

# ON COINCIDENCE OF CLASSES OF FUNCTIONS DEFINED BY A GENERALISED MODULUS OF SMOOTHNESS AND THE APPROPRIATE INVERSE THEOREM

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**ABSTRACT.** We give the theorem of coincidence of a class of functions defined by a generalised modulus of smoothness with a class of functions defined by the order of the best approximation by algebraic polynomials. We also prove the appropriate inverse theorem in approximation theory.

**0.** In [4], an asymmetric operator of generalised translation was introduced, by means of it the generalised modulus of continuity was defined, and the theorem of coincidence of a class of functions defined by that modulus with a class of functions with given order of the best approximation by algebraic polynomials was proved.

In our paper the analogous results are obtained for a generalised modulus of smoothness of order  $r$ . In addition, in the present paper we prove a theorem inverse to the Jackson's theorem related to that modulus of smoothness.

**1.** By  $L_p$  we denote the set of functions  $f$  measurable on the segment  $[-1, 1]$  such that for  $1 \leq p < \infty$

$$\|f\|_p = \left( \int_{-1}^1 |f(x)|^p dx \right)^{1/p} < \infty,$$

and for  $p = \infty$

$$\|f\|_\infty = \operatorname{ess\,sup}_{-1 \leq x \leq 1} |f(x)| < \infty.$$

Denote by  $L_{p,\alpha}$  the set of functions  $f$  such that  $f(x)(1-x^2)^\alpha \in L_p$ , and put

$$\|f\|_{p,\alpha} = \|f(x)(1-x^2)^\alpha\|_p.$$

By  $E_n(f)_{p,\alpha}$  we denote the best approximation of the function  $f \in L_{p,\alpha}$  by algebraic polynomials of degree not greater than  $n-1$ , in  $L_{p,\alpha}$  metrics, i.e.

$$E_n(f)_{p,\alpha} = \inf_{P_n \in \mathbb{P}_n} \|f - P_n\|_{p,\alpha},$$

where  $\mathbb{P}_n$  is the set of algebraic polynomials of degree not greater than  $n-1$ .

By  $E(p, \alpha, \lambda)$  we denote the class of functions  $f \in L_{p,\alpha}$  satisfying the condition

$$E_n(f)_{p,\alpha} \leq Cn^{-\lambda},$$

where  $\lambda > 0$  and  $C$  is a constant not depending on  $n$ .

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For a function  $f$  we define the operator of generalised translation  $\hat{T}_t(f, x)$  by

$$\begin{aligned} \hat{T}_t(f, x) = \frac{1}{\pi(1-x^2)} \int_0^\pi & \left( 1 - \left( x \cos t - \sqrt{1-x^2} \sin t \cos \varphi \right)^2 \right. \\ & \left. - 2 \sin^2 t \sin^2 \varphi + 4(1-x^2) \sin^2 t \sin^4 \varphi \right) \\ & \times f(x \cos t - \sqrt{1-x^2} \sin t \cos \varphi) d\varphi. \end{aligned}$$

By means of that operator of generalised translation we define the generalised difference of order  $r$  by

$$\begin{aligned} \Delta_t^1(f, x) &= \Delta_t(f, x) = \hat{T}_t(f, x) - f(x), \\ \Delta_{t_1, \dots, t_r}^r(f, x) &= \Delta_{t_r} \left( \Delta_{t_1, \dots, t_{r-1}}^{r-1}(f, x), x \right) \quad (r = 2, 3, \dots), \end{aligned}$$

and the generalised modulus of smoothness of order  $r$  by

$$\hat{\omega}_r(f, \delta)_{p, \alpha} = \sup_{\substack{|t_j| \leq \delta \\ j=1, 2, \dots, r}} \left\| \Delta_{t_1, \dots, t_r}^r(f, x) \right\|_{p, \alpha} \quad (r = 1, 2, \dots).$$

Denote by  $H(p, \alpha, r, \lambda)$  the class of functions  $f \in L_{p, \alpha}$  satisfying the condition

$$\hat{\omega}_r(f, \delta)_{p, \alpha} \leq C \delta^\lambda,$$

where  $\lambda > 0$  and  $C$  is a constant not depending on  $\delta$ .

**2.** Put  $y = \cos t$ ,  $z = \cos \varphi$  in the operator  $\hat{T}_t(f, x)$ , we denote it by  $T_y(f, x)$  and rewrite it in the form

$$\begin{aligned} T_y(f, x) = \frac{1}{\pi(1-x^2)} \int_{-1}^1 & (1 - R^2 - 2(1-y^2)(1-z^2) \\ & + 4(1-x^2)(1-y^2)(1-z^2)^2) f(R) \frac{dz}{\sqrt{1-z^2}}, \end{aligned}$$

where  $R = xy - z\sqrt{1-x^2}\sqrt{1-y^2}$ .

We define the operator of generalised translation of order  $r$  by

$$\begin{aligned} T_y^1(f, x) &= T_y(f, x), \\ T_{y_1, \dots, y_r}^r(f, x) &= T_{y_r} \left( T_{y_1, \dots, y_{r-1}}^{r-1}(f, x), x \right) \quad (r = 2, 3, \dots). \end{aligned}$$

By  $P_\nu^{(\alpha, \beta)}(x)$  ( $\nu = 0, 1, \dots$ ) we denote the Jacobi's polynomials, i.e. algebraic polynomials of degree  $\nu$  orthogonal with the weight function  $(1-x)^\alpha(1+x)^\beta$  on the segment  $[-1, 1]$  and normed by the condition  $P_\nu^{(\alpha, \beta)}(1) = 1$  ( $\nu = 0, 1, \dots$ ).

Denote by  $a_n(f)$  the Fourier–Jacobi coefficients of a function  $f$ , integrable with the weight function  $(1-x^2)^2$  on the segment  $[-1, 1]$ , with respect to the system of Jacobi polynomials  $\left\{ P_n^{(2, 2)}(x) \right\}_{n=0}^\infty$ ; i.e.,

$$a_n(f) = \int_{-1}^1 f(x) P_n^{(2, 2)}(x) (1-x^2)^2 dx \quad (n = 0, 1, \dots).$$

We define the following operators, having an auxiliary role later on

$$\begin{aligned} T_{1; y}(f, x) &= \frac{1}{\pi(1-x^2)} \int_{-1}^1 (1 - R^2 - 2(1-y^2)(1-z^2)) f(R) \frac{dz}{\sqrt{1-z^2}}, \\ T_{2; y}(f, x) &= \frac{8}{3\pi} \int_{-1}^1 (1-z^2)^2 f(R) \frac{dz}{\sqrt{1-z^2}}, \end{aligned}$$

where  $R = xy - z\sqrt{1-x^2}\sqrt{1-y^2}$ , and the corresponding operators of order  $r$

$$\begin{aligned} T_{k;y}^1(f, x) &= T_{k;y}(f, x), \\ T_{k;y_1, \dots, y_r}^r(f, x) &= T_{k;y_r} \left( T_{k;y_1, \dots, y_{r-1}}^{r-1}(f, x), x \right) \quad (r = 2, 3, \dots) \end{aligned}$$

for  $k = 1, 2$ .

