from the smokestack. What is the relationship between the rate of change of r with respect to time and the rate of change of D with respect to time?

20. The mass of a circular oil slick of radius r is $M = K\,(r - \ln(1+r))$, where K is a positive constant. What is the relationship between the rate of change of the radius with respect to t and the rate of change of the mass with respect to time?

21. Gasoline is pouring into a cylindrical tank of radius 3 feet. When the depth of the gasoline is 4 feet, the depth is increasing at 0.2 ft/sec. How fast is the volume of gasoline changing at that instant?

22. The metal frame of a rectangular box has a square base. The horizontal rods in the base are made out of one metal and the vertical rods out of a different metal. If the horizontal rods expand at a rate of 0.001 cm/hr and the vertical rods expand at a rate of 0.002 cm/hr, at what rate is the volume of the box expanding when the base has an area of $9\,\text{cm}^2$ and the volume is $180\,\text{cm}^3$?

23. Point P moves around the unit circle.[8] (See Figure 4.82.) The angle θ, in radians, changes with time as shown in Figure 4.83.

(a) Estimate the coordinates of P when $t = 2$.
(b) When $t = 2$, approximately how fast is the point P moving in the x-direction? In the y-direction?

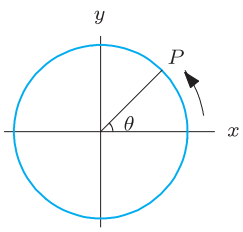


Figure 4.82

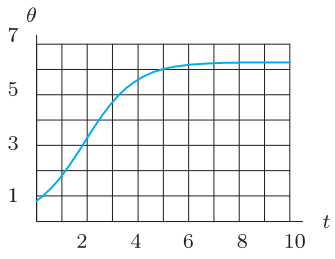


Figure 4.83

24. Figure 4.84 shows the number of gallons, G, of gasoline used on a trip of M miles.

(a) The function f is linear on each of the intervals $0 < M < 70$ and $70 < M < 100$. What is the slope of these lines? What are the units of these slopes?
(b) What is gas consumption (in miles per gallon) during the first 70 miles of this trip? During the next 30 miles?
(c) Figure 4.85 shows distance traveled, M (in miles), as a function of time t, in hours since the start of the trip. Describe this trip in words. Give a possible explanation for what happens one hour into the trip. What do your answers to part (b) tell you about the trip?
(d) If we let $G = k(t) = f(h(t))$, estimate $k(0.5)$ and interpret your answer in terms of the trip.
(e) Find $k'(0.5)$ and $k'(1.5)$. Give units and interpret your answers.

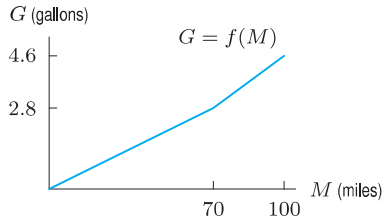


Figure 4.84

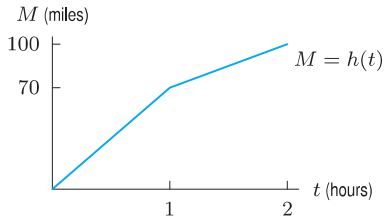


Figure 4.85:

25. On February 16, 2007, paraglider Eva Wisnierska[9] was caught in a freak thunderstorm over Australia and carried upward at a speed of about 3000 ft/min. Table 4.4 gives the temperature at various heights. Approximately how fast (in °F/ per minute) was her temperature decreasing when she was at 4000 feet?

Table 4.4

y (thousand ft)	2	4	6	8	10	12	14	16
H (°F)	60	52	38	31	23	16	9	2

26. The circulation time of a mammal (that is, the average time it takes for all the blood in the body to circulate once and return to the heart) is proportional to the fourth root of the body mass of the mammal. The constant of proportionality is 17.40 if circulation time is in seconds and body mass is in kilograms. The body mass of a growing child is 45 kg and is increasing at a rate of 0.1 kg/month. What is the rate of change of the circulation time of the child?

[8] Based on an idea from Caspar Curjel.
[9] www.sciencedaily.com, accessed May 15, 2007.

27. Coroners estimate time of death using the rule of thumb that a body cools about 2°F during the first hour after death and about 1°F for each additional hour. Assuming an air temperature of 68°F and a living body temperature of 98.6°F, the temperature $T(t)$ in $^\circ$F of a body at a time t hours since death is given by

$$T(t) = 68 + 30.6e^{-kt}.$$

(a) For what value of k will the body cool by 2°F in the first hour?
(b) Using the value of k found in part (a), after how many hours will the temperature of the body be decreasing at a rate of 1°F per hour?
(c) Using the value of k found in part (a), show that, 24 hours after death, the coroner's rule of thumb gives approximately the same temperature as the formula.

28. A certain quantity of gas occupies a volume of 20 cm^3 at a pressure of 1 atmosphere. The gas expands without the addition of heat, so, for some constant k, its pressure, P, and volume, V, satisfy the relation

$$PV^{1.4} = k.$$

(a) Find the rate of change of pressure with volume. Give units.
(b) The volume is increasing at 2 cm^3/min when the volume is 30 cm^3. At that moment, is the pressure increasing or decreasing? How fast? Give units.

29. **(a)** A hemispherical bowl of radius 10 cm contains water to a depth of h cm. Find the radius of the surface of the water as a function of h.
(b) The water level drops at a rate of 0.1 cm per hour. At what rate is the radius of the water decreasing when the depth is 5 cm?

30. A cone-shaped coffee filter of radius 6 cm and depth 10 cm contains water, which drips out through a hole at the bottom at a constant rate of 1.5 cm^3 per second.

(a) If the filter starts out full, how long does it take to empty?
(b) Find the volume of water in the filter when the depth of the water is h cm.
(c) How fast is the water level falling when the depth is 8 cm?

31. A spherical snowball is melting. Its radius is decreasing at 0.2 cm per hour when the radius is 15 cm. How fast is its volume decreasing at that time?

32. A ruptured oil tanker causes a circular oil slick on the surface of the ocean. When its radius is 150 meters, the radius of the slick is expanding by 0.1 meter/minute and its thickness is 0.02 meter. At that moment:

(a) How fast is the area of the slick expanding?
(b) The circular slick has the same thickness everywhere, and the volume of oil spilled remains fixed. How fast is the thickness of the slick decreasing?

33. A potter forms a piece of clay into a cylinder. As he rolls it, the length, L, of the cylinder increases and the radius, r, decreases. If the length of the cylinder is increasing at 0.1 cm per second, find the rate at which the radius is changing when the radius is 1 cm and the length is 5 cm.

34. A gas station stands at the intersection of a north-south road and an east-west road. A police car is traveling toward the gas station from the east, chasing a stolen truck which is traveling north away from the gas station. The speed of the police car is 100 mph at the moment it is 3 miles from the gas station. At the same time, the truck is 4 miles from the gas station going 80 mph. At this moment:

(a) Is the distance between the car and truck increasing or decreasing? How fast? (Distance is measured along a straight line joining the car and the truck.)
(b) How does your answer change if the truck is going 70 mph instead of 80 mph?

35. A train is traveling at 0.8 km/min along a long straight track, moving in the direction shown in Figure 4.86. A movie camera, 0.5 km away from the track, is focused on the train.

(a) Express z, the distance between the camera and the train, as a function of x.
(b) How fast is the distance from the camera to the train changing when the train is 1 km from the camera? Give units.
(c) How fast is the camera rotating (in radians/min) at the moment when the train is 1 km from the camera?

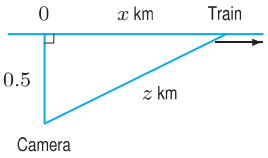


Figure 4.86

36. A lighthouse is 2 km from the long, straight coastline shown in Figure 4.87. Find the rate of change of the distance of the spot of light from the point O with respect to the angle θ.

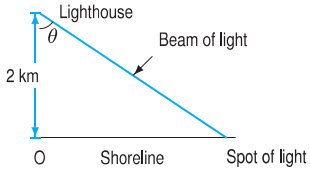


Figure 4.87

37. A train is heading due west from St. Louis. At noon, a plane flying horizontally due north at a fixed altitude of 4 miles passes directly over the train. When the train has traveled another mile, it is going 80 mph, and the plane has traveled another 5 miles and is going 500 mph. At that moment, how fast is the distance between the train and the plane increasing?

38. The radius of a spherical balloon is increasing by 2 cm/sec. At what rate is air being blown into the balloon at the moment when the radius is 10 cm? Give units in your answer.

39. A spherical cell is growing at a constant rate of 400 μm^3/day (1 μm$= 10^{-6}$ m). At what rate is its radius increasing when the radius is 10 μm?

40. A raindrop is a perfect sphere with radius r cm and surface area S cm^2. Condensation accumulates on the raindrop at a rate equal to kS, where $k = 2$ cm/sec. Show that the radius of the raindrop increases at a constant rate and find that rate.

41. The length of each side of a cube is increased at a constant rate. Which is greater, the relative rate of change of the volume of the cube, $\frac{1}{V}\frac{dV}{dt}$, or the relative change of the surface area of the cube, $\frac{1}{A}\frac{dA}{dt}$?

42. Sand falls from a hopper at a rate of 0.1 cubic meters per hour and forms a conical pile beneath. If the side of the cone makes an angle of $\pi/6$ radians with the vertical, find the rate at which the height of the cone increases. At what rate does the radius of the base increase? Give both answers in terms of h, the height of the pile in meters.

43. A circular region is irrigated by a 20 meter long pipe, fixed at one end and rotating horizontally, spraying water. One rotation takes 5 minutes. A road passes 30 meters from the edge of the circular area. See Figure 4.88.

(a) How fast is the end of the pipe, P, moving?
(b) How fast is the distance PQ changing when θ is $\pi/2$? When θ is 0?

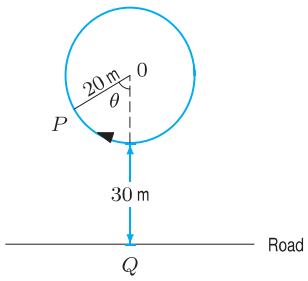


Figure 4.88

44. A water tank is in the shape of an inverted cone with depth 10 meters and top radius 8 meters. Water is flowing into the tank at 0.1 cubic meters/min but leaking out at a rate of $0.004h^2$ cubic meters/min, where h is the depth of the water in meters. Can the tank ever overflow?

45. For the amusement of the guests, some hotels have elevators on the outside of the building. One such hotel is 300 feet high. You are standing by a window 100 feet above the ground and 150 feet away from the hotel, and the elevator descends at a constant speed of 30 ft/sec, starting at time $t = 0$, where t is time in seconds. Let θ be the angle between the line of your horizon and your line of sight to the elevator. (See Figure 4.89.)

(a) Find a formula for $h(t)$, the elevator's height above the ground as it descends from the top of the hotel.
(b) Using your answer to part (a), express θ as a function of time t and find the rate of change of θ with respect to t.
(c) The rate of change of θ is a measure of how fast the elevator appears to you to be moving. At what height is the elevator when it appears to be moving fastest?

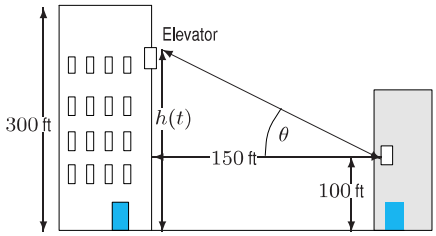


Figure 4.89

46. In a romantic relationship between Angela and Brian, who are unsuited for each other, $a(t)$ represents the affection Angela has for Brian at time t days after they meet, while $b(t)$ represents the affection Brian has for Angela at time t. If $a(t) > 0$ then Angela likes Brian; if $a(t) < 0$ then Angela dislikes Brian; if $a(t) = 0$ then Angela neither likes nor dislikes Brian. Their affection for each other is given by the relation $a^2(t) + b^2(t) = c$, where c is a constant.

(a) Show that $a(t) \cdot a'(t) = -b(t) \cdot b'(t)$.
(b) At any time during their relationship, the rate per day at which Brian's affection for Angela changes is $b'(t) = -a(t)$. Explain what this means if Angela

(i) Likes Brian, (ii) Dislikes Brian.

(c) Use parts (a) and (b) to show that $a'(t) = b(t)$. Explain what this means if Brian

(i) Likes Angela, (ii) Dislikes Angela.

