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SOME REVERSE l_p-TYPE INEQUALITIES INVOLVING CERTAIN QUASI MONOTONE SEQUENCES

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Abstract. In this paper, we give some l_p-type inequalities about sequences satisfying certain quasi monotone type properties. As special cases, reverse l_p-type inequalities for non-negative decreasing sequences are obtained. The inequalities are closely related to Copson's and Leindler's inequalities, but the sign of the inequalities is reversed.

1. Introduction

For non-negative number sequences the following, classical inequalities of Hardy and Littlewood, are well known [3, p. 255, Th 346].

Let $\{b_\nu\}_{\nu=1}^\infty$ be a sequence of non-negative numbers, $\alpha > 0$, m and n positive integers such that $n < m$. The following inequalities hold true:

$$\sum_{\mu=n}^{m} \mu^{\alpha-1} \left(\sum_{\nu=\mu}^{m} b_\nu \right)^p \le C \sum_{\mu=n}^{m} \mu^{\alpha-1} (\mu b_\mu)^p, \tag{1.1}$$

$$\sum_{\mu=n}^{m} \mu^{-\alpha-1} \left(\sum_{\nu=n}^{\mu} b_\nu \right)^p \le C \sum_{\mu=n}^{m} \mu^{-\alpha-1} (\mu b_\mu)^p \tag{1.2}$$

for $p \ge 1$; and

$$\sum_{\mu=n}^{m} \mu^{\alpha-1} \left(\sum_{\nu=\mu}^{m} b_\nu \right)^p \ge C \sum_{\mu=n}^{m} \mu^{\alpha-1} (\mu b_\mu)^p, \tag{1.3}$$

$$\sum_{\mu=n}^{m} \mu^{-\alpha-1} \left(\sum_{\nu=n}^{\mu} b_\nu \right)^p \ge C \sum_{\mu=n}^{m} \mu^{-\alpha-1} (\mu b_\mu)^p \tag{1.4}$$

for $0 < p \le 1$, where positive constant C depends only on numbers α and p, and does not depend on integers m, n, and the sequence $\{b_\nu\}_{\nu=1}^\infty$.

Closely related to these inequalities are classical Copson inequalities [1], Leindler's inequalities [5, 6, 7], and those proved or used in [8, 9, 10, 2].

In the paper we prove some related inequalities which involve non-negative sequences satisfying certain monotone-type properties. As special cases, inequalities converse to (1.1), (1.2), (1.3) and (1.4) for the case of non-negative monotone decreasing number sequences are deduced.

In order to prove the inequalities we need the following

Theorem 1.1. *Let $\{b_\nu\}_{\nu=1}^\infty$ be a sequence of non-negative numbers, $0 < \alpha < \beta$, m and n positive integers such that $n < m$. Then the following inequality holds*

$$\left(\sum_{\mu=n}^{m} b_\mu^\beta \right)^{1/\beta} \le \left(\sum_{\mu=n}^{m} b_\mu^\alpha \right)^{1/\alpha}.$$

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The proof of the theorem is due to Jensen [3, p. 28, Th. 19].

We call a number sequence $\{a_\nu\}_{\nu=1}^{\infty}$ non-negative monotone decreasing (or increasing), and denote it by $a_\nu \downarrow$ (or $a_\nu \uparrow$), if for each positive integer ν the following conditions are satisfied

(1) $a_\nu \geq 0$,
(2) $a_{\nu+1} \leq a_\nu$ (or $a_{\nu+1} \geq a_\nu$, respectively).