### 3.

**Lemma 3.1.** *Let  $P_n(x)$  be an algebraic polynomial of degree not greater than  $n-1$ ,  $1 \leq p \leq \infty$ ,  $\alpha > -\frac{1}{p}$  and  $\rho \geq 0$ . Then the following inequalities hold true*

$$\begin{aligned} \|P_n'(x)\|_{p, \alpha + \frac{1}{2}} &\leq C_1 n \|P_n\|_{p, \alpha}, \\ \|P_n\|_{p, \alpha} &\leq C_2 n^{2\rho} \|P_n\|_{p, \alpha + \rho}, \end{aligned}$$

where the constants  $C_1$  and  $C_2$  do not depend on  $n$ .

Lemma is proved in [2].

**Lemma 3.2.** *The operators  $T_{1;y}$  and  $T_{2;y}$  have the following properties*

$$\begin{aligned} T_{1;y} \left( P_\nu^{(2,2)}, x \right) &= P_\nu^{(2,2)}(x) P_{\nu+2}^{(0,0)}(y), \\ T_{2;y} \left( P_\nu^{(2,2)}, x \right) &= P_\nu^{(2,2)}(x) P_\nu^{(2,2)}(y) \end{aligned}$$

for  $\nu = 0, 1, \dots$ .

Lemma 3.2 is proved in [4].

**Lemma 3.3.** *Let  $g(x)T_{k;y}(f, x) \in L_{1,2}$  for every  $y$ . Then for  $k = 1, 2$  the following equality holds true*

$$\int_{-1}^1 f(x) T_{k;y}(g, x) (1-x^2)^2 dx = \int_{-1}^1 g(x) T_{k;y}(f, x) (1-x^2)^2 dx.$$

*Proof.* Let  $k = 1$  and

$$\begin{aligned} I_1 &= \int_{-1}^1 f(x) T_{1;y}(g, x) (1-x^2)^2 dx \\ &= \frac{1}{\pi} \int_{-1}^1 \int_{-1}^1 f(x) g(R) (1-R^2 - 2(1-y^2)(1-z^2)) (1-x^2) \frac{dz dx}{\sqrt{1-z^2}}, \end{aligned}$$

where  $R = xy - z\sqrt{1-x^2}\sqrt{1-y^2}$ . Performing change of variables in the double integral by the formulas

$$\begin{aligned} x &= Ry + V\sqrt{1-R^2}\sqrt{1-y^2}, \\ (3.1) \quad z &= -\frac{R\sqrt{1-y^2} - Vy\sqrt{1-R^2}}{\sqrt{1 - \left(Ry + V\sqrt{1-R^2}\sqrt{1-y^2}\right)^2}}, \end{aligned}$$

we get

$$\begin{aligned} I_1 &= \frac{1}{\pi} \int_{-1}^1 \int_{-1}^1 (1-R^2) f \left( Ry + V\sqrt{1-R^2}\sqrt{1-y^2} \right) g(R) \\ &\quad \times \left( 1 - \left( Ry + V\sqrt{1-R^2}\sqrt{1-y^2} \right)^2 - 2(1-y^2)(1-V^2) \right) \frac{dV dR}{\sqrt{1-V^2}} \\ &= \int_{-1}^1 g(R) T_{1;y}(f, R) (1-R^2)^2 dR, \end{aligned}$$

which proves the equality of the lemma for  $k = 1$ .

Let  $k = 2$  and

$$\begin{aligned} I_2 &= \int_{-1}^1 f(x) T_{2;y}(g, x) (1 - x^2)^2 dx \\ &= \frac{8}{3\pi} \int_{-1}^1 \int_{-1}^1 f(x) g(R) (1 - x^2)^2 (1 - z^2)^2 \frac{dz dx}{\sqrt{1 - z^2}}. \end{aligned}$$

Performing change of variables in that double integral by the formulas (3.1) we get

$$\begin{aligned} I_2 &= \frac{8}{3\pi} \int_{-1}^1 \int_{-1}^1 f(Ry + V\sqrt{1 - R^2}\sqrt{1 - y^2}) g(R) (1 - R^2)^2 \\ &\quad \times (1 - V^2)^2 \frac{dV dR}{\sqrt{1 - V^2}} = \int_{-1}^1 g(R) T_{2;y}(f, R) (1 - R^2)^2 dR. \end{aligned}$$

Lemma 3.3 is proved.  $\square$

**Corollary 3.1.** *If  $f \in L_{1,2}$ , then for every natural number  $r$  we have  $T_{k;y}^r(f, x) \in L_{1,2}$  ( $k = 1, 2$ ).*

*Proof.* Put  $g(x) \equiv 1$  on  $[-1, 1]$ , considering that by Lemma 3.2 (see [1, vol. II, p. 180])

$$\begin{aligned} T_{1;y}(1, x) &= T_{1;y}(P_0^{(2,2)}, x) = P_0^{(2,2)}(x) P_2^{(0,0)}(y) = \frac{3}{2}y^2 - \frac{1}{2}, \\ T_{2;y}(1, x) &= 1, \end{aligned}$$

we have  $f(x)T_{k;y}(1, x) \in L_{1,2}$  ( $k = 1, 2$ ). Hence, applying Lemma 3.3 we derive

$$\int_{-1}^1 T_{k;y}(f, x) (1 - x^2)^2 dx = \int_{-1}^1 f(x) T_{k;y}(1, x) (1 - x^2)^2 dx \quad (k = 1, 2).$$

Therefrom it follows that  $T_{k;y}(f, x) \in L_{1,2}$ . Now the corollary is proved by induction.  $\square$

**Lemma 3.4.** *Let  $f \in L_{1,2}$ . For every natural number  $n$  the following equality holds true*

$$\int_{-1}^1 T_{1;y}(f, x) P_n^{(1,1)}(y) dy = \sum_{m=0}^{n-2} a_m(f) \gamma_m(x),$$

where  $\gamma_m(x)$  is an algebraic polynomial of degree not greater than  $n-2$ , and  $\gamma_m(x) \equiv 0$  for  $n = 0$  or  $n = 1$ .