(d) If $a(0) = 1$ and $b(0) = 1$ who first dislikes the other?

47. In a 19^{th} century sea-battle, the number of ships on each side remaining t hours after the start are given by $x(t)$ and $y(t)$. If the ships are equally equipped, the relation between them is $x^2(t) - y^2(t) = c$, where c is a positive constant. The battle ends when one side has no ships remaining.

(a) If, at the start of the battle, 50 ships on one side oppose 40 ships on the other, what is the value of c?
(b) If $y(3) = 16$, what is $x(3)$? What does this represent in terms of the battle?
(c) There is a time T when $y(T) = 0$. What does this T represent in terms of the battle?
(d) At the end of the battle, how many ships remain on the victorious side?
(e) At any time during the battle, the rate per hour at which y loses ships is directly proportional to the number of x ships, with constant of proportionality k. Write an equation that represents this. Is k positive or negative?
(f) Show that the rate per hour at which x loses ships is directly proportional to the number of y ships, with constant of proportionality k.
(g) Three hours after the start of the battle, x is losing ships at the rate of 32 ships per hour. What is k? At what rate is y losing ships at this time?

4.7 L'HOPITAL'S RULE, GROWTH, AND DOMINANCE

Suppose we want to calculate the exact value of the limit

$$\lim_{x \to 0} \frac{e^{2x} - 1}{x}.$$

Substituting $x = 0$ gives us $0/0$, which is undefined:

$$\frac{e^{2(0)} - 1}{0} = \frac{1 - 1}{0} = \frac{0}{0}.$$

Substituting values of x near 0 gives us an approximate value for the limit.

However, the limit can be calculated exactly using local linearity. Suppose we let $f(x)$ be the numerator, so $f(x) = e^{2x} - 1$, and $g(x)$ be the denominator, so $g(x) = x$. Then $f(0) = 0$ and $f'(x) = 2e^{2x}$, so $f'(0) = 2$. When we zoom in on the graph of $f(x) = e^{2x} - 1$ near the origin, we see its tangent line $y = 2x$ shown in Figure 4.90. We are interested in the ratio $f(x)/g(x)$, which is approximately the ratio of the y-values in Figure 4.90. So, for x near 0,

$$\frac{f(x)}{g(x)} = \frac{e^{2x} - 1}{x} \approx \frac{2x}{x} = \frac{2}{1} = \frac{f'(0)}{g'(0)}.$$

As $x \to 0$, this approximation gets better, and we have

$$\lim_{x \to 0} \frac{e^{2x} - 1}{x} = 2.$$

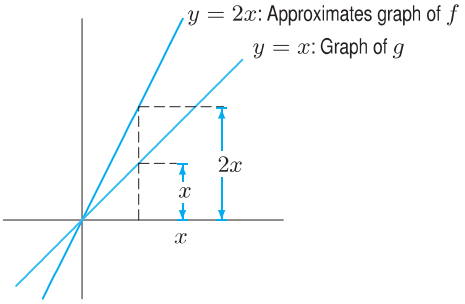


Figure 4.90: Ratio $(e^{2x} - 1)/x$ is approximated by ratio of slopes as we zoom in near the origin

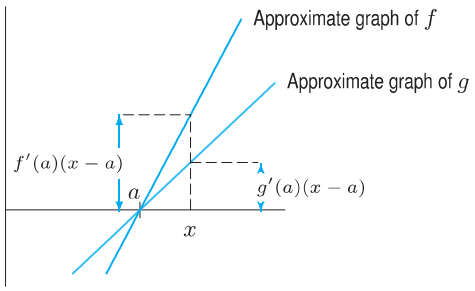


Figure 4.91: Ratio $f(x)/g(x)$ is approximated by ratio of slopes, $f'(a)/g'(a)$, as we zoom in at a

L'Hopital's Rule

If $f(a) = g(a) = 0$, we can use the same method to investigate limits of the form

$$\lim_{x \to a} \frac{f(x)}{g(x)}.$$

As in the previous case, we zoom in on the graphs of $f(x)$ and $g(x)$. Figure 4.91 shows that both graphs cross the x-axis at $x = a$. This suggests that the limit of $f(x)/g(x)$ as $x \to a$ is the ratio of slopes, giving the following result:

L'Hopital's rule:[10] If f and g are differentiable, $f(a) = g(a) = 0$, and $g'(a) \neq 0$, then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{f'(a)}{g'(a)}.$$

Example 1 Use l'Hopital's rule to confirm that $\lim_{x \to 0} \dfrac{\sin x}{x} = 1$.

Solution Let $f(x) = \sin x$ and $g(x) = x$. Then $f(0) = g(0) = 0$ and $f'(x) = \cos x$ and $g'(x) = 1$. Thus,

$$\lim_{x \to 0} \frac{\sin x}{x} = \frac{\cos 0}{1} = 1.$$

If we also have $f'(a) = g'(a) = 0$, then we can use the following result:

More general form of l'Hopital's rule: If f and g are differentiable and $f(a) = g(a) = 0$, then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)},$$

provided the limit on the right exists.

Example 2 Calculate $\lim_{t \to 0} \dfrac{e^t - 1 - t}{t^2}$.

Solution Let $f(t) = e^t - 1 - t$ and $g(t) = t^2$. Then $f(0) = e^0 - 1 - 0 = 0$ and $g(0) = 0$, and $f'(t) = e^t - 1$ and $g'(t) = 2t$. So

$$\lim_{t \to 0} \frac{e^t - 1 - t}{t^2} = \lim_{t \to 0} \frac{e^t - 1}{2t}.$$

Since $f'(0) = g'(0) = 0$, the ratio $f'(0)/g'(0)$ is not defined. So we use l'Hopital's rule again:

$$\lim_{t \to 0} \frac{e^t - 1 - t}{t^2} = \lim_{t \to 0} \frac{e^t - 1}{2t} = \lim_{t \to 0} \frac{e^t}{2} = \frac{1}{2}.$$

[10] Marquis de l'Hopital (1661–1704) was a French nobleman who wrote the first calculus text.

We can also use l'Hopital's rule in cases involving infinity.

L'Hopital's rule applies to limits involving infinity, provided f and g are differentiable. For a any real number or $\pm\infty$:

- When $\lim_{x\to a} f(x) = \pm\infty$ and $\lim_{x\to a} g(x) = \pm\infty$,

or

- When $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} g(x) = 0$.

it can be shown that:

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)},$$

provided the limit on the right-hand side exists.

Notice that we cannot evaluate $f'(x)/g'(x)$ directly when $a = \pm\infty$. The next example shows how this version of l'Hopital's rule is used.

Example 3 Calculate $\lim_{x\to\infty} \dfrac{5x + e^{-x}}{7x}$.

Solution Let $f(x) = 5x + e^{-x}$ and $g(x) = 7x$. Then $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} g(x) = \infty$, and $f'(x) = 5 - e^{-x}$ and $g'(x) = 7$, so

$$\lim_{x\to\infty} \frac{5x + e^{-x}}{7x} = \lim_{x\to\infty} \frac{(5 - e^{-x})}{7} = \frac{5}{7}.$$

Dominance: Powers, Polynomials, Exponentials, and Logarithms

In Chapter 1, we saw that some functions were much larger than others as $x \to \infty$. We say that g *dominates* f as $x \to \infty$ if $\lim_{x\to\infty} \dfrac{f(x)}{g(x)} = 0$. L'Hopital's rule gives us an easy way of checking this.

Example 4 Check that $x^{1/2}$ dominates $\ln x$ as $x \to \infty$.

Solution We apply l'Hopital's rule to $(\ln x)/x^{1/2}$:

$$\lim_{x\to\infty} \frac{\ln x}{x^{1/2}} = \lim_{x\to\infty} \frac{1/x}{\frac{1}{2}x^{-1/2}}.$$

To evaluate this limit, we simplify and get

$$\lim_{x\to\infty} \frac{1/x}{\frac{1}{2}x^{-1/2}} = \lim_{x\to\infty} \frac{2x^{1/2}}{x} = \lim_{x\to\infty} \frac{2}{x^{1/2}} = 0.$$

Therefore we have

$$\lim_{x\to\infty} \frac{\ln x}{x^{1/2}} = 0,$$

which tells us that $x^{1/2}$ dominates $\ln x$ as $x \to \infty$.

Example 5 Check that any exponential function of the form e^{kx} (with $k > 0$) dominates any power function of the form Ax^p (with A and p positive) as $x \to \infty$.

Solution We apply l'Hopital's rule repeatedly to Ax^p/e^{kx}:

$$\lim_{x\to\infty} \frac{Ax^p}{e^{kx}} = \lim_{x\to\infty} \frac{Apx^{p-1}}{ke^{kx}} = \lim_{x\to\infty} \frac{Ap(p-1)x^{p-2}}{k^2e^{kx}} = \cdots$$

Keep applying l'Hopital's rule until the power of x is no longer positive. Then the limit of the numerator must be a finite number, while the limit of the denominator must be ∞. Therefore we have

$$\lim_{x\to\infty} \frac{Ax^p}{e^{kx}} = 0,$$

so e^{kx} dominates Ax^p.

Recognizing the Form of a Limit

Although expressions like $0/0$ and ∞/∞ have no numerical value, they are useful in describing the form of a limit. We can also use l'Hopital's rule to calculate some limits of the form $\lim_{x\to\infty} f(x)g(x) = 0 \cdot \infty$, providing we rewrite them appropriately.

Example 6 Calculate $\lim_{x\to\infty} xe^{-x}$.

Solution Since $\lim_{x\to\infty} x = \infty$ and $\lim_{x\to\infty} e^{-x} = 0$, we see that

$$xe^{-x} \to \infty \cdot 0 \qquad \text{as } x \to \infty.$$

Rewriting

$$\infty \cdot 0 \quad \text{as} \quad \frac{\infty}{\frac{1}{0}} = \frac{\infty}{\infty}$$

gives a form whose value can be determined using l'Hopital's rule, so we rewrite the function xe^{-x} as

$$xe^{-x} = \frac{x}{e^x} \to \frac{\infty}{\infty} \qquad \text{as } x \to \infty.$$

Taking $f(x) = x$ and $g(x) = e^x$ gives $f'(x) = 1$ and $g'(x) = e^x$, so

$$\lim_{x\to\infty} xe^{-x} = \lim_{x\to\infty} \frac{x}{e^x} = \lim_{x\to\infty} \frac{1}{e^x} = 0.$$

A Famous Limit

In the following example, l'Hopital's rule is applied to calculate a limit that can be used to define e.

Example 7 Evaluate $\lim_{x\to\infty} \left(1 + \frac{1}{x}\right)^x$.

Solution As $x \to \infty$, we see that $\left(1 + \frac{1}{x}\right)^x \to 1^\infty$, a form whose value in this context is to be determined. Since $\ln 1^\infty = \infty \cdot \ln 1 = \infty \cdot 0$, we write

$$y = \left(1 + \frac{1}{x}\right)^x$$

and find the limit of $\ln y$:

$$\lim_{x\to\infty} \ln y = \lim_{x\to\infty} \ln \left(1 + \frac{1}{x}\right)^x = \lim_{x\to\infty} x \ln \left(1 + \frac{1}{x}\right) = \infty \cdot 0.$$

As in the previous example, we rewrite

$$\infty \cdot 0 \quad \text{as} \quad \frac{0}{\frac{1}{\infty}} = \frac{0}{0},$$

which suggests rewriting

$$\lim_{x\to\infty} x\ln(1+\frac{1}{x}) \quad \text{as} \quad \lim_{x\to\infty} \frac{\ln(1+\frac{1}{x})}{\frac{1}{x}}.$$

Since $\lim_{x\to\infty} \ln(1+1/x) = 0$ and $\lim_{x\to\infty}(1/x) = 0$, we can use l'Hopital's rule with $f(x) = \ln(1+1/x)$ and $g(x) = 1/x$. We have

$$f'(x) = \frac{1}{1+1/x}\left(-\frac{1}{x^2}\right) \qquad \text{and} \qquad g'(x) = -\frac{1}{x^2},$$

so

$$\lim_{x\to\infty} \ln y = \lim_{x\to\infty} \frac{\ln(1+1/x)}{1/x} = \lim_{x\to\infty} \frac{1}{1+1/x}\left(-\frac{1}{x^2}\right) \Big/ \left(-\frac{1}{x^2}\right)$$
$$= \lim_{x\to\infty} \frac{1}{1+1/x} = 1.$$

Since $\lim_{x\to\infty} \ln y = 1$, we have

$$\lim_{x\to\infty} y = e^1 = e.$$

Example 8 Put the following limits in a form that can be evaluated using l'Hopital's rule:

(a) $\lim_{x\to 0^+} x\ln x$ (b) $\lim_{x\to\infty} x^{1/x}$ (c) $\lim_{x\to 0^+} x^x$ (d) $\lim_{x\to 0} \frac{1}{x} - \frac{1}{\sin x}$

Solution

(a) We have

$$\lim_{x\to 0^+} x\ln x = 0\cdot\infty.$$

We can rewrite

$$0\cdot\infty \quad \text{as} \quad \frac{\infty}{\frac{1}{0}} = \frac{\infty}{\infty}.$$

This corresponds to rewriting

$$\lim_{x\to 0^+} x\ln x \quad \text{as} \quad \lim_{x\to 0^+} \frac{\ln x}{\frac{1}{x}}.$$

This is an ∞/∞ form that can be evaluated using l'Hopital's rule. (Note that we could also have written

$$0\cdot\infty \quad \text{as} \quad \frac{0}{\frac{1}{\infty}} = \frac{0}{0},$$

but this leads to rewriting

$$\lim_{x\to 0^+} x\ln x \quad \text{as} \quad \lim_{x\to 0^+} \frac{x}{\frac{1}{\ln x}}.$$

It turns out that l'Hopital's rule fails to simplify this limit.)