We call a number sequence $\{\lambda_\nu\}_{\nu=1}^{\infty}$ quasi lacunary monotone if it is a non-negative monotone sequence (i.e., $\lambda_\nu \downarrow$ or $\lambda_\nu \uparrow$) and there are positive constants K_1 and K_2 such that for each positive integer ν the following condition is satisfied

$$K_1 \lambda_{2^\nu} \leq \lambda_{2^{\nu+1}} \leq K_2 \lambda_{2^\nu}.$$

Finally, we call a non-negative number sequence $\{\lambda_\mu\}_{\mu=1}^{\infty}$ quasi geometrically increasing if there is a positive constant K such that for each positive integer m the following condition is satisfied

$$\sum_{\mu=1}^{m} \lambda_\mu \leq K \lambda_m.$$

2. Inequalities for quasi monotone sequences

In this section we give several reverse inequalities of l_p-type involving non-negative decreasing sequences, quasi lacunary monotone sequences or quasi geometrically increasing sequences.

Theorem 2.1. *Let a sequence $\{a_\nu\}_{\nu=1}^{\infty}$ be such that $a_\nu \downarrow$, $\{\lambda_\mu\}_{\mu=1}^{\infty}$ and $\{\gamma_\nu\}_{\nu=1}^{\infty}$ be lacunary monotone sequences, m and n positive integers such that $m \geq 16n$. If $p \geq 1$, then the following inequality holds*

$$\sum_{\mu=n}^{m} \lambda_\mu \left(\sum_{\nu=n}^{\mu} a_\nu \gamma_\nu \right)^p \geq C \sum_{\mu=4n}^{m} \lambda_\mu (\mu a_\mu \gamma_\mu)^p. \tag{2.1}$$

If, in addition, $\{2^\mu \lambda_{2^\mu}\}_{\mu=1}^{\infty}$ is a quasi geometrically increasing sequence, then the following inequality holds

$$\sum_{\mu=n}^{m} \lambda_\mu \left(\sum_{\nu=\mu}^{m} a_\nu \gamma_\nu \right)^p \geq C \sum_{\mu=8n}^{m} \lambda_\mu (\mu a_\mu \gamma_\mu)^p. \tag{2.2}$$

Here and later on C and C_i will denote positive constants depending only on p and the sequences $\{\lambda_\mu\}_{\mu=1}^{\infty}$ and $\{\gamma_\nu\}_{\nu=1}^{\infty}$, and not depending on m, n and the sequence $\{a_\nu\}_{\nu=1}^{\infty}$.

Theorem 2.2. *Let a sequence $\{a_\nu\}_{\nu=1}^{\infty}$ be such that $a_\nu \downarrow$, $\{\lambda_\mu\}_{\mu=1}^{\infty}$ and $\{\gamma_\nu\}_{\nu=1}^{\infty}$ be lacunary monotone sequences, m and n positive integers such that $m \geq 4n$. If $0 < p \leq 1$, then the following inequality holds*

$$\sum_{\mu=4n}^{m} \lambda_\mu \left(\sum_{\nu=4n}^{\mu} a_\nu \gamma_\nu \right)^p \leq C \sum_{\mu=n}^{m} \lambda_\mu (\mu a_\mu \gamma_\mu)^p. \tag{2.3}$$

If, in addition, $\{2^\mu \lambda_{2^\mu}\}_{\mu=1}^{\infty}$ is a quasi geometrically increasing sequence, then the following inequality holds

$$\sum_{\mu=4n}^{m} \lambda_\mu \left(\sum_{\nu=\mu}^{m} a_\nu \gamma_\nu \right)^p \leq C \sum_{\mu=n}^{m} \lambda_\mu (\mu a_\mu \gamma_\mu)^p. \tag{2.4}$$

3. Proof of Theorem 2.1

We prove inequality (2.2). For given n and m we choose positive integers N and M such that $2^{N-1} < n \le 2^N$ and $2^M \le m < 2^{M+1}$. Then the following inequality holds

$$I = \sum_{\mu=n}^{m} \lambda_\mu \Big(\sum_{\nu=\mu}^{m} a_\nu \gamma_\nu\Big)^p \ge \sum_{\mu=2^N}^{2^M} \lambda_\mu \Big(\sum_{\nu=\mu}^{2^M} a_\nu \gamma_\nu\Big)^p.$$