Lemma 3.4 is proved in [4].

**Lemma 3.5.** *Let  $q$  and  $m$  given natural numbers and let  $f \in L_{1,2}$ . For every natural numbers  $l$  and  $r$  ( $l \leq r$ ) the function*

$$Q_1^{(l)}(x) = \int_0^\pi \cdots \int_0^\pi T_{1;\cos t_1, \dots, \cos t_l}^l(f, x) \prod_{s=1}^r \left( \frac{\sin \frac{mt_s}{2}}{\sin \frac{t_s}{2}} \right)^{2q+4} \sin^3 t_s dt_1 \dots dt_r$$

is an algebraic polynomial of degree not greater than  $(q+2)(m-1)$ .

*Proof.* Since

$$A_s = \left( \frac{\sin \frac{mt_s}{2}}{\sin \frac{t_s}{2}} \right)^{2q+4} = \sum_{k=0}^{(q+2)(m-1)} a_k \cos kt_s = \sum_{k=0}^{(q+2)(m-1)} b_k (\cos t_s)^k,$$

it follows that

$$\begin{aligned} A_s \sin^2 t_s &= \sum_{k=0}^{(q+2)(m-1)} b_k (\cos t_s)^k (1 - \cos^2 t_s) = \sum_{k=0}^{(q+2)(m-1)+2} c_k (\cos t_s)^k \\ &= \sum_{k=0}^{(q+2)(m-1)+2} \alpha_k P_k^{(1,1)}(\cos t_s) \quad (s = 1, 2, \dots, r). \end{aligned}$$

Hence we have

$$\begin{aligned} Q_1^{(l)}(x) &= \sum_{k=0}^{(q+2)(m-1)+2} \alpha_k \int_0^\pi \dots \int_0^\pi \prod_{\substack{s=1 \\ s \neq l}}^r \left( \frac{\sin \frac{m t_s}{2}}{\sin \frac{t_s}{2}} \right)^{2q+4} \\ &\quad \times \sin^3 t_s dt_1 \dots dt_{l-1} dt_{l+1} \dots dt_r \int_0^\pi T_{1;\cos t_1, \dots, \cos t_l}^l(f, x) P_k^{(1,1)}(\cos t_l) \sin t_l dt_l. \end{aligned}$$

Let

$$\varphi_{l,k}(x) = \int_0^\pi T_{1;\cos t_1, \dots, \cos t_l}^l(f, x) P_k^{(1,1)}(\cos t_l) \sin t_l dt_l.$$

Substituting  $y = \cos t_l$  we obtain

$$\varphi_{l,k}(x) = \int_{-1}^1 T_{1;y}^{l-1} \left( T_{1;\cos t_1, \dots, \cos t_{l-1}}^{l-1}(f, x), x \right) P_k^{(1,1)}(y) dy.$$

Using Lemma 3.4 we get

$$\varphi_{l,k}(x) = \sum_{m=0}^{k-2} \gamma_m(x) \int_{-1}^1 T_{1;\cos t_1, \dots, \cos t_{l-1}}^{l-1}(f, R) P_m^{(2,2)}(R) (1 - R^2)^2 dR.$$

Considering Corollary 3.1 we have that  $T_{1;\cos t_1, \dots, \cos t_{l-1}}^{l-1}(f, R) \in L_{1,2}$ . Applying  $l-1$  times Lemma 3.3 and Lemma 3.2 we obtain

$$\begin{aligned} \varphi_{l,k}(x) &= \sum_{m=0}^{k-2} \gamma_m(x) \int_{-1}^1 T_{1;\cos t_1, \dots, \cos t_{l-2}}^{l-2}(f, R) T_{1;\cos t_{l-1}} \left( P_m^{(2,2)}, R \right) \\ &\quad \times (1 - R^2)^2 dR = \sum_{m=0}^{k-2} \gamma_m(x) P_{m+2}^{(0,0)}(\cos t_{l-1}) \\ &\quad \times \int_{-1}^1 T_{1;\cos t_1, \dots, \cos t_{l-2}}^{l-2}(f, R) P_m^{(2,2)}(R) (1 - R^2)^2 dR \\ &= \sum_{m=0}^{k-2} \gamma_m(x) P_{m+2}^{(0,0)}(\cos t_1) \dots P_{m+2}^{(0,0)}(\cos t_{l-1}) \\ &\quad \times \int_{-1}^1 f(R) P_m^{(2,2)}(R) (1 - R^2)^2 dR = \sum_{m=0}^{k-2} \gamma_m(x) a_m(f) \prod_{s=1}^{l-1} P_{m+2}^{(0,0)}(\cos t_s), \end{aligned}$$

where  $a_m(f)$  is the Fourier-Jacobi coefficient of the function  $f$  with respect to the system  $\{P_m^{(2,2)}(x)\}_{m=0}^\infty$ . Substituting  $\varphi_{l,k}(x)$  in the expression for  $Q_1^{(l)}(x)$  we get

$$Q_1^{(l)}(x) = \sum_{k=0}^{(q+2)(m-1)+2} \alpha_k \sum_{m=0}^{k-2} \beta_m \gamma_m(x).$$

Since  $\gamma_m(x)$  is an algebraic polynomial of degree not greater than  $k-2$  for  $k \geq 2$  and  $\gamma_m(x) \equiv 0$  for  $k=0$  and  $k=1$ , then the last equality yields that  $Q_1^{(l)}(x)$  is an algebraic polynomial of degree not greater than  $(q+2)(m-1)$ .