(b) In this case we have a ∞^0 form, so we take the logarithm and get a $0\cdot\infty$ form:

$$\lim_{x\to\infty} \ln(x^{1/x}) = \lim_{x\to\infty} \frac{1}{x}\ln x = \lim_{x\to\infty} \frac{\ln x}{x}.$$

This is an ∞/∞ form that can be evaluated using l'Hopital's rule. Once we get the answer, we exponentiate it to get the original limit.

(c) Since $\lim_{x\to 0^+} x = 0$, this is a 0^0 form. If we take the logarithm, we get

$$\ln 0^0 = 0\cdot\ln 0 = 0\cdot\infty.$$

This corresponds to the limit

$$\lim_{x\to 0^+} \ln x^x = \lim_{x\to 0^+} x \ln x,$$

which considered in part (a).

(d) We have

$$\lim_{x\to 0} \frac{1}{x} - \frac{1}{\sin x} = \frac{1}{0} - \frac{1}{0} = \infty - \infty.$$

Limits like this can often be calculated by adding the fractions:

$$\lim_{x\to 0} \frac{1}{x} - \frac{1}{\sin x} = \lim_{x\to 0} \frac{\sin x - x}{x \sin x},$$

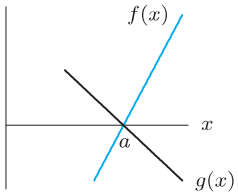
a $0/0$ form that can be evaluated using l'Hopital's rule twice.

Exercises and Problems for Section 4.7

Exercises

For Exercises 1–4, find the sign of $\lim_{x\to a} \frac{f(x)}{g(x)}$ from the figure.

1.



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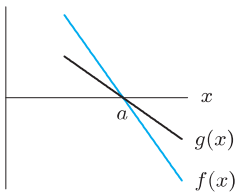

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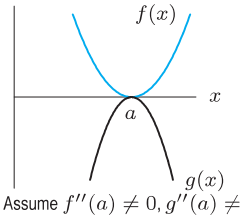
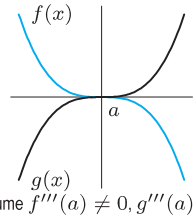


Assume $f''(a) \neq 0, g''(a) \neq 0$

4.



Assume $f'''(a) \neq 0, g'''(a) \neq 0$

Based on your knowledge of the behavior of the numerator and denominator, predict the value of the limits in Exercises 5–8. Then find each limit using l'Hopital's rule.

5. $\lim_{x\to 0} \frac{x^2}{\sin x}$

6. $\lim_{x\to 0} \frac{\sin^2 x}{x}$

7. $\lim_{x\to 0} \frac{\sin x}{x^{1/3}}$

8. $\lim_{x\to 0} \frac{x}{(\sin x)^{1/3}}$

In Exercises 9–12, which function dominates as $x \to \infty$?

9. x^5 and $0.1x^7$

10. $0.01x^3$ and $50x^2$

11. $\ln(x+3)$ and $x^{0.2}$

12. x^{10} and $e^{0.1x}$

Problems

13. Find the horizontal asymptote of $f(x) = \frac{2x^3 + 5x^2}{3x^3 - 1}$.

14. **(a)** What is the slope of $f(x) = \sin(3x)$ at $x = 0$?
(b) What is the slope of $g(x) = 5x$ at $x = 0$?
(c) Use the results of parts (a) and (b) to calculate $\lim_{x\to 0} \frac{\sin(3x)}{5x}$.

15. The functions f and g and their tangent lines at $(4, 0)$ are shown in Figure 4.92. Find $\lim_{x\to 4} \frac{f(x)}{g(x)}$.

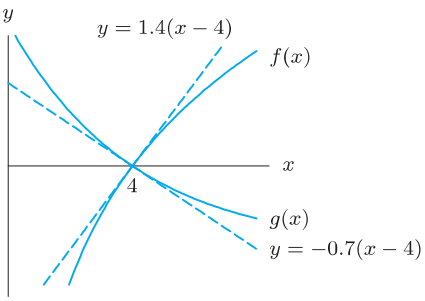


Figure 4.92

In Problems 16–21 describe the form of the limit ($0/0$, ∞/∞, $\infty\cdot 0$, $\infty-\infty$, 1^∞, 0^0, ∞^0, or none of these). Does l'Hopital's rule apply? If so, explain how.

16. $\lim_{x\to\infty} \dfrac{x}{e^x}$

17. $\lim_{x\to 1} \dfrac{x}{x-1}$

18. $\lim_{t\to\infty} \left(\dfrac{1}{t} - \dfrac{2}{t^2}\right)$

19. $\lim_{t\to 0^+} \dfrac{1}{t} - \dfrac{1}{e^t - 1}$

20. $\lim_{x\to 0} (1+x)^x$

21. $\lim_{x\to\infty} (1+x)^{1/x}$

In Problems 22–35 determine whether the limit exists, and where possible evaluate it.

22. $\lim_{x\to 1} \dfrac{\ln x}{x^2-1}$

23. $\lim_{t\to\pi} \dfrac{\sin^2 t}{t-\pi}$

24. $\lim_{n\to\infty} \sqrt[n]{n}$

25. $\lim_{x\to 0^+} x \ln x$

26. $\lim_{x\to 0} \dfrac{\sinh(2x)}{x}$

27. $\lim_{x\to 0} \dfrac{1-\cosh(3x)}{x}$

28. $\lim_{x\to 1^-} \dfrac{\cos^{-1} x}{x-1}$

29. $\lim_{x\to 0} \left(\dfrac{1}{x} - \dfrac{1}{\sin x}\right)$

30. $\lim_{t\to 0^+} \left(\dfrac{2}{t} - \dfrac{1}{e^t-1}\right)$

31. $\lim_{t\to 0} \left(\dfrac{1}{t} - \dfrac{1}{e^t-1}\right)$

32. $\lim_{x\to\infty} \left(1+\sin\left(\dfrac{3}{x}\right)\right)^x$

33. $\lim_{t\to 0} \dfrac{\sin^2 At}{\cos At - 1}$, $A \neq 0$

34. $\lim_{t\to\infty} e^t - t^n$, where n is a positive integer

35. $\lim_{x\to 0^+} x^a \ln x$, where a is a positive constant.

In Problems 36–38, explain why l'Hopital's rule cannot be used to calculate the limit. Then evaluate the limit if it exists.

36. $\lim_{x\to 1} \dfrac{\sin(2x)}{x}$

37. $\lim_{x\to 0} \dfrac{\cos x}{x}$

38. $\lim_{x\to\infty} \dfrac{e^{-x}}{\sin x}$

39. If a, b, x, and y are all positive and $a+b=1$, evaluate

(a) $\lim_{p\to 0} \dfrac{\ln(ax^p + by^p)}{p}$ **(b)** $\lim_{p\to 0} (ax^p + by^p)^{1/p}$

In Problems 40–42, evaluate the limit using the fact that

$$\lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n = e.$$

40. $\lim_{x\to 0^+} (1+x)^{1/x}$

41. $\lim_{n\to\infty} \left(1+\dfrac{2}{n}\right)^n$

42. $\lim_{x\to 0^+} (1+kx)^{t/x}$; $k>0$

43. Show that $\lim_{n\to\infty} \left(1-\dfrac{1}{n}\right)^n = e^{-1}$.

44. Use the result of Problem 43 to evaluate

$$\lim_{n\to\infty} \left(1-\frac{\lambda}{n}\right)^n.$$

Evaluate the limits in Problems 45–47 where

$$f(t) = \left(\frac{3^t + 5^t}{2}\right)^{1/t} \quad \text{for } t \neq 0.$$

45. $\lim_{t\to-\infty} f(t)$

46. $\lim_{t\to+\infty} f(t)$

47. $\lim_{t\to 0} f(t)$

In Problems 48–51, evaluate the limits as x approaches 0.

48. $\dfrac{\sinh(2x)}{x}$

49. $\dfrac{1-\cosh(3x)}{x}$

50. $\dfrac{1-\cosh(5x)}{x^2}$

51. $\dfrac{x-\sinh(x)}{x^3}$

In Problems 52–54, compute a limit that Euler used to illustrate the use of l'Hopital's rule.

52. $\lim_{x\to 0} \dfrac{e^x - 1 - \ln(1+x)}{x^2}$

53. $\lim_{x\to\pi/2} \dfrac{1-\sin x+\cos x}{\sin x + \cos x - 1}$

54. $\lim_{x\to 1} \dfrac{x^x - x}{1-x+\ln x}$

4.8 PARAMETRIC EQUATIONS

Representing Motion in the Plane

To represent the motion of a particle in the xy-plane we use two equations, one for the x-coordinate of the particle, $x = f(t)$, and another for the y-coordinate, $y = g(t)$. Thus at time t the particle is at the point $(f(t), g(t))$. The equation for x describes the right-left motion; the equation for y describes the up-down motion. The two equations for x and y are called *parametric equations* with *parameter* t.

Example 1 Describe the motion of the particle whose coordinates at time t are $x = \cos t$ and $y = \sin t$.

Solution Since $(\cos t)^2 + (\sin t)^2 = 1$, we have $x^2 + y^2 = 1$. That is, at any time t the particle is at a point (x, y) on the unit circle $x^2 + y^2 = 1$. We plot points at different times to see how the particle moves on the circle. (See Figure 4.93 and Table 4.5.) The particle moves at a uniform speed, completing one full trip counterclockwise around the circle every 2π units of time. Notice how the x-coordinate goes repeatedly back and forth from -1 to 1 while the y-coordinate goes repeatedly up and down from -1 to 1. The two motions combine to trace out a circle.

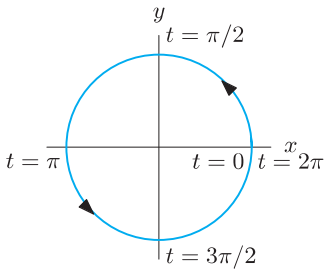


Figure 4.93: The circle parameterized by $x = \cos t$, $y = \sin t$

Table 4.5 *Points on the circle with* $x = \cos t$, $y = \sin t$

t	x	y
0	1	0
$\pi/2$	0	1
π	-1	0
$3\pi/2$	0	-1
2π	1	0

Example 2 Figure 4.94 shows the graphs of two functions, $f(t)$ and $g(t)$. Describe the motion of the particle whose coordinates at time t are $x = f(t)$ and $y = g(t)$.