By splitting the first sum into blocks of length 2^i, we obtain

$$I \ge \sum_{i=N+1}^{M} \sum_{\mu=2^{i-1}+1}^{2^i} \lambda_\mu \Big(\sum_{\nu=\mu}^{2^M} a_\nu \gamma_\nu\Big)^p.$$

By bounding the third sum from below, taking into account that $\{\lambda_\mu\}_{\mu=1}^\infty$ is a quasi lacunary monotone sequence, we have

$$I \ge \sum_{i=N+1}^{M} \Big(\sum_{\nu=2^i}^{2^M} a_\nu \gamma_\nu\Big)^p \sum_{\mu=2^{i-1}+1}^{2^i} \lambda_\mu \ge C_1 \sum_{i=N+1}^{M} 2^{i-1} \lambda_{2^{i-1}} \Big(\sum_{\nu=2^i}^{2^M} a_\nu \gamma_\nu\Big)^p,$$

where C_1 depends only on the sequence $\{\lambda_\mu\}_{\mu=1}^\infty$. Now, we split the second sum into blocks of length 2^{i-1}, remove the terms with index $i = M$, and taking into consideration that $a_\nu \downarrow$ and $\{\lambda_\mu\}_{\mu=1}^\infty$, $\{\gamma_\nu\}_{\nu=1}^\infty$ are lacunary monotone sequences, we get

$$I \ge C_2 \sum_{i=N+1}^{M-1} 2^i \lambda_{2^i} \Big(\sum_{j=i}^{M-1} a_{2^{j+1}} \sum_{\nu=2^j+1}^{2^{j+1}} \gamma_\nu\Big)^p \ge C_3 \sum_{i=N+1}^{M-1} 2^i \lambda_{2^i} \Big(\sum_{j=i}^{M-1} a_{2^{j+1}} 2^j \gamma_{2^j}\Big)^p.$$

By applying Theorem 1.1 to this inequality taking into account that $1 \le p$, then changing the order of summation, we have

$$\begin{aligned} I &\ge C_3 \sum_{i=N+1}^{M-1} 2^i \lambda_{2^i} \sum_{j=i}^{M-1} a_{2^{j+1}}^p 2^{jp} \gamma_{2^j}^p \ge C_3 \sum_{j=N+1}^{M-1} a_{2^{j+1}}^p 2^{jp} \gamma_{2^j}^p \sum_{i=N+1}^{j} 2^i \lambda_{2^i} \\ &\ge C_3 \sum_{j=N+1}^{M-1} a_{2^{j+1}}^p 2^{j(p+1)} \gamma_{2^j}^p \lambda_{2^j}. \end{aligned}$$

Since $a_\nu \downarrow$, taking into consideration that $\{\gamma_\nu\}_{\nu=1}^\infty$ and $\{\lambda_\mu\}_{\mu=1}^\infty$ are quasi lacunary monotone and quasi geometrically increasing sequences, respectively, we obtain

$$I \ge C_4 \sum_{j=N+1}^{M-1} \sum_{\mu=2^{j+1}+1}^{2^{j+2}} a_\mu^p \gamma_\mu^p \mu^p \lambda_\mu.$$

We rewrite the above inequality in the form

$$I \ge C_4 \sum_{\mu=2^{N+2}+1}^{2^{M+1}} \lambda_\mu (\mu a_\mu \gamma_\mu)^p,$$

and since $2^{N+2} < 8n$, we obtain

$$I \ge C_4 \sum_{\mu=8n}^{n} \lambda_\mu (\mu a_\mu \gamma_\mu)^p.$$

Thus, we have proved inequality (2.2) assuming that $N+1 \le M-1$. In fact, for $m \ge 16n$ we get $2^{N-1} < n \le \frac{m}{16} \le 2^{M-3}$, yielding that the condition $N+1 \le M-1$ is satisfied.