Lemma 3.5 is proved.  $\square$

**Lemma 3.6.** *Let  $q$  and  $m$  given natural numbers. Let  $f \in L_{1,2}$ . For every natural numbers  $l$  and  $r$  ( $l \leq r$ ) the function*

$$Q_2^{(l)}(x) = \int_0^\pi \cdots \int_0^\pi T_{2;\cos t_1, \dots, \cos t_l}^l(f, x) \prod_{s=1}^r \left( \frac{\sin \frac{mt_s}{2}}{\sin \frac{t_s}{2}} \right)^{2q+4} \sin^5 t_s dt_1 \dots dt_r$$

*is an algebraic polynomial of degree not greater than  $(q+2)(m-1)$ .*

*Proof.* As shown in Lemma 3.5

$$\begin{aligned} A_s &= \left( \frac{\sin \frac{mt_s}{2}}{\sin \frac{t_s}{2}} \right)^{2q+4} = \sum_{k=0}^{(q+2)(m-1)} b_k (\cos t_s)^k \\ &= \sum_{k=0}^{(q+2)(m-1)} \beta_k P_k^{(2,2)}(\cos t_s) \quad (s = 1, 2, \dots, r). \end{aligned}$$

Hence

$$\begin{aligned} Q_2^{(l)}(x) &= \sum_{k=0}^{(q+2)(m-1)} \beta_k \int_0^\pi \cdots \int_0^\pi \prod_{\substack{s=1 \\ s \neq l}}^r \left( \frac{\sin \frac{mt_s}{2}}{\sin \frac{t_s}{2}} \right)^{2q+4} \\ &\times \sin^5 t_s dt_1 \dots dt_{l-1} dt_{l+1} \dots dt_r \int_0^\pi T_{2;\cos t_1, \dots, \cos t_l}^l(f, x) P_k^{(2,2)}(\cos t_l) \sin^5 t_l dt_l. \end{aligned}$$

Let

$$\begin{aligned} \psi_{l,k}(x) &= \int_0^\pi T_{2;\cos t_1, \dots, \cos t_l}^l(f, x) P_k^{(2,2)}(\cos t_l) \sin^5 t_l dt_l \\ &= \int_0^\pi T_{2;\cos t_l} \left( T_{2;\cos t_1, \dots, \cos t_{l-1}}^{l-1}(f, x), x \right) P_k^{(2,2)}(\cos t_l) \sin^5 t_l dt_l \end{aligned}$$

Substituting  $y = \cos t_l$  we obtain

$$\psi_{l,k}(x) = \int_{-1}^1 T_{2;y} \left( T_{2;\cos t_1, \dots, \cos t_{l-1}}^{l-1}(f, x), x \right) P_k^{(2,2)}(y) (1-y^2)^2 dy.$$

Since operator  $T_{2;y}(f, x)$  is symmetrical on  $x$  and  $y$ , i.e. for every function  $g$  holds  $T_{2;y}(g, x) = T_{2;x}(g, y)$ , we have

$$\psi_{l,k}(x) = \int_{-1}^1 T_{2;x} \left( T_{2;\cos t_1, \dots, \cos t_{l-1}}^{l-1}(f, y), y \right) P_k^{(2,2)}(y) (1-y^2)^2 dy.$$

Since Corollary 3.1 yields  $T_{2;\cos t_1, \dots, \cos t_{l-1}}^{l-1}(f, y) \in L_{1,2}$ , applying Lemma 3.3 we obtain

$$\psi_{l,k}(x) = \int_{-1}^1 T_{2;\cos t_1, \dots, \cos t_{l-1}}^{l-1}(f, y) T_{2;x} \left( P_k^{(2,2)}, y \right) (1-y^2)^2 dy.$$

Considering the property of the operator  $T_{2;x}$  from Lemma 3.2 we get

$$\psi_{l,k}(x) = P_k^{(2,2)}(x) \int_{-1}^1 T_{2;\cos t_1, \dots, \cos t_{l-1}}^{l-1}(f, y) P_k^{(2,2)}(y) (1-y^2)^2 dy.$$

Applying  $l-1$  times Lemma 3.3 and Lemma 3.2 we obtain

$$\begin{aligned} \psi_{l,k}(x) &= P_k^{(2,2)}(x) P_k^{(2,2)}(\cos t_1) \dots P_k^{(2,2)}(\cos t_{l-1}) \\ &\times \int_{-1}^1 f(y) P_k^{(2,2)}(y) (1-y^2)^2 dy = P_k^{(2,2)}(x) a_k(f) \prod_{s=1}^{l-1} P_k^{(2,2)}(\cos t_s). \end{aligned}$$

where  $a_k(f)$  is the Fourier–Jacobi coefficient of the function  $f$  with respect to the system  $\left\{P_k^{(2,2)}(x)\right\}_{k=0}^{\infty}$ . Substituting  $\psi_{l,k}(x)$  in the expression for  $Q_2^{(l)}(x)$  we get

$$Q_2^{(l)}(x) = \sum_{k=0}^{(q+2)(m-1)} \delta_k P_k^{(2,2)}(x).$$

Since  $P_k^{(2,2)}(x)$  is an algebraic polynomial of degree not greater than  $k$ , the last equality implies that  $Q_2^{(l)}(x)$  is an algebraic polynomial of degree not greater than  $(q+2)(m-1)$ .

Lemma is proved.  $\square$

**Lemma 3.7.** *Operator  $T_y$  has the following properties*

- (1) *The operator  $T_y(f, x)$  is linear on  $f$ ;*
- (2)  *$T_1(f, x) = f(x)$ ;*
- (3)  *$T_y\left(P_n^{(2,2)}, x\right) = P_n^{(2,2)}(x)R_n(y)$  ( $n = 0, 1, \dots$ ),*  
*where  $R_n(y) = P_{n+2}^{(0,0)}(y) + \frac{3}{2}(1-y^2)P_n^{(2,2)}(y)$ ;*
- (4)  *$T_y(1, x) = 1$ ;*
- (5)  *$a_k(T_y(f, x)) = R_k(y)a_k(f)$  ( $k = 0, 1, \dots$ ).*

Lemma 3.7 is proved in [4].

**Corollary 3.2.** *If  $P_n(x)$  is an algebraic polynomial of degree not greater than  $n-1$ , then for every natural number  $r$ , for fixed  $y_1, y_2, \dots, y_r$ , functions  $T_{y_1, \dots, y_r}^r(P_n, x)$  and  $\Delta_{y_1, \dots, y_r}^r(P_n, x)$  are algebraic polynomials on  $x$  of degree not greater than  $n-1$ .*

**Lemma 3.8.** *If  $-1 \leq x \leq 1$ ,  $-1 \leq z \leq 1$ ,  $0 \leq t \leq \pi$  and  $R = xy + z\sqrt{1-x^2} \times \sqrt{1-y^2}$ , then  $-1 \leq R \leq 1$  and*

$$\begin{aligned} \left(x\sqrt{1-y^2} + yz\sqrt{1-x^2}\right)^2 &\leq (1-R^2), \\ \left(\sqrt{1-x^2}y + xz\sqrt{1-y^2}\right)^2 &\leq (1-R^2), \\ (1-x^2)(1-z^2) &\leq (1-R^2), \\ (1-y^2)(1-z^2) &\leq (1-R^2). \end{aligned}$$

Lemma 3.8 is proved in [4] and [3].