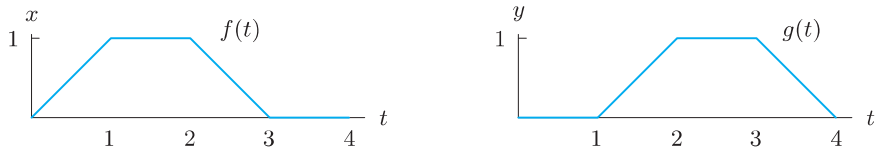


Figure 4.94: Graphs of $x = f(t)$ and $y = g(t)$ used to trace out the path $(f(t), g(t))$ in Figure 4.95

Solution Between times $t = 0$ and $t = 1$, the x-coordinate goes from 0 to 1, while the y-coordinate stays fixed at 0. So the particle moves along the x-axis from $(0, 0)$ to $(1, 0)$. Then, between times $t = 1$ and $t = 2$, the x-coordinate stays fixed at $x = 1$, while the y-coordinate goes from 0 to 1. Thus, the particle moves along the vertical line from $(1, 0)$ to $(1, 1)$. Similarly, between times $t = 2$ and $t = 3$, it moves horizontally back to $(0, 1)$, and between times $t = 3$ and $t = 4$ it moves down the y-axis to $(0, 0)$. Thus, it traces out the square in Figure 4.95.

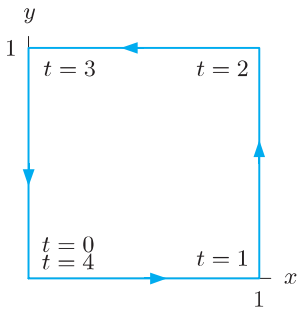


Figure 4.95: The square parameterized by $(f(t), g(t))$

Different Motions Along the Same Path

Example 3 Describe the motion of the particle whose x and y coordinates at time t are given by the equations

$$x = \cos(3t), \quad y = \sin(3t).$$

Solution Since $(\cos(3t))^2 + (\sin(3t))^2 = 1$, we have $x^2 + y^2 = 1$, giving motion around the unit circle. But from Table 4.6, we see that the particle in this example is moving three times as fast as the particle in Example 1. (See Figure 4.96.)

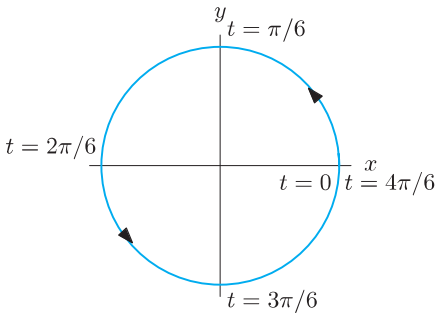


Figure 4.96: The circle parameterized by $x = \cos(3t)$, $y = \sin(3t)$

Table 4.6 *Points on circle with* $x = \cos(3t)$, $y = \sin(3t)$

t	x	y
0	1	0
$\pi/6$	0	1
$2\pi/6$	-1	0
$3\pi/6$	0	-1
$4\pi/6$	1	0

Example 3 is obtained from Example 1 by replacing t by $3t$; this is called a *change in parameter*. If we make a change in parameter, the particle traces out the same curve (or a part of it) but at a different speed or in a different direction.

Example 4 Describe the motion of the particle whose x and y coordinates at time t are given by

$$x = \cos(e^{-t^2}), \quad y = \sin(e^{-t^2}).$$

Solution As in Examples 1 and 3, we have $x^2 + y^2 = 1$ so the motion lies on the unit circle. As time t goes from $-\infty$ (way back in the past) to 0 (the present) to ∞ (way off in the future), e^{-t^2} goes from near 0 to 1 back to near 0. So $(x, y) = (\cos(e^{-t^2}), \sin(e^{-t^2}))$ goes from near $(1, 0)$ to $(\cos 1, \sin 1)$ and back to near $(1, 0)$. The particle does not actually reach the point $(1, 0)$. (See Figure 4.97 and Table 4.7.)

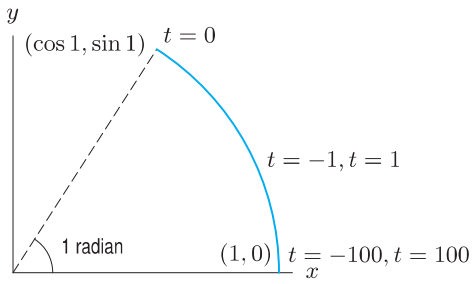


Figure 4.97: The circle parameterized by $x = \cos(e^{-t^2})$, $y = \sin(e^{-t^2})$

Table 4.7 *Points on circle with* $x = \cos(e^{-t^2})$, $y = \sin(e^{-t^2})$

t	x	y
-100	~ 1	~ 0
-1	0.93	0.36
0	0.54	0.84
1	0.93	0.36
100	~ 1	~ 0

Motion in a Straight Line

An object moves with constant speed along a straight line through the point (x_0, y_0). Both the x- and y-coordinates have a constant rate of change. Let $a = dx/dt$ and $b = dy/dt$. Then at time t the object has coordinates $x = x_0 + at$, $y = y_0 + bt$. (See Figure 4.98.) Notice that a represents the change in x in one unit of time, and b represents the change in y. Thus the line has slope $m = b/a$.

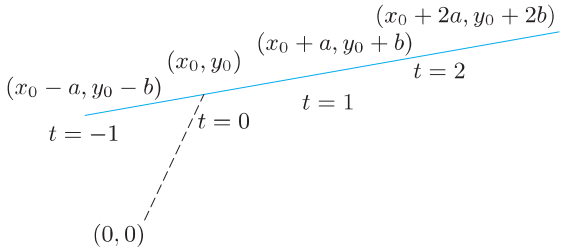


Figure 4.98: The line $x = x_0 + at$, $y = y_0 + bt$

This yields the following:

Parametric Equations for a Straight Line

An object moving along a line through the point (x_0, y_0), with $dx/dt = a$ and $dy/dt = b$, has parametric equations

$$x = x_0 + at, \quad y = y_0 + bt.$$

The slope of the line is $m = b/a$.

Example 5 Find parametric equations for:

(a) The line passing through the points $(2, -1)$ and $(-1, 5)$.
(b) The line segment from $(2, -1)$ to $(-1, 5)$.

Solution

(a) Imagine an object moving with constant speed along a straight line from $(2, -1)$ to $(-1, 5)$, making the journey from the first point to the second in one unit of time. Then $dx/dt = ((-1) - 2)/1 = -3$ and $dy/dt = (5 - (-1))/1 = 6$. Thus the parametric equation are

$$x = 2 - 3t, \quad y = -1 + 6t.$$

(b) In the parameterization in part (a), $t = 0$ corresponds to the point $(2, -1)$ and $t = 1$ corresponds to the point $(-1, 5)$. So the parameterization of the segment is

$$x = 2 - 3t, \quad y = -1 + 6t, \qquad 0 \le t \le 1.$$

There are many other possible parametric equations for this line.

Speed and Velocity

An object moves along a straight line at a constant speed, with $dx/dt = a$ and $dy/dt = b$. In one unit of time, the object moves a units horizontally and b units vertically. Thus, by the Pythagorean Theorem, it travels a distance $\sqrt{a^2 + b^2}$. So its speed is

$$\text{Speed} = \frac{\text{Distance traveled}}{\text{Time taken}} = \frac{\sqrt{a^2 + b^2}}{1} = \sqrt{a^2 + b^2}.$$

For general motion along a curve with varying speed, we make the following definition:

The **instantaneous speed** of a moving object is defined to be

$$v = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}.$$

The quantity $v_x = dx/dt$ is the **instantaneous velocity** in the x-direction; $v_y = dy/dt$ is the **instantaneous velocity** in the y-direction. The **velocity vector** $\vec{v}$ is written $\vec{v} = v_x\vec{i} + v_y\vec{j}$.

The quantities v_x and v_y are called the *components* of the velocity in the x- and y-directions. The velocity vector $\vec{v}$ is a useful way to keep track of the velocities in both directions using one mathematical object. The symbols $\vec{i}$ and $\vec{j}$ represent vectors of length one in the positive x and y-directions, respectively. For more about vectors, see Appendix D.

Example 6 A particle moves along a curve in the xy-plane with $x(t) = 2t + e^t$ and $y(t) = 3t - 4$, where t is time. Find the velocity vector and speed of the particle when $t = 1$.

Solution Differentiating gives

$$\frac{dx}{dt} = 2 + e^t \quad \frac{dy}{dt} = 3.$$

When $t = 1$ we have $v_x = 2 + e, v_y = 3$. So the velocity vector is $\vec{v} = (2 + e)\vec{i} + 3\vec{j}$ and the speed is $\sqrt{(2 + e)^2 + 3^2} = \sqrt{13 + 4e + e^2} = 5.591$.

Example 7 A particle moves in the xy-plane with $x = 2t^3 - 9t^2 + 12t$ and $y = 3t^4 - 16t^3 + 18t^2$, where t is time.

(a) At what times is the particle
  (i) Stopped
  (ii) Moving parallel to the x- or y- axis?

(b) Find the speed of the particle at time t.

Solution (a) Differentiating gives

$$\frac{dx}{dt} = 6t^2 - 18t + 12 \quad \frac{dy}{dt} = 12t^3 - 48t^2 + 36t.$$

We are interested in the points at which $dx/dt = 0$ or $dy/dt = 0$. Solving gives

$$\frac{dx}{dt} = 6(t^2 - 3t + 2) = 6(t - 1)(t - 2) \quad \text{so} \quad \frac{dx}{dt} = 0 \quad \text{if } t = 1 \text{ or } t = 2.$$

$$\frac{dy}{dt} = 12t(t^2 - 4t + 3) = 12t(t - 1)(t - 3) \quad \text{so} \quad \frac{dy}{dt} = 0 \quad \text{if } t = 0, t = 1, \text{ or } t = 3.$$

(i) The particle is stopped if both dx/dt and dy/dt are 0, which occurs at $t = 1$.

(ii) The particle is moving parallel to the x-axis if $dy/dt = 0$ but $dx/dt \neq 0$. This occurs at $t = 0$ and $t = 3$. The particle is moving parallel to the y-axis if $dx/dt = 0$ but $dy/dt \neq 0$. This occurs at $t = 2$.

(b) We have

$$\text{Speed} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \sqrt{(6t^2 - 18t + 12)^2 + (12t^3 - 48t^2 + 36t)^2}$$

$$= 6\sqrt{4t^6 - 32t^5 + 89t^4 - 102t^3 + 49t^2 - 12t + 4}.$$

Example 8 A child is sitting on a ferris wheel of diameter 10 meters, making one revolution every 2 minutes. Find the speed of the child

(a) Using geometry. (b) Using a parameterization of the motion.

Solution (a) The child moves at a constant speed around a circle of radius 5 meters, completing one revolution every 2 minutes. One revolution around a circle of radius 5 is a distance of 10π, so the child's speed is $10\pi/2 = 5\pi \approx 15.7$ m/min. See Figure 4.99.

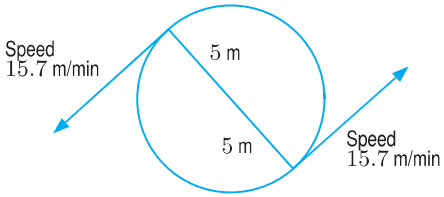


Figure 4.99: Motion of a child on a ferris wheel at two different times is represented by the arrows. The direction of each arrow is the direction of motion at that time.

(b) The ferris wheel has radius 5 meters and completes 1 revolution counterclockwise every 2 minutes. If the origin is at the center of the circle and we measure x and y in meters, the motion is parameterized by equations of the form

$$x = 5\cos(\omega t), \quad y = 5\sin(\omega t),$$

where ω is chosen to make the period 2 minutes. Since the period of $\cos(\omega t)$ and $\sin(\omega t)$ is $2\pi/\omega$, we must have

$$\frac{2\pi}{\omega} = 2, \quad \text{so} \quad \omega = \pi.$$

Thus, for t in minutes, the motion is described by the equations

$$x = 5\cos(\pi t), \quad y = 5\sin(\pi t).$$

So the speed is given by

$$v = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$$

$$= \sqrt{(-5\pi)^2\sin^2(\pi t) + (5\pi)^2\cos^2(\pi t)} = 5\pi\sqrt{\sin^2(\pi t) + \cos^2(\pi t)} = 5\pi \approx 15.7 \text{ m/min},$$

which agrees with the speed we calculated in part (a).

Tangent Lines

To find the tangent line at a point (x_0, y_0) to a curve given parametrically, we find the straight line motion through (x_0, y_0) with the same velocity in the x and y directions as a particle moving along the curve.