**Lemma 3.9.** *Let given numbers  $p$  and  $\alpha$  be such that  $1 \leq p \leq \infty$ ;*

$$\begin{aligned} \frac{1}{2} < \alpha \leq 1 & \quad \text{for } p = 1, \\ 1 - \frac{1}{2p} < \alpha < \frac{3}{2} - \frac{1}{2p} & \quad \text{for } 1 < p < \infty, \\ 1 \leq \alpha < \frac{3}{2} & \quad \text{for } p = \infty. \end{aligned}$$

*Let  $f \in L_{p,\alpha}$ . The following inequality holds true*

$$\|T_y(f, x)\|_{p,\alpha} \leq C \|f\|_{p,\alpha},$$

*where the constant  $C$  does not depend on  $f$  and  $y$ .*

Lemma 3.9 is also proved in [4].

**Corollary 3.3.** *Let given numbers  $p$  and  $\alpha$  be such that  $1 \leq p \leq \infty$ ;*

$$\begin{aligned} \frac{1}{2} < \alpha \leq 1 & \quad \text{for } p = 1, \\ 1 - \frac{1}{2p} < \alpha < \frac{3}{2} - \frac{1}{2p} & \quad \text{for } 1 < p < \infty, \\ 1 \leq \alpha < \frac{3}{2} & \quad \text{for } p = \infty. \end{aligned}$$

*Let  $f \in L_{p,\alpha}$ . The following inequality holds true*

$$\|T_{y_1, \dots, y_r}^r(f, x)\|_{p,\alpha} \leq C \|f\|_{p,\alpha},$$

*where the constant  $C$  does not depend on  $f$  and  $y_j$  ( $j = 1, 2, \dots, r$ ).*

The corollary is proved by applying  $r$  times Lemma 3.9 taking into consideration Corollary 3.1 (see [4]).

#### 4.

**Theorem 4.1.** *Let  $q$ ,  $m$  and  $r$  given natural numbers and let  $f \in L_{1,2}$ . The function*

$$\begin{aligned} Q(x) = \frac{1}{(\gamma_m)^r} \int_0^\pi \dots \int_0^\pi (\Delta_{t_1, \dots, t_r}^r(f, x) - (-1)^r f(x)) \\ \times \prod_{s=1}^r \left( \frac{\sin \frac{mt_s}{2}}{\sin \frac{t_s}{2}} \right)^{2q+4} \sin^3 t_s dt_1 \dots dt_r, \end{aligned}$$

*where*

$$\gamma_m = \int_0^\pi \left( \frac{\sin \frac{mt}{2}}{\sin \frac{t}{2}} \right)^{2q+4} \sin^3 t dt,$$

*is an algebraic polynomial of degree not greater than  $(q+2)(m-1)$ .*

*Proof.* To prove the theorem it is sufficient to show that for every  $l = 1, 2, \dots, r$  the function

$$Q^{(l)}(x) = \frac{1}{(\gamma_m)^r} \int_0^\pi \dots \int_0^\pi T_{\cos t_1, \dots, \cos t_l}^l(f, x) \prod_{s=1}^r \left( \frac{\sin \frac{mt_s}{2}}{\sin \frac{t_s}{2}} \right)^{2q+4} \sin^3 t_s dt_1 \dots dt_r$$

*is an algebraic polynomial of degree not greater than  $(q+2)(m-1)$ .*

It is obvious that the function  $Q^{(l)}(x)$  can be written in the form

$$Q^{(l)}(x) = \frac{1}{(\gamma_m)^r} \left( Q_1^{(l)}(x) + \frac{3}{2} Q_2^{(l)}(x) \right),$$

where  $Q_1^{(l)}(x)$  and  $Q_2^{(l)}(x)$  are the functions from Lemmas 3.5 and 3.6 respectively. But, then Lemmas 3.5 and 3.6 yield that  $Q^{(l)}(x)$  is an algebraic polynomial of degree not greater than  $(q+2)(m-1)$ .

Theorem is proved.  $\square$

**Theorem 4.2.** *Let given numbers  $p$ ,  $\alpha$ ,  $r$  and  $\lambda$  be such that  $1 \leq p \leq \infty$ ,  $\lambda > 0$ ,  $r \in \mathbb{N}$ ;*

$$\begin{aligned} \alpha \leq 2 & \quad \text{for } p = 1, \\ \alpha < 3 - \frac{1}{p} & \quad \text{for } 1 < p \leq \infty. \end{aligned}$$

*Let  $f \in L_{p,\alpha}$  and*

$$\hat{\omega}_r(f, \delta)_{p,\alpha} \leq M \delta^\lambda.$$



Then

$$E_n(f)_{p,\alpha} \leq CMn^{-\lambda},$$

where the constant  $C$  does not depend on  $f$ ,  $M$  and  $n$ .

*Proof.* It can easily be proved that under the conditions of the theorem, if  $f \in L_{p,\alpha}$ , then  $f \in L_{1,2}$ .

We choose a natural number  $q$  such that  $2q > \lambda$ , and for each natural number  $n$  we choose the natural number  $m$  satisfying the condition

$$(4.1) \quad \frac{n-1}{q+2} < m \leq \frac{n-1}{q+2} + 1.$$

For those  $q$  and  $m$  polynomial  $Q(x)$  defined in Theorem 4.1 is an algebraic polynomial of degree not greater than  $n-1$ . Hence

$$\begin{aligned} E_n(f)_{p,\alpha} &\leq \|f(x) - (-1)^{r+1}Q(x)\|_{p,\alpha} \\ &= \left\| \frac{1}{(\gamma_m)^r} \int_0^\pi \cdots \int_0^\pi \Delta_{t_1, \dots, t_r}^r(f, x) \right. \\ &\quad \times \prod_{s=1}^r \left( \frac{\sin \frac{mt_s}{2}}{\sin \frac{t_s}{2}} \right)^{2q+4} \sin^3 t_s dt_1 \dots dt_r \left. \right\|_{p,\alpha}. \end{aligned}$$

Applying the generalised inequality of Minkowski we obtain

$$\begin{aligned} E_n(f)_{p,\alpha} &\leq \frac{1}{(\gamma_m)^r} \int_0^\pi \cdots \int_0^\pi \|\Delta_{t_1, \dots, t_r}^r(f, x)\|_{p,\alpha} \\ &\quad \times \prod_{s=1}^r \left( \frac{\sin \frac{mt_s}{2}}{\sin \frac{t_s}{2}} \right)^{2q+4} \sin^3 t_s dt_1 \dots dt_r \\ &\leq \frac{1}{(\gamma_m)^r} \int_0^\pi \cdots \int_0^\pi \hat{\omega}_r \left( f, \sum_{j=1}^r t_j \right)_{p,\alpha} \prod_{s=1}^r \left( \frac{\sin \frac{mt_s}{2}}{\sin \frac{t_s}{2}} \right)^{2q+4} \sin^3 t_s dt_1 \dots dt_r. \end{aligned}$$

Hence, considering the conditions of the theorem we have (see [?, p. 31])

$$\begin{aligned} E_n(f)_{p,\alpha} &\leq \frac{M}{(\gamma_m)^r} \int_0^\pi \cdots \int_0^\pi \left( \sum_{j=1}^r t_j \right)^\lambda \\ &\quad \times \prod_{s=1}^r \left( \frac{\sin \frac{mt_s}{2}}{\sin \frac{t_s}{2}} \right)^{2q+4} \sin^3 t_s dt_1 \dots dt_r \\ &\leq C_1 M \sum_{j=1}^r \frac{1}{(\gamma_m)^r} \int_0^\pi \cdots \int_0^\pi t_j^\lambda \prod_{s=1}^r \left( \frac{\sin \frac{mt_s}{2}}{\sin \frac{t_s}{2}} \right)^{2q+4} \sin^3 t_s dt_1 \dots dt_r. \end{aligned}$$

Applying the standard evaluation of the Jackson's kernel, considering inequality (4.1), we obtain

$$E_n(f)_{p,\alpha} \leq C_2 M m^{-\lambda} \leq C_3 M n^{-\lambda}.$$

Theorem 4.2 is proved.  $\square$

**Theorem 4.3.** *Let given numbers  $p, \alpha, r$  and  $\lambda$  be such that  $1 \leq p \leq \infty, r \in \mathbb{N}, 0 < \lambda < 2r$ ;*

$$\begin{aligned} \frac{1}{2} < \alpha &\leq 1 && \text{for } p = 1, \\ 1 - \frac{1}{2p} < \alpha &< \frac{3}{2} - \frac{1}{2p} && \text{for } 1 < p < \infty, \\ 1 \leq \alpha &< \frac{3}{2} && \text{for } p = \infty. \end{aligned}$$

If  $f \in L_{p,\alpha}$  and

$$E_n(f)_{p,\alpha} \leq \frac{M}{n^\lambda},$$

then

$$\hat{\omega}_r(f, \delta)_{p,\alpha} \leq CM\delta^\lambda,$$

where the constant  $C$  does not depend on  $f, M$  and  $\delta$ .

*Proof.* Let  $P_n(x)$  be the polynomial of degree not greater than  $n - 1$  such that

$$\|f - P_n\|_{p,\alpha} = E_n(f)_{p,\alpha} \quad (n = 1, 2, \dots).$$

We construct the polynomials  $Q_k(x)$  by

$$Q_k(x) = P_{2^k}(x) - P_{2^{k-1}}(x) \quad (k = 1, 2, \dots)$$

and  $Q_0(x) = P_1(x)$ . Since for  $k \geq 1$  we have

$$\begin{aligned} \|Q_k\|_{p,\alpha} &= \|P_{2^k} - P_{2^{k-1}}\|_{p,\alpha} \leq \|P_{2^k} - f\|_{p,\alpha} + \|f - P_{2^{k-1}}\|_{p,\alpha} \\ &= E_{2^k}(f)_{p,\alpha} + E_{2^{k-1}}(f)_{p,\alpha}, \end{aligned}$$

then under the conditions of the theorem it follows that

$$\|Q_k\|_{p,\alpha} \leq C_1 M 2^{-k\lambda}.$$

It is obvious that without lost in generality we may assume that  $t_s \neq 0$  ( $s = 1, 2, \dots, r$ ). For  $0 < |t_s| < \delta$  ( $s = 1, 2, \dots, r$ ) we estimate

$$I = \|\Delta_{t_1, \dots, t_r}^r(f, x)\|_{p,\alpha}.$$

For every natural number  $N$ , considering that linearity of the operator  $\hat{T}_{t_1}(f, x)$  implies the linearity of the operator  $\hat{T}_{t_1, \dots, t_r}^r(f, x)$ , i.e. the linearity of the difference  $\Delta_{t_1, \dots, t_r}^r(f, x)$ , we have

$$I \leq \|\Delta_{t_1, \dots, t_r}^r(f - P_{2^N}, x)\|_{p,\alpha} + \|\Delta_{t_1, \dots, t_r}^r(P_{2^N}, x)\|_{p,\alpha}.$$

Since  $P_{2^N}(x) = \sum_{k=0}^N Q_k(x)$ , we get

$$I \leq \|\Delta_{t_1, \dots, t_r}^r(f - P_{2^N}, x)\|_{p,\alpha} + \sum_{k=1}^N \|\Delta_{t_1, \dots, t_r}^r(Q_k, x)\|_{p,\alpha}.$$

Applying Corollary 3.3 we have

$$I \leq C_2 E_{2^N}(f)_{p,\alpha} + \sum_{k=1}^N I_k.$$

Let  $N$  be chosen so that

$$(4.2) \quad \frac{\pi}{2^N} < \delta \leq \frac{\pi}{2^{N-1}}.$$

We prove that the following inequality holds true

$$(4.3) \quad I_k \leq C_3 M \delta^{2r} 2^{k(2r-\lambda)}.$$

Let

$$\psi_k(x) = \Delta_{t_1, \dots, t_r}^r(Q_k, x).$$

It can be proved that

$$(4.4) \quad \psi_k(x) = \frac{1}{2\pi(1-x^2)} \int_0^{t_r} \int_{-u}^u \int_0^\pi \left( A(v)(R'_v)^2 \frac{d^2}{dR_v^2} \Delta_{t_1, \dots, t_{r-1}}^{r-1}(Q_k, R_v) \right. \\ \left. - (A(v)R_v - 2A'(v)R'_v) \frac{d}{dR_v} \Delta_{t_1, \dots, t_{r-1}}^{r-1}(Q_k, R_v) \right. \\ \left. + A''(v) \Delta_{t_1, \dots, t_{r-1}}^{r-1}(Q_k, R_v) \right) d\varphi dv du,$$

where  $R_v = x \cos v - \sqrt{1-x^2} \cos \varphi \sin v$ ,

$$A(v) = 1 - R_v^2 - 2 \sin^2 v \sin^2 \varphi + 4(1-x^2) \sin^2 v \sin^4 \varphi.$$