Example 9 Find the tangent line at the point $(1, 2)$ to the curve defined by the parametric equation

$$x = t^3, \quad y = 2t.$$

Solution At time $t = 1$ the particle is at the point $(1, 2)$. The velocity in the x-direction at time t is $v_x = dx/dt = 3t^2$, and the velocity in the y-direction is $v_y = dy/dt = 2$. So at $t = 1$ the velocity in the x-direction is 3 and the velocity in the y-direction is 2. Thus the tangent line has parametric equations

$$x = 1 + 3t, \quad y = 2 + 2t.$$

Parametric Representations of Curves in the Plane

Sometimes we are more interested in the curve traced out by the particle than we are in the motion itself. In that case we will call the parametric equations a *parameterization* of the curve. As we can see by comparing Examples 1 and 3, two different parameterizations can describe the same curve in the xy-plane. Though the parameter, which we usually denote by t, may not have physical meaning, it is often helpful to think of it as time.

Example 10 Give a parameterization of the semicircle of radius 1 shown in Figure 4.100.

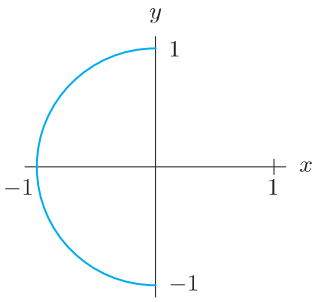


Figure 4.100: Parameterization of semicircle for Example 10

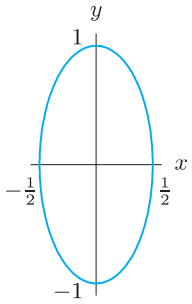


Figure 4.101: Parameterization of the ellipse $4x^2 + y^2 = 1$ for Example 11

Solution We can use the equations $x = \cos t$ and $y = \sin t$ for counterclockwise motion in a circle, from Example 1 on page 235. The particle passes $(0, 1)$ at $t = \pi/2$, moves counterclockwise around the circle, and reaches $(0, -1)$ at $t = 3\pi/2$. So a parameterization is

$$x = \cos t, \ y = \sin t, \quad \frac{\pi}{2} \le t \le \frac{3\pi}{2}.$$

To find the xy-equation of a curve given parametrically, we eliminate the parameter t in the parametric equations. In the previous example, we use the Pythagorean identity, so

$$\cos^2 t + \sin^2 t = 1 \quad \text{gives} \quad x^2 + y^2 = 1.$$

Example 11 Give a parameterization of the ellipse $4x^2 + y^2 = 1$ shown in Figure 4.101.

Solution Since $(2x)^2 + y^2 = 1$, we adapt the parameterization of the circle in Example 1. Replacing x by $2x$ gives the equations $2x = \cos t, y = \sin t$. A parameterization of the ellipse is thus

$$x = \tfrac{1}{2}\cos t, \qquad y = \sin t, \qquad 0 \le t \le 2\pi.$$

We usually require that the parameterization of a curve go from one end of the curve to the other without retracing any portion of the curve. This is different from parameterizing the motion of a particle, where, for example, a particle may move around the same circle many times.

Parameterizing the Graph of a Function

The graph of any function $y = f(x)$ can be parameterized by letting the parameter t be x:

$$x = t, \quad y = f(t).$$

Example 12 Give parametric equations for the curve $y = x^3 - x$. In which direction does this parameterization trace out the curve?

Solution Let $x = t, \ y = t^3 - t$. Thus, $y = t^3 - t = x^3 - x$. Since $x = t$, as time increases the x-coordinate moves from left to right, so the particle traces out the curve $y = x^3 - x$ from left to right.

Curves Given Parametrically

Some complicated curves can be graphed more easily using parametric equations; the next example shows such a curve.

Example 13 Assume t is time in seconds. Sketch the curve traced out by the particle whose motion is given by

$$x = \cos(3t), \quad y = \sin(5t).$$

Solution The x-coordinate oscillates back and forth between 1 and -1, completing 3 oscillations every 2π seconds. The y-coordinate oscillates up and down between 1 and -1, completing 5 oscillations every 2π seconds. Since both the x- and y-coordinates return to their original values every 2π seconds, the curve is retraced every 2π seconds. The result is a pattern called a Lissajous figure. (See Figure 4.102.) Problems 55–58 concern Lissajous figures $x = \cos(at)$, $y = \sin(bt)$ for other values of a and b.

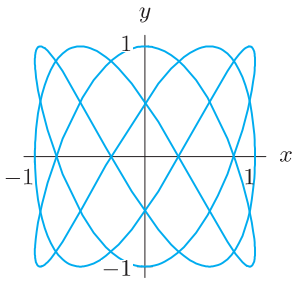


Figure 4.102: A Lissajous figure: $x = \cos(3t), y = \sin(5t)$

Slope and Concavity of Parametric Curves

Suppose we have a curve traced out by the parametric equations $x = f(t)$, $y = g(t)$. To find the slope at a point on the curve, we could, in theory, eliminate the parameter t and then differentiate the function we obtain. However, the chain rule gives us an easier way.

Suppose the curve traced out by the parametric equations is represented by $y = h(x)$. (It may be represented by an implicit function.) Thinking of x and y as functions of t, the chain rule gives

$$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt},$$

so we obtain the slope of the curve as a function of t:

$$\text{Slope of curve} = \frac{dy}{dx} = \frac{dy/dt}{dx/dt}.$$

We can find the second derivative, d^2y/dx^2, by a similar method and use it to investigate the concavity of the curve. The chain rule tells us that if w is any differentiable function of x, then

$$\frac{dw}{dx} = \frac{dw/dt}{dx/dt}.$$

For $w = dy/dx$, we have

$$\frac{dw}{dx} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2},$$

so the chain rule gives the second derivative at any point on a parametric curve:

$$\frac{d^2y}{dx^2} = \frac{d}{dt}\left(\frac{dy}{dx}\right) \Big/ \frac{dx}{dt}.$$

Example 14 If $x = \cos t,\ y = \sin t$, find the point corresponding to $t = \pi/4$, the slope of the curve at the point, and d^2y/dx^2 at the point.

Solution The point corresponding to $t = \pi/4$ is $(\cos(\pi/4), \sin(\pi/4)) = (1/\sqrt{2}, 1/\sqrt{2})$.

To find the slope, we use

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{\cos t}{-\sin t},$$

so when $t = \pi/4$,

$$\text{Slope} = \frac{\cos(\pi/4)}{-\sin(\pi/4)} = -1.$$

Thus, the curve has slope -1 at the point $(1/\sqrt{2}, 1/\sqrt{2})$. This is as we would expect, since the curve traced out is the circle of Example 9.

To find d^2y/dx^2, we use $w = dy/dx = -(\cos t)/(\sin t)$, so

$$\frac{d^2y}{dx^2} = \frac{d}{dt}\left(-\frac{\cos t}{\sin t}\right) \Big/ (-\sin t) = -\frac{(-\sin t)(\sin t) - (\cos t)(\cos t)}{\sin^2 t} \cdot \left(-\frac{1}{\sin t}\right) = -\frac{1}{\sin^3 t}.$$

Thus, at $t = \pi/4$

$$\left.\frac{d^2y}{dx^2}\right|_{t=\pi/4} = -\frac{1}{(\sin(\pi/4))^3} = -2\sqrt{2}.$$

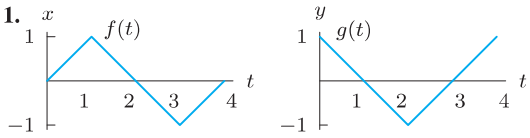
The concavity is negative since the point is on the top half of the circle where the graph is bending downward.

Exercises and Problems for Section 4.8

Exercises

For Exercises 1–4, use the graphs of f and g to describe the motion of a particle whose position at time t is given by $x = f(t), y = g(t)$.

1.



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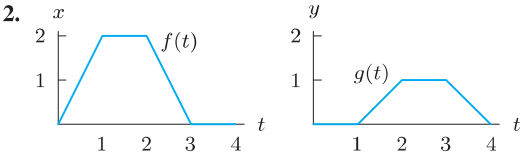

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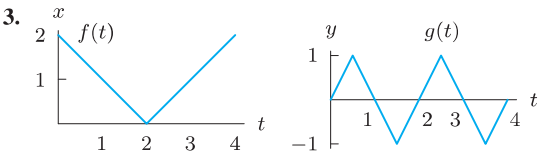

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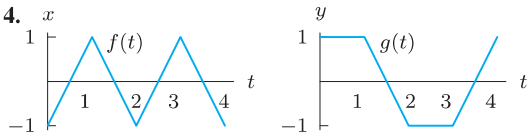


Exercises 5–10 give parameterizations of the unit circle or a part of it. Describe in words how the circle is traced out, including when and where the particle is moving clockwise and when and where the particle is moving counterclockwise.

5. $x = \sin t, \quad y = \cos t$

6. $x = \cos t, \quad y = -\sin t$

7. $x = \cos(t^2), \quad y = \sin(t^2)$

8. $x = \cos(t^3 - t), \quad y = \sin(t^3 - t)$

9. $x = \cos(\ln t), \quad y = \sin(\ln t)$

10. $x = \cos(\cos t), \quad y = \sin(\cos t)$

For Exercises 11–14, find the speed for the given motion of a particle. Find any times when the particle comes to a stop.

11. $x = t^2, \quad y = t^3$

12. $x = \cos(t^2), \quad y = \sin(t^2)$

13. $x = \cos 2t, \quad y = \sin t$

14. $x = t^2 - 2t, y = t^3 - 3t$

15. Find parametric equations for the tangent line at $t = 2$ for Problem 11.

In Exercises 16–22, write a parameterization for the curves in the xy-plane.

16. A circle of radius 3 centered at the origin and traced out clockwise.

17. A vertical line through the point $(-2, -3)$.

18. A circle of radius 5 centered at the point $(2, 1)$ and traced out counterclockwise.

19. A circle of radius 2 centered at the origin traced clockwise starting from $(-2, 0)$ when $t = 0$.

20. The line through the points $(2, -1)$ and $(1, 3)$.

21. An ellipse centered at the origin and crossing the x-axis at ± 5 and the y-axis at ± 7.

22. An ellipse centered at the origin, crossing the x-axis at ± 3 and the y-axis at ± 7. Start at the point $(-3, 0)$ and trace out the ellipse counterclockwise.

In Exercises 23–25, find an equation of the tangent line to the curve for the given value of t.

23. $x = t^3 - t, \quad y = t^2 \quad$ when $t = 2$

24. $x = t^2 - 2t, \quad y = t^2 + 2t \quad$ when $t = 1$

25. $x = \sin(3t), \quad y = \sin(4t) \quad$ when $t = \pi$

What curves do the parametric equations in Exercises 26–28 trace out? Find the Cartesian equation for each curve.

26. $x = 2 + \cos t, \; y = 2 - \sin t$

27. $x = 2 + \cos t, \; y = 2 - \cos t$

28. $x = 2 + \cos t, \; y = \cos^2 t$

In Exercises 29–34, describe the motion of a particle moving according to the parametric equations. Find the equation of the curve along which the particle moves.

29. $x = 3t + 1$, $y = t - 4$

30. $x = t^2 + 3$, $y = t^2 - 2$

31. $x = t + 4$, $y = t^2 - 3$

32. $x = \cos 3t$, $y = \sin 3t$

33. $x = 3\cos t$, $y = 3\sin t$

34. $x = 2 + 5\cos t$, $y = 7 + 5\sin t$

Problems

35. A line is parameterized by $x = 10 + t$ and $y = 2t$.

(a) What part of the line do we get by restricting t to $t < 0$?

(b) What part of the line do we get by restricting t to $0 \le t \le 1$?

36. A line is parameterized by $x = 2 + 3t$ and $y = 4 + 7t$.

(a) What part of the line is obtained by restricting t to nonnegative numbers?

(b) What part of the line is obtained if t is restricted to $-1 \le t \le 0$?

(c) How should t be restricted to give the part of the line to the left of the y-axis?

37. **(a)** Explain how you know that the following two pairs of equations parameterize the same line:

$$x = 2 + t, y = 4 + 3t \text{ and } x = 1 - 2t, y = 1 - 6t.$$

(b) What are the slope and y intercept of this line?

38. Describe the similarities and differences among the motions in the plane given by the following three pairs of parametric equations:

(a) $x = t, \quad y = t^2$ **(b)** $x = t^2, \quad y = t^4$

(c) $x = t^3, \quad y = t^6$.

39. What can you say about the values of a, b and k if the equations

$$x = a + k\cos t, \quad y = b + k\sin t, \qquad 0 \le t \le 2\pi,$$

trace out the following circles in Figure 4.103?

(a) C_1 **(b)** C_2 **(c)** C_3

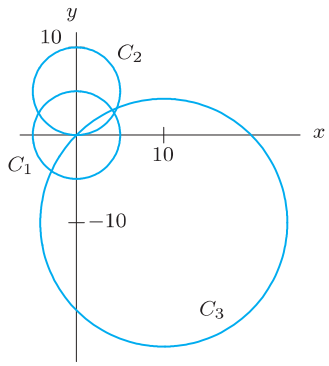


Figure 4.103

40. Suppose $a, b, c, d, m, n, p, q > 0$. Match each pair of parametric equations with one of the lines l_1, l_2, l_3, l_4 in Figure 4.104.

I. $\begin{cases} x = a + ct, \\ y = -b + dt. \end{cases}$ II. $\begin{cases} x = m + pt, \\ y = n - qt. \end{cases}$

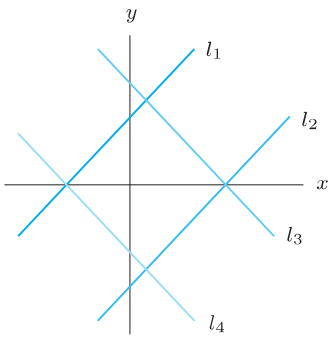


Figure 4.104

41. Describe in words the curve represented by the parametric equations

$$x = 3 + t^3, \quad y = 5 - t^3.$$

42. **(a)** Sketch the parameterized curve $x = t\cos t$, $y = t\sin t$ for $0 \le t \le 4\pi$.
(b) By calculating the position at $t = 2$ and $t = 2.01$, estimate the speed at $t = 2$.
(c) Use derivatives to calculate the speed at $t = 2$ and compare your answer to part (b).

43. The position of a particle at time t is given by $x = e^t$ and $y = 2e^{2t}$.
(a) Find dy/dx in terms of t.
(b) Eliminate the parameter and write y in terms of x.
(c) Using your answer to part (b), find dy/dx in terms of x.

44. For x and y in meters, the motion of the particle given by

$$x = t^3 - 3t, \quad y = t^2 - 2t,$$

where the y-axis is vertical and the x-axis is horizontal.
(a) Does the particle ever come to a stop? If so, when and where?
(b) Is the particle ever moving straight up or down? If so, when and where?
(c) Is the particle ever moving straight horizontally right or left? If so, when and where?

45. At time t, the position of a particle moving on a curve is given by $x = e^{2t} - e^{-2t}$ and $y = 3e^{2t} + e^{-2t}$.
(a) Find all values of t at which the curve has
(i) A horizontal tangent.
(ii) A vertical tangent.
(b) Find dy/dx in terms of t.
(c) Find $\lim_{t\to\infty} dy/dx$.

46. Figure 4.105 shows the graph of a parameterized curve $x = f(t)$, $y = f'(t)$ for a function $f(t)$.
(a) Is $f(t)$ an increasing or decreasing function?
(b) As t increases, is the curve traced from P to Q or from Q to P?
(c) Is $f(t)$ concave up or concave down?

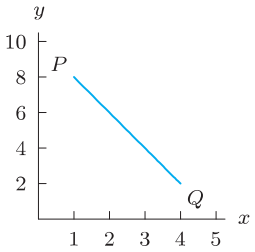


Figure 4.105

47. At time t, the position of a particle is $x(t) = 5\sin(2t)$ and $y(t) = 4\cos(2t)$, with $0 \le t < 2\pi$.
(a) Graph the path of the particle for $0 \le t < 2\pi$, indicating the direction of motion.
(b) Find the position and velocity of the particle when $t = \pi/4$.
(c) How many times does the particle pass through the point found in part (b)?
(d) What does your answer to part (b) tell you about the direction of the motion relative to the coordinate axes when $t = \pi/4$?
(e) What is the speed of the particle at time $t = \pi$?

48. At time t, a projectile launched with angle of elevation α and initial velocity v_0 has position $x(t) = (v_0\cos\alpha)t$ and $y(t) = (v_0\sin\alpha)t - \frac{1}{2}gt^2$, where g is the acceleration due to gravity.
(a) A football player kicks a ball at an angle of 36° above the ground with an initial velocity of 60 feet per second. Write the parametric equations for the position of the football at time t seconds. Use $g = 32\text{ft/sec}^2$.
(b) Graph the path that the football follows.
(c) How long does it take for the football to hit the ground? How far is it from the spot where the football player kicked it?
(d) What is the maximum height the football reaches during its flight?
(e) At what speed is the football traveling 1 second after it was kicked?

49. Two particles move in the xy-plane. At time t, the position of particle A is given by $x(t) = 4t - 4$ and $y(t) = 2t - k$, and the position of particle B is given by $x(t) = 3t$ and $y(t) = t^2 - 2t - 1$.
(a) If $k = 5$, do the particles ever collide? Explain.
(b) Find k so that the two particles do collide.
(c) At the time that the particles collide in part (b), which particle is moving faster?

50. **(a)** Find d^2y/dx^2 for $x = t^3 + t, y = t^2$.
(b) Is the curve concave up or down at $t = 1$?

51. **(a)** An object moves along the path $x = 3t$ and $y = \cos(2t)$, where t is time. Write the equation for the line tangent to this path at $t = \pi/3$.
(b) Find the smallest positive value of t for which the y-coordinate is a local maximum.
(c) Find d^2y/dx^2 when $t = 2$. What does this tell you about the concavity of the graph at $t = 2$?

52. The position of a particle at time t is given by $x = e^t + 3$ and $y = e^{2t} + 6e^t + 9$.

(a) Find dy/dx in terms of t.
(b) Find d^2y/dx^2. What does this tell you about the concavity of the graph?
(c) Eliminate the parameter and write y in terms of x.
(d) Using your answer from part (c), find dy/dx and d^2y/dx^2 in terms of x. Show that these answers are the same as the answers to parts (a) and (b).

53. A particle moves in the xy-plane so that its position at time t is given by $x = \sin t$ and $y = \cos(2t)$ for $0 \le t < 2\pi$.

(a) At what time does the particle first touch the x-axis? What is the speed of the particle at that time?
(b) Is the particle ever at rest?
(c) Discuss the concavity of the graph.

54. Derive the general formula for the second derivative d^2y/dx^2 of a parametrically defined curve:

$$\frac{d^2y}{dx^2} = \frac{(dx/dt)(d^2y/dt^2) - (dy/dt)(d^2x/dt^2)}{(dx/dt)^3}.$$

Graph the Lissajous figures in Problems 55–58 using a calculator or computer.

55. $x = \cos 2t, \quad y = \sin 5t$

56. $x = \cos 3t, \quad y = \sin 7t$

57. $x = \cos 2t, \quad y = \sin 4t$

58. $x = \cos 2t, \quad y = \sin \sqrt{3}t$

59. A hypothetical moon orbits a planet which in turn orbits a star. Suppose that the orbits are circular and that the moon orbits the planet 12 times in the time it takes for the planet to orbit the star once. In this problem we will investigate whether the moon could come to a stop at some instant. (See Figure 4.106.)

(a) Suppose the radius of the moon's orbit around the planet is 1 unit and the radius of the planet's orbit around the star is R units. Explain why the motion of the moon relative to the star can be described by the parametric equations

$$x = R\cos t + \cos(12t), \quad y = R\sin t + \sin(12t).$$

(b) Find values for R and t such that the moon stops relative to the star at time t.
(c) On a graphing calculator, plot the path of the moon for the value of R you obtained in part (b). Experiment with other values for R.

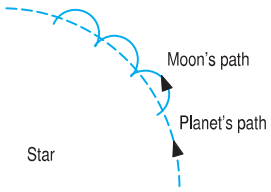


Figure 4.106

CHAPTER SUMMARY (see also Ready Reference at the end of the book)

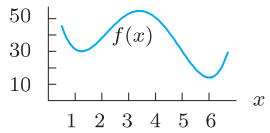
- **Local extrema**
  Maximum, minimum, critical point, tests for local maxima/minima.
- **Using the second derivative**
  Concavity, inflection point.
- **Families of curves**
  Role of parameters
- **Optimization**
  Global extremum, modeling problems, graphical optimization, upper and lower bounds, extreme value theorem.
- **Marginality**
  Cost/revenue functions, marginal cost/marginal revenue functions.
- **L'Hopital's Rule**
  Limits, growth, dominance
- **Rates and Related Rates**
- **Parametric equations**
  Motion of a particle in the plane, parametric equations for lines and other curves, velocity, tangent lines, slope, concavity.

REVIEW EXERCISES AND PROBLEMS FOR CHAPTER FOUR

Exercises

For Exercises 1–2, indicate all critical points on the given graphs. Which correspond to local minima, local maxima, global maxima, global minima, or none of these? (Note that the graphs are on closed intervals.)

1.



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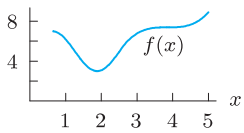


In Exercises 3–6, do the following:

(a) Find f' and f''.
(b) Find the critical points of f.
(c) Find any inflection points.
(d) Evaluate f at the critical points and the endpoints. Identify the global maxima and minima of f.
(e) Sketch f. Indicate clearly where f is increasing or decreasing, and its concavity.

3. $f(x) = x^3 - 3x^2 \quad (-1 \le x \le 3)$

4. $f(x) = x + \sin x \quad (0 \le x \le 2\pi)$

5. $f(x) = e^{-x} \sin x \quad (0 \le x \le 2\pi)$

6. $f(x) = x^{-2/3} + x^{1/3} \quad (1.2 \le x \le 3.5)$

In Exercises 7–9, find the limits as x tends to $+\infty$ and $-\infty$, and then proceed as in Exercises 3–6. (That is, find f', etc.).

7. $f(x) = 2x^3 - 9x^2 + 12x + 1$

8. $f(x) = \dfrac{4x^2}{x^2+1}$

9. $f(x) = xe^{-x}$

In Exercises 10–11, find the global maximum and minimum for the function on the closed interval.

10. $f(x) = x - \sin(2x), \quad 0 \le x \le \pi/2$

11. $f(x) = x^2 - 2|x|, \quad -3 \le x \le 4$

In Exercises 12–14, find the exact global maximum and minimum values of the function.

12. $h(z) = \dfrac{1}{z} + 4z^2$ for $z > 0$

13. $g(t) = \dfrac{1}{t^3+1}$ for $t \ge 0$

14. $f(x) = \dfrac{1}{(x-1)^2+2}$

In Exercises 15–20, use derivatives to identify local maxima and minima and points of inflection. Graph the function. Confirm your answers using a calculator or computer.

15. $f(x) = x^3 + 3x^2 - 9x - 15$

16. $f(x) = x^5 - 15x^3 + 10$

17. $f(x) = x - 2\ln x \quad$ for $x > 0$

18. $f(x) = x^2 e^{5x}$

19. $f(x) = e^{-x^2}$

20. $f(x) = \dfrac{x^2}{x^2+1}$

In Exercises 21–22, describe the motion of a particle moving according to the parametric equations. Find the equation of the curve along which the particle moves.

21. $x = 4 - 2t$, $y = 4t + 1$

22. $x = 2\sin t$, $y = 2\cos t$

23. **(a)** Find all critical points and all inflection points of the function $f(x) = x^4 - 2ax^2 + b$. Assume a and b are positive constants.
(b) Find values of the parameters a and b if f has a critical point at the point $(2, 5)$.
(c) If there is a critical point at $(2, 5)$, where are the inflection points?

24. **(a)** For a a positive constant, find all critical points of $f(x) = x - a\sqrt{x}$.
(b) What value of a gives a critical point at $x = 5$? Does $f(x)$ have a local maximum or a local minimum at this critical point?

25. The mass of a cube in grams is $M = x^3 + 0.1x^4$, where x is the length of one side in centimeters. If the length is increasing at a rate of 0.02 cm/hr, at what rate is the mass of the cube increasing when its length is 5 cm?