Applying estimates from Lemma 3.8 and performing change of variables  $z = \cos \varphi$  we obtain

$$|\psi_k(x)| \leq \frac{C_4}{1-x^2} \int_0^{t_r} \int_{-u}^u \int_{-1}^1 B(R_v) \frac{dz}{\sqrt{1-z^2}} dv du,$$

where

$$B(R_v) = (1-R_v^2)^2 \left| \frac{d^2}{dR_v^2} \Delta_{t_1, \dots, t_{r-1}}^{r-1}(Q_k, R_v) \right| \\ + (1-R_v^2) \left| \frac{d}{dR_v} \Delta_{t_1, \dots, t_{r-1}}^{r-1}(Q_k, R_v) \right| \\ + \left| \Delta_{t_1, \dots, t_{r-1}}^{r-1}(Q_k, R_v) \right| = B_1(R_v) + B_2(R_v) + B_3(R_v).$$

Therefore using the generalised Minkowski's inequality we get

$$(4.5) \quad I_k = \|\psi_k(x)\|_{p, \alpha} \leq C_4 \int_0^{t_r} \int_{-u}^u \int_{-1}^1 \left\| \frac{B(R_v)}{1-x^2} \right\|_{p, \alpha} \frac{dz}{\sqrt{1-z^2}} dv du.$$

Let  $p = 1$ . Considering that  $\alpha \leq 1$  we obtain

$$I_k \leq \int_0^{t_r} \int_{-u}^u \int_{-1}^1 |B(R_v)| (1-x^2)^{\alpha-1} (1-z^2)^{\alpha-1} \frac{dx dz}{\sqrt{1-z^2}} dv du.$$

Let  $1 < p < \infty$ . Applying the Hölder's inequality in the inside integral in equation (4.5), considering that  $\alpha < \frac{3}{2} - \frac{1}{2p}$  we obtain

$$I_k \leq C_4 \int_0^{t_r} \int_{-u}^u \int_{-1}^1 \left\{ \int_{-1}^1 |B(R_v)|^p (1-x^2)^{p(\alpha-1)} (1-z^2)^{p(\alpha-1)} dx \right\}^{\frac{1}{p}} \\ \times (1-z^2)^{-\frac{1}{2p}} (1-z^2)^{-\alpha + \frac{1}{2p} + \frac{1}{2}} dz dv du \\ \leq C_5 \int_0^{t_r} \int_{-u}^u \left\{ \int_{-1}^1 \int_{-1}^1 |B(R_v)|^p (1-x^2)^{p(\alpha-1)} \right. \\ \left. \times (1-z^2)^{p(\alpha-1)} \frac{dx dz}{\sqrt{1-z^2}} \right\}^{\frac{1}{p}} dv du.$$

Thus, under the conditions of the theorem, for  $1 \leq p < \infty$  we have

$$I_k \leq C_6 \int_0^{t_r} \int_{-u}^u \left\{ \int_{-1}^1 \int_{-1}^1 |B(R_v)|^p (1-x^2)^{p(\alpha-1)} \right. \\ \left. \times (1-z^2)^{p(\alpha-1)} \frac{dx dz}{\sqrt{1-z^2}} \right\}^{\frac{1}{p}} dv du.$$

Performing the change of variables in double integral by the formulas

$$\begin{aligned} R &= x \cos v - z \sqrt{1 - x^2} \sin v, \\ V &= \frac{x \sin v + z \sqrt{1 - x^2} \cos v}{\sqrt{1 - (x \cos v - z \sqrt{1 - x^2} \sin v)^2}}, \end{aligned}$$

we obtain

$$\begin{aligned} I_k &\leq C_6 \int_0^{t_r} \int_{-u}^u \left\{ \int_{-1}^1 \int_{-1}^1 |B(R)|^p (1 - R^2)^{p(\alpha-1)} \right. \\ &\quad \left. \times (1 - V^2)^{p(\alpha-1)-\frac{1}{2}} dR dV \right\}^{\frac{1}{p}} dv du. \end{aligned}$$

Since, under the conditions of theorem  $\alpha > 1 - \frac{1}{2p}$ , it follows that

$$\begin{aligned} I_k &\leq C_7 \int_0^{t_r} \int_{-u}^u \left\{ \int_{-1}^1 |B(R)|^p (1 - R^2)^{p(\alpha-1)} dR \right\}^{\frac{1}{p}} dv du \\ &\leq C_8 t_r^2 \|B(R)\|_{p, \alpha-1}. \end{aligned}$$

Let now  $p = \infty$ . Considering the estimates from Lemma 3.8 and that  $\alpha \geq 1$ , inequality (4.5) yields

$$\begin{aligned} I_k &\leq C_4 \int_0^{t_r} \int_{-u}^u \int_{-1}^1 \operatorname{ess\,sup}_{-1 \leq x \leq 1} |B(R_v)| (1 - x^2)^{\alpha-1} \frac{dz}{\sqrt{1 - z^2}} dv du \\ &\leq C_4 \|B(x)\|_{\infty, \alpha-1} \int_0^{t_r} \int_{-u}^u \int_{-1}^1 (1 - z^2)^{-\alpha+\frac{1}{2}} dz dv du. \end{aligned}$$