26. A particle lies on a line perpendicular to a thin circular ring and through its center. The radius of the ring is a, and the distance from the point to the center of the ring is y. For a positive constant K, the gravitational force exerted by the ring on the particle is given by

$$F = \frac{Ky}{(a^2+y^2)^{3/2}},$$

If y is increasing, how is F changing with respect to time, t, when

(a) $y = 0$ **(b)** $y = a/\sqrt{2}$ **(c)** $y = 2a$

27. If θ is the angle between a line through the origin and the positive x-axis, the area, in cm ^{2}, of part of a cardioid is

$$A = 3\theta + 4\sin\theta + \frac{1}{2}\sin(2\theta).$$

If the angle θ is increasing at a rate of 0.3 radians per minute, at what rate is the area changing when $\theta = \pi/2$?

Problems

28. For the function, f, graphed in Figure 4.107:

(a) Sketch $f'(x)$.
(b) Where does $f'(x)$ change its sign?
(c) Where does $f'(x)$ have local maxima or minima?

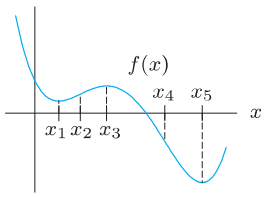


Figure 4.107

29. Using your answer to Problem 28 as a guide, write a short paragraph (using complete sentences) which describes the relationships between the following features of a function f:

- The local maxima and minima of f.
- The points at which the graph of f changes concavity.
- The sign changes of f'.
- The local maxima and minima of f'.

30. Figure 4.108 is a graph of f'. For what values of x does f have a local maximum? A local minimum?

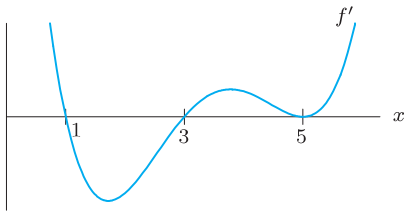


Figure 4.108: Graph of f' (not f)

31. On the graph of f' in Figure 4.109, indicate the x-values that are critical points of the function f itself. Are they local maxima, local minima, or neither?

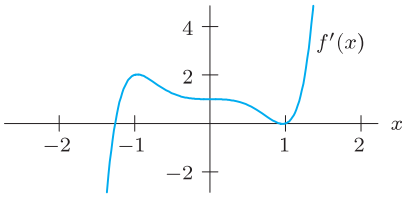


Figure 4.109: Graph of f' (not f)

32. Figure 4.110 is the graph of f', the derivative of a function f. At which of the points $0, x_1, x_2, x_3, x_4, x_5$, is the function f:

(a) At a local maximum value?
(b) At a local minimum value?
(c) Climbing fastest?
(d) Falling most steeply?

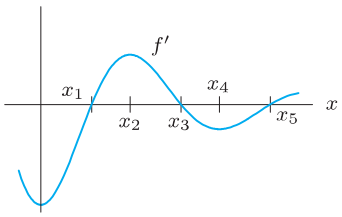


Figure 4.110: Graph of f' not f

Find formulas for the functions described in Problems 33–37.

33. A cubic polynomial with a local maximum at $x = 1$, a local minimum at $x = 3$, a y-intercept of 5, and an x^3 term whose coefficient is 1.

34. A quartic polynomial whose graph is symmetric about the y-axis and has local maxima at $(-1, 4)$ and $(1, 4)$ and a y-intercept of 3.

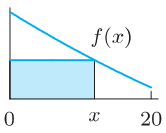
35. A function of the form $y = ax^b \ln x$, where a and b are nonzero constants, which has a local maximum at the point $(e^2, 6e^{-1})$.

36. A function of the form $y = A\sin(Bx) + C$ with a maximum at $(5, 2)$, a minimum at $(15, 1.5)$, and no critical points between these two points.

37. A function of the form $y = axe^{-bx^2}$ with a global maximum at $(1, 2)$ and a global minimum at $(-1, -2)$.

In Problems 38–39, find the x-value maximizing the shaded area. One vertex is on the graph of $f(x) = x^2/3 - 50x + 1000$.

38.



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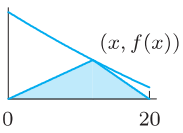

For the solids in Problems 40–43, find the dimensions giving the maximum volume, given that the surface area is 8 cm^2.

40. A closed rectangular box, with a square base x by x cm and height h cm.

41. A open-topped rectangular box, with a square base x by x cm and height h cm.

42. A closed cylinder with radius r cm and height h cm.

43. A cylinder open at one end with radius r cm and height h cm.

Find the best possible bounds for the functions in Problems 44–46.

44. $e^{-x}\sin x, \quad \text{for } x \geq 0$

45. $x\sin x, \quad \text{for } 0 \leq x \leq 2\pi$

46. $x^3 - 6x^2 + 9x + 5 \quad \text{for } 0 \leq x \leq 5$

47. Find the value(s) of m, if any, that give the global maximum and minimum of P as a function of m where

$$6jm^2 + P = 4jk - 5km, \quad \text{for positive constants } j \text{ and } k.$$

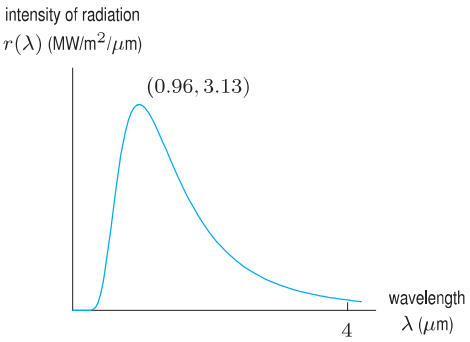

Figure 4.111

48. Find values of a and b so that the function $y = axe^{-bx}$ has a local maximum at the point $(2, 10)$.

49. **(a)** Find all critical points of $f(t) = at^2e^{-bt}$, assuming a and b are nonzero constants.
(b) Find values of the parameters a and b so that f has a critical point at the point $(5, 12)$.
(c) Identify each critical point as a local maximum or local minimum of the function f in part (b).

50. What effect does increasing the value of a have on the graph of $f(x) = x^2 + 2ax$? Consider roots, maxima and minima, and both positive and negative values of a.

51. Sketch several members of the family $y = x^3 - ax^2$ on the same axes. Show that the critical points lie on the curve $y = -\frac{1}{2}x^3$.

52. A drug is injected into a patient at a rate given by $r(t) = ate^{-bt}$ ml/sec, where t is in seconds since the injection started, $0 \leq t \leq 5$, and a and b are constants. The maximum rate of 0.3 ml/sec occurs half a second after the injection starts. Find a formula for a and b.

53. Ice is being formed in the shape of a circular cylinder with inner radius 1 cm and height 3 cm. The outer radius of the ice is increasing at 0.03 cm per hour when the outer radius is 1.5 cm. How fast is the volume of the ice increasing at this time?

54. Any body radiates energy at various wavelengths. Figure 4.111 shows the intensity of the radiation of a black body at a temperature $T = 3000°$ kelvin as a function of the wavelength. The intensity of the radiation is highest in the infrared range, that is, at wavelengths longer than that of visible light (0.4–0.7μm). Max Planck's radiation law, announced to the Berlin Physical Society on October 19, 1900, states that

$$r(\lambda) = \frac{a}{\lambda^5(e^{b/\lambda} - 1)}.$$

Find constants a and b so that the formula fits the graph. (Later in 1900 Planck showed from theory that $a = 2\pi c^2 h$ and $b = \frac{hc}{Tk}$ where c = speed of light, h = Planck's constant, and k = Boltzmann's constant.)

55. A right triangle has one vertex at the origin and one vertex on the curve $y = e^{-x/3}$ for $1 \leq x \leq 5$. One of the two perpendicular sides is along the x-axis; the other is parallel to the y-axis. Find the maximum and minimum areas for such a triangle.

56. A rectangle has one side on the x-axis and two corners on the top half of the circle of radius 1 centered at the origin. Find the maximum area of such a rectangle. What are the coordinates of its vertices?

57. A business sells an item at a constant rate of r units per month. It reorders in batches of q units, at a cost of $a + bq$ dollars per order. Storage costs are k dollars per item per month, and, on average, $q/2$ items are in storage, waiting to be sold. [Assume r, a, b, k are positive constants.]

(a) How often does the business reorder?
(b) What is the average monthly cost of reordering?
(c) What is the total monthly cost, C of ordering and storage?
(d) Obtain Wilson's lot size formula, the optimal batch size which minimizes cost.

58. You run a small furniture business. You sign a deal with a customer to deliver up to 400 chairs, the exact number to be determined by the customer later. The price will be \$90 per chair up to 300 chairs, and above 300, the price will be reduced by \$0.25 per chair (on the whole order) for every additional chair over 300 ordered. What are the largest and smallest revenues your company can make under this deal?

59. A landscape architect plans to enclose a 3000 square foot rectangular region in a botanical garden. She will use shrubs costing \$45 per foot along three sides and fencing costing \$20 per foot along the fourth side. Find the minimum total cost.

60. A ship is steaming due north at 12 knots (1 knot = 1.85 kilometers/hour) and sights a large tanker 3 kilometers away northwest steaming at 15 knots due east. For reasons of safety, the ships want to maintain a distance of at least 100 meters between them. Use a calculator or computer to determine the shortest distance between them if they remain on their current headings, and hence decide if they need to change course.

76. A spherical balloon is inflated so that its radius is increasing at a constant rate of 1 cm per second. At what rate is air being blown into the balloon when its radius is 5 cm?

77. When the growth of a spherical cell depends on the flow of nutrients through the surface, it is reasonable to assume that the growth rate, dV/dt, is proportional to the surface area, S. Assume that for a particular cell $dV/dt = \frac{1}{3} \cdot S$. At what rate is its radius r increasing?

78. A horizontal disk of radius a centered at the origin in the xy-plane is rotating about a vertical axis through the center. The angle between the positive x-axis and a radial line painted on the disk is θ radians.

(a) What does $d\theta/dt$ represent?
(b) What is the relationship between $d\theta/dt$ and the speed v of a point on the rim?

79. A voltage, V volts, applied to a resistor of R ohms produces an electric current of I amps where $V = IR$. As the current flows the resistor heats up and its resistance falls. If 100 volts is applied to a resistor of 1000 ohms the current is initially 0.1 amps but rises by 0.001 amps/minute. At what rate is the resistance falling if the voltage remains constant?

80. A radio navigation system used by aircraft gives a cockpit readout of the distance, s, in miles, between a fixed ground station and the aircraft. The system also gives a readout of the instantaneous rate of change, ds/dt, of this distance in miles/hour. An aircraft on a straight flight path at a constant altitude of 10,560 feet (2 miles) has passed directly over the ground station and is now flying away from it. What is the speed of this aircraft along its constant altitude flight path when the cockpit readouts are $s = 4.6$ miles and $ds/dt = 210$ miles/hour?

Problems 81–82 involve Boyle's Law which states that for a fixed quantity of gas at a constant temperature, the pressure, P, and the volume, V, are inversely related. Thus, for some constant k

$$PV = k.$$

81. A fixed quantity of gas is allowed to expand at constant temperature. Find the rate of change of pressure with respect to volume.

82. A certain quantity of gas occupies 10 cm^3 at a pressure of 2 atmospheres. The pressure is increased, while keeping the temperature constant.

(a) Does the volume increase or decrease?
(b) If the pressure is increasing at a rate of 0.05 atmospheres/minute when the pressure is 2 atmospheres, find the rate at which the volume is changing at that moment. What are the units of your answer?

CAS Challenge Problems

83. A population, P, in a restricted environment may grow with time, t, according to the *logistic function*

$$P = \frac{L}{1 + Ce^{-kt}}$$

where L is called the carrying capacity and L, C and k are positive constants.

(a) Find $\lim_{t \to \infty} P$. Explain why L is called the carrying capacity.
(b) Using a computer algebra system, show that the graph of P has an inflection point at $P = L/2$.

84. For positive a, consider the family of functions

$$y = \arctan\left(\frac{\sqrt{x} + \sqrt{a}}{1 - \sqrt{ax}}\right), \qquad x > 0.$$

(a) Graph several curves in the family and describe how the graph changes as a varies.
(b) Use a computer algebra system to find dy/dx, and graph the derivative for several values of a. What do you notice?
(c) Do your observations in part (b) agree with the answer to part (a)? Explain. [Hint: Use the fact that $\sqrt{ax} = \sqrt{a}\sqrt{x}$ for $a > 0$, $x > 0$.]

85. The function $\operatorname{arcsinh} x$ is the inverse function of $\sinh x$.

(a) Use a computer algebra system to find a formula for the derivative of $\operatorname{arcsinh} x$.
(b) Derive the formula by hand by differentiating both sides of the equation

$$\sinh(\operatorname{arcsinh} x) = x.$$

[Hint: Use the identity $\cosh^2 x - \sinh^2 x = 1$.]

86. The function $\operatorname{arccosh} x$, for $x \geq 0$, is the inverse function of $\cosh x$, for $x \geq 0$.

(a) Use a computer algebra system to find the derivative of $\operatorname{arccosh} x$.
(b) Derive the formula by hand by differentiating both sides of the equation

$$\cosh(\operatorname{arccosh} x) = x, \quad x \geq 1.$$

[Hint: Use the identity $\cosh^2 x - \sinh^2 x = 1$.]

87. Consider the family of functions

$$f(x) = \frac{\sqrt{a + x}}{\sqrt{a} + \sqrt{x}}, \quad x \geq 0, \quad \text{for positive } a.$$

(a) Using a computer algebra system, find the local maxima and minima of f.

(b) On one set of axes, graph this function for several values of a. How does varying a affect the shape of the graph? Explain your answer in terms of the answer to part (a).

(c) Use your computer algebra system to find the inflection points of f when $a = 2$.

88. (a) Use a computer algebra system to find the derivative of

$$y = \arctan\left(\sqrt{\frac{1-\cos x}{1+\cos x}}\right).$$

(b) Graph the derivative. Does the graph agree with the answer you got in part (a)? Explain using the identity $\cos(x) = \cos^2(x/2) - \sin^2(x/2)$.

89. In 1696, the first calculus textbook was published by the Marquis de l'Hopital. The following problem is a simplified version of a problem from this text.

In Figure 4.113, two ropes are attached to the ceiling at points $\sqrt{3}$ meters apart. The rope on the left is 1 meter long and has a pulley attached at its end. The rope on the right is 3 meters long; it passes through the pulley and has a weight tied to its end. When the weight is released, the ropes and pulley arrange themselves so that the distance from the weight to the ceiling is maximized.

(a) Show that the maximum distance occurs when the weight is exactly halfway between the the points where the ropes are attached to the ceiling. [Hint: Write the vertical distance from the weight to the ceiling in terms of its horizontal distance to the point at which the left rope is tied to the ceiling. A computer algebra system will be useful.]

(b) Does the weight always end up halfway between the ceiling anchor points no matter how long the left-hand rope is? Explain.

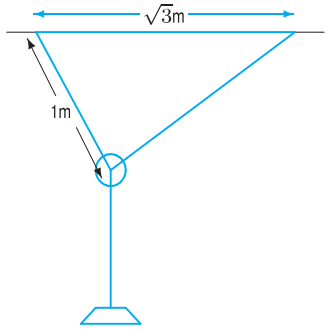

Figure 4.113

CHECK YOUR UNDERSTANDING

Are the statements in Problems 1–8 true or false for a function f whose domain is all real numbers? If a statement is true, explain how you know. If a statement is false, give a counterexample.

1. A local minimum of f occurs at a critical point of f.

2. If $x = p$ is not a critical point of f, then $x = p$ is not a local maximum of f.

3. A local maximum of f occurs at a point where

$$f'(x) = 0.$$

4. If $x = p$ is not a local maximum of f, then $x = p$ is not a critical point of f.

5. If $f'(p) = 0$, then $f(x)$ has a local minimum or local maximum at $x = p$.

6. If $f'(x)$ is continuous and $f(x)$ has no critical points, then f is everywhere increasing or everywhere decreasing.

7. If $f''(p) = 0$, then the graph of f has an inflection point at $x = p$.

8. If $f''(x)$ is continuous and the graph of f has an inflection point at $x = p$, then $f''(p) = 0$.

In Problems 9–10, give an example of function(s) with the given properties.

9. A family of functions, $f(x)$, depending on a parameter a, such that each member of the family has exactly one critical point.

10. A family of functions, $g(x)$, depending on two parameters, a and b, such that each member of the family has exactly two critical points and one inflection point. You may want to restrict a and b.

11. Let $f(x) = x^2$. Decide if the following statements are true or false. Explain your answer.

(a) f has an upper bound on the interval $(0, 2)$.

(b) f has a global maximum on the interval $(0, 2)$.

(c) f does not have a global minimum on the interval $(0, 2)$.

(d) f does not have a global minimum on any interval (a, b).

(e) f has a global minimum on any interval $[a, b]$.

12. Which of the following statements is implied by the statement "If f is continuous on $[a, b]$ then f has a global maximum on $[a, b]$?"

(a) If f has a global maximum on $[a, b]$ then f must be continuous on $[a, b]$.

(b) If f is not continuous on $[a, b]$ then f does not have a global maximum on $[a, b]$.

(c) If f does not have a global maximum on $[a, b]$ then f is not continuous on $[a, b]$.

Are the statements in Problems 13–24 true of false? Give an explanation for your answer.

13. A critical point of f must be a local maximum or minimum of f.

14. Since the function $f(x) = 1/x$ is continuous for $x > 0$ and the interval $(0, 1)$ is bounded, f has a maximum on the interval $(0, 1)$.

15. Every cubic polynomial has an inflection point.

16. The Extreme Value Theorem says that only continuous functions have global maxima and minima on every closed, bounded interval.

17. The global maximum of $f(x) = x^2$ on every closed interval is at one of the endpoints of the interval.

18. A function can have two different upper bounds.

19. If a differentiable function $f(x)$ has a global maximum on the interval $0 \le x \le 10$ at $x = 0$, then $f'(0) \le 0$.

20. If the radius of a circle is increasing at a constant rate, then so is the circumference.

21. If the radius of a circle is increasing at a constant rate, then so is the area.

22. The curve given parametrically by $x = f(t)$ and $y = g(t)$ has no sharp corners if f and g are differentiable.

23. If a curve is given parametrically by $x = \cos(t^2)$, $y = \sin(t^2)$, then its slope is $\tan(t^2)$.

24. If $g'(a) \neq 0$, then $\lim_{x \to a} \dfrac{f(x)}{g(x)} = \dfrac{f'(a)}{g'(a)}$.

In Problems 25–30, give an example of a function f that makes the statement true, or say why such an example is impossible. Assume that f'' exists everywhere.

25. f is concave up and $f(x)$ is positive for all x.

26. f is concave down and $f(x)$ is positive for all x.

27. f is concave down and $f(x)$ is negative for all x.

28. f is concave up and $f(x)$ is negative for all x.

29. $f(x)f''(x) < 0$ for all x.

30. $f(x)f'(x)f''(x)f'''(x) < 0$ for all x.[12]

PROJECTS FOR CHAPTER FOUR

1. Building a Greenhouse

Your parents are going to knock out the bottom of the entire length of the south wall of their house and turn it into a greenhouse by replacing the bottom portion of the wall with a huge sloped piece of glass (which is expensive). They have already decided they are going to spend a certain fixed amount. The triangular ends of the greenhouse will be made of various materials they already have lying around.[13]

The floor space in the greenhouse is only considered usable if they can both stand up in it, so part of it will be unusable. They want to choose the dimensions of the greenhouse to get the most usable floor space. What should the dimensions of the greenhouse be and how much usable space will your parents get?

2. Fitting a Line to Data

(a) The line which best fits the data points $(x_1, y_1), (x_2, y_2) \ldots (x_n, y_n)$ is the one which minimizes the sum of the squares of the vertical distances from the points to the line. These are the distances marked in Figure 4.114. Find the best fitting line of the form $y = mx$ for the points $(2, 3.5), (3, 6.8), (5, 9.1)$.

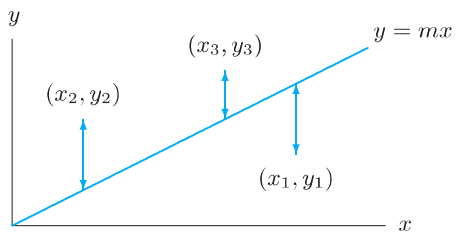

Figure 4.114

[12] From the 1998 William Lowell Putnam Mathematical Competition, by permission of the Mathematical Association of America.

[13] Adapted from M. Cohen, E. Gaughan, A. Knoebel, D. Kurtz, D. Pengelley, *Student Research Projects in Calculus* (Washington DC: Mathematical Association of America, 1992).

(b) A cone with height and radius both equal to r has volume, V, proportional to r^3; that is, $V = kr^3$ for some constant k. A lab experiment is done to measure the volume of several cones; the results are in the following table. Using the method of part (a), determine the best value of k. [Note: Since the volumes were determined experimentally, the values may not be accurate. Assume that the radii were measured accurately.]

Radius (cm)	2	5	7	8
Volume (cm^3)	8.7	140.3	355.8	539.2

(c) Using the method of part (a), show that the best fitting line of the form $y = mx$ for the points $(x_1, y_1), (x_2, y_2) \ldots (x_n, y_n)$ has

$$m = \frac{x_1y_1 + x_2y_2 + \cdots + x_ny_n}{x_1^2 + x_2^2 + \cdots + x_n^2}.$$

3. Firebreaks

The summer of 2000 was devastating for forests in the western US: over 3.5 million acres of trees were lost to fires, making this the worst fire season in 30 years. This project studies a fire management technique called *firebreaks*, which reduce the damage done by forest fires. A firebreak is a strip where trees have been removed in a forest so that a fire started on one side of the strip will not spread to the other side. Having many firebreaks helps confine a fire to a small area. On the other hand, having too many firebreaks involves removing large swaths of trees.[14]

(a) A forest in the shape of a 50 km by 50 km square has firebreaks in rectangular strips 50 km by 0.01 km. The trees between two firebreaks are called a stand of trees. All firebreaks in this forest are parallel to each other and to one edge of the forest, with the first firebreak at the edge of the forest. The firebreaks are evenly spaced throughout the forest. (For example, Figure 4.115 shows four firebreaks.) The total area lost in the case of a fire is the area of the stand of trees in which the fire started plus the area of all the firebreaks.

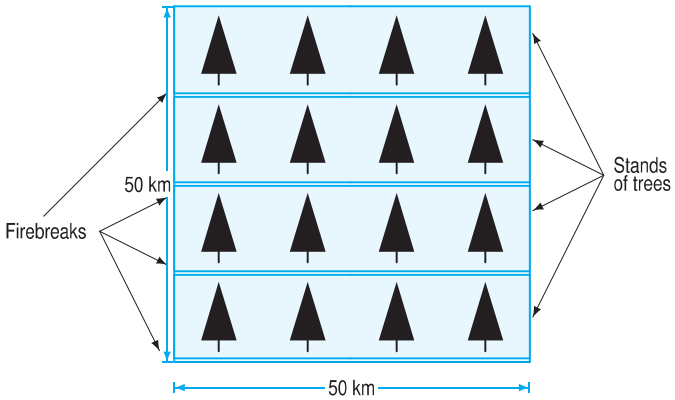

Figure 4.115

(i) Find the number of firebreaks that minimizes the total area lost to the forest in the case of a fire.

(ii) If a firebreak is 50 km by b km, find the optimal number of firebreaks as a function of b. If the width, b, of a firebreak is quadrupled, how does the optimal number of firebreaks change?

[14] Adapted from D. Quinney and R. Harding, *Calculus Connections* (New York: John Wiley & Sons, 1996).

(b) Now suppose firebreaks are arranged in two equally spaced sets of parallel lines, as shown in Figure 4.116. The forest is a 50 km by 50 km square, and each firebreak is a rectangular strip 50 km by 0.01 km. Find the number of firebreaks in each direction that minimizes the total area lost to the forest in the case of a fire.

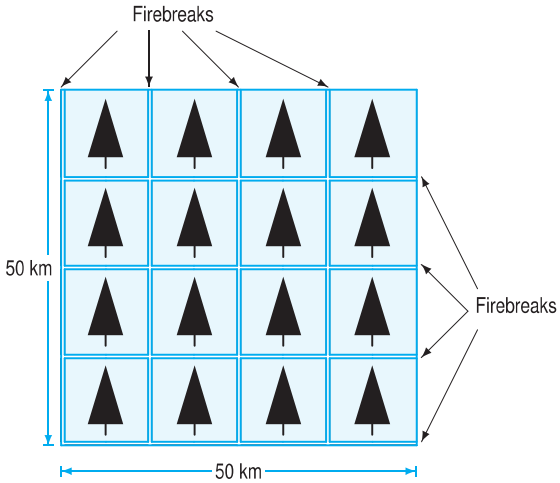

Figure 4.116