Hence, considering that  $\alpha < \frac{3}{2}$  we get

$$I_k \leq C_9 t_r^2 \|B(x)\|_{\infty, \alpha-1}.$$

Thus for all  $1 \leq p \leq \infty$  we proved that

$$I_k \leq C_{10} t_r^2 \|B(x)\|_{p, \alpha-1}.$$

Applying Lemma 3.1 and Corollaries 3.2 and 3.3 under the conditions of the theorem we obtain

$$\begin{aligned} I_k &= \|\Delta_{t_1, \dots, t_r}^r(Q_k, x)\|_{p, \alpha} \leq C_{10} t_r^2 \|B(x)\|_{p, \alpha-1} \\ &\leq C_{10} t_r^2 \left( \|B_1(x)\|_{p, \alpha-1} + \|B_2(x)\|_{p, \alpha-1} + \|B_3(x)\|_{p, \alpha-1} \right) \\ &= C_{10} t_r^2 \left\{ \left\| \frac{d^2}{dx^2} \Delta_{t_1, \dots, t_{r-1}}^{r-1}(Q_k, x) \right\|_{p, \alpha+1} \right. \\ &\quad \left. + \left\| \frac{d}{dx} \Delta_{t_1, \dots, t_{r-1}}^{r-1}(Q_k, x) \right\|_{p, \alpha} + \left\| \Delta_{t_1, \dots, t_{r-1}}^{r-1}(Q_k, x) \right\|_{p, \alpha-1} \right\} \\ &\leq C_{11} t_r^2 2^{2k} \left\| \Delta_{t_1, \dots, t_{r-1}}^{r-1}(Q_k, x) \right\|_{p, \alpha}. \end{aligned}$$

Applying  $r$  times this inequality it follows that

$$I_k \leq C_{12} t_1^2 \dots t_r^2 2^{2kr} \|Q_k\|_{p, \alpha}.$$

Therefore we have

$$I_k \leq C_{13} M \delta^{2r} 2^{k(2r-\lambda)}.$$

Inequality (4.3) is proved.

Inequalities (4.3) and (4.2) yield

$$I \leq C_{14}M \left( \delta^\lambda + \delta^{2r} \sum_{k=1}^N 2^{k(2r-\lambda)} \right) \leq C_{15}M \left( \delta^\lambda + \delta^{2r} 2^{N(2r-\lambda)} \right) \leq C_{16}M \delta^\lambda.$$

Theorem 4.3 is completed.  $\square$

Now we formulate the theorem of coincidence of the class  $H(p, \alpha, r, \lambda)$  with the class  $E(p, \alpha, \lambda)$ , and the inverse theorem.

**Theorem 4.4.** *Let given numbers  $p, \alpha, r$  and  $\lambda$  be such that  $1 \leq p \leq \infty, 0 < \lambda < 2r, r \in \mathbb{N}$ ;*

$$\begin{aligned} \frac{1}{2} < \alpha \leq 1 & \quad \text{for } p = 1, \\ 1 - \frac{1}{2p} < \alpha < \frac{3}{2} - \frac{1}{2p} & \quad \text{for } 1 < p < \infty, \\ 1 \leq \alpha < \frac{3}{2} & \quad \text{for } p = \infty. \end{aligned}$$

*The class  $H(p, \alpha, r, \lambda)$  coincides with the class  $E(p, \alpha, \lambda)$ .*

Theorem 4.4 is implied by Theorems 4.2 and 4.3 proved above.

**Theorem 4.5.** *Let given numbers  $p, \alpha, r$  and  $\lambda$  be such that  $1 \leq p \leq \infty, 0 < \lambda < 2r, r \in \mathbb{N}$ ;*

$$\begin{aligned} \frac{1}{2} < \alpha \leq 1 & \quad \text{for } p = 1, \\ 1 - \frac{1}{2p} < \alpha < \frac{3}{2} - \frac{1}{2p} & \quad \text{for } 1 < p < \infty, \\ 1 \leq \alpha < \frac{3}{2} & \quad \text{for } p = \infty. \end{aligned}$$

*If  $f \in L_{p,\alpha}$ , then the following inequality holds*

$$\hat{\omega}_r \left( f, \frac{1}{n} \right)_{p,\alpha} \leq \frac{C_1}{n^{2r}} \sum_{\nu=1}^n \nu^{2r-1} E_\nu(f)_{p,\alpha},$$

*where the constant  $C$  does not depend on  $f$  and  $n$ .*

*Proof.* Let  $P_n(x)$  be the polynomial of degree not greater than  $n-1$  such that

$$\|f - P_n\|_{p,\alpha} = E_n(f)_{p,\alpha} \quad (n = 1, 2, \dots),$$

and

$$Q_k(x) = P_{2^k}(x) - P_{2^{k-1}}(x) \quad (k = 1, 2, \dots),$$

$$Q_0(x) = P_1(x).$$

For given  $n$  we chose the natural number  $N$  such that

$$\frac{n}{2} < 2^N \leq n+1.$$

By the proof of Theorem 4.3 it follows that

$$\begin{aligned}
\hat{\omega}_r \left( f, \frac{1}{n} \right)_{p,\alpha} &\leq C_2 \left( E_{2^N}(f)_{p,\alpha} + \frac{1}{n^{2r}} \sum_{\mu=1}^N 2^{2\mu r} \|Q_k\|_{p,\alpha} \right) \\
&\leq 2C_2 \left( E_{2^N}(f)_{p,\alpha} + \frac{1}{n^{2r}} \sum_{\mu=1}^N 2^{2\mu r} \left( E_{2^\mu}(f)_{p,\alpha} + E_{2^{\mu-1}}(f)_{p,\alpha} \right) \right) \\
&\leq 4C_2 \left( E_{2^N}(f)_{p,\alpha} + \frac{1}{n^{2r}} \sum_{\mu=0}^{N-1} 2^{2(\mu+1)r} E_{2^\mu}(f)_{p,\alpha} \right) \\
&\leq \frac{C_3}{n^{2r}} \sum_{\mu=0}^N 2^{2(\mu+1)r} E_{2^\mu}(f)_{p,\alpha} .
\end{aligned}$$

Considering that for  $\mu \geq 1$  we have

$$\sum_{\nu=2^{\mu-1}}^{2^\mu-1} \nu^{2r-1} E_\nu(f)_{p,\alpha} \geq E_{2^\mu}(f)_{p,\alpha} 2^{\mu-1} 2^{(\mu-1)(2r-1)} \geq C_4 2^{2(\mu+1)r} E_{2^\mu}(f)_{p,\alpha} ,$$

it follows that

$$\begin{aligned}
\hat{\omega}_r \left( f, \frac{1}{n} \right)_{p,\alpha} &\leq \frac{C_5}{n^{2r}} \left( 2^{2r} E_1(f)_{p,\alpha} + \sum_{\mu=1}^N \sum_{\nu=2^{\mu-1}}^{2^\mu-1} \nu^{2r-1} E_\nu(f)_{p,\alpha} \right) \\
&\leq \frac{C_6}{n^{2r}} \sum_{\nu=1}^n \nu^{2r-1} E_\nu(f)_{p,\alpha} .
\end{aligned}$$

Theorem 4.5 is proved.  $\square$

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