In order to prove inequality (2.1), put

$$J=\sum_{\mu=n}^{m}\lambda_\mu\Bigl(\sum_{\nu=n}^{\mu}a_\nu\gamma_\nu\Bigr)^p,$$

in a similar manner, but by making use of the fact that $\{\lambda_\mu\}_{\mu=1}^\infty$ is solely a quasi lacunary monotone sequence (i.e. without a quasi geometrically increasing sequence assumption), we obtain

$$J\ge C_5\sum_{\mu=2^{N+1}}^{2^M-1}\lambda_\mu(\mu a_\mu\gamma_\mu)^p.$$

Thus

$$\sum_{\mu=2^{N+1}}^{2^M-1}\lambda_\mu(\mu a_\mu\gamma_\mu)^p\le C_6J. \tag{3.1}$$

Since $a_\nu\downarrow$ and $\{\lambda_\mu\}_{\mu=1}^\infty$, $\{\gamma_\nu\}_{\nu=1}^\infty$ are lacunary monotone sequences, we have

$$J_1=\sum_{\mu=2^M}^{2^{M+1}}\lambda_\mu(\mu a_\mu\gamma_\mu)^p\le a_{2^M}^p\sum_{\mu=2^M}^{2^{M+1}}\lambda_\mu(\mu a_\mu\gamma_\mu)^p\le C_7\sum_{\mu=2^{M-1}}^{2^M-1}\lambda_\mu(\mu a_\mu\gamma_\mu)^p.$$

Hence, for $N+1\le M-1$ we obtain

$$J_1\le C_7\sum_{\mu=2^{N+1}}^{2^M-1}\lambda_\mu(\mu a_\mu\gamma_\mu)^p.$$

Thus, inequality (3.1) yields

$$J_1\le C_8J;$$

or

$$\sum_{\mu=2^M}^{2^{M+1}}\lambda_\mu(\mu a_\mu\gamma_\mu)^p.\le C_8J. \tag{3.2}$$

Adding inequalities (3.1) and (3.2) together, we obtain

$$\sum_{\mu=2^{N+1}}^{2^{M+1}}\lambda_\mu(\mu a_\mu\gamma_\mu)^p\le C_9\sum_{\mu=n}^{m}\lambda_\mu\Bigl(\sum_{\nu=n}^{\mu}a_\nu\gamma_\nu\Bigr)^p.$$

Since $2^{N+1}\le 4n$, $2^{M+1}>m$, the above inequality implies the inequality (2.1). This completes the proof of Theorem 2.1.

4. Proof of Theorem 2.2

We prove the inequality (2.4). Let positive integers N and M be defined by the inequalities $2^{N-1}<n\le 2^N$ and $2^M\le m<2^{M+1}$. This yields

$$I=\sum_{\mu=4n}^{m}\lambda_\mu\Bigl(\sum_{\nu=\mu}^{m}a_\nu\gamma_\nu\Bigr)^p\le\sum_{\mu=2^{N+1}+1}^{2^{M+1}}\lambda_\mu\Bigl(\sum_{\nu=\mu}^{2^{M+1}}a_\nu\gamma_\nu\Bigr)^p$$
$$=\sum_{i=N+1}^{M}\sum_{\mu=2^i+1}^{2^{i+1}}\lambda_\mu\Bigl(\sum_{\nu=\mu}^{2^{M+1}}a_\nu\gamma_\nu\Bigr)^p.$$

By bounding the third sum from above, taking into account that $\{\lambda_\mu\}_{\mu=1}^\infty$ is a quasi lacunary monotone sequence, we get

$$I \le \sum_{i=N+1}^{M} \left(\sum_{\nu=2^i+1}^{2^{M+1}} a_\nu \gamma_\nu \right)^p \sum_{\mu=2^i+1}^{2^{i+1}} \lambda_\mu \le C_1 \sum_{i=N+1}^{M} 2^i \lambda_{2^i} \left(\sum_{\nu=2^i+1}^{2^{M+1}} a_\nu \gamma_\nu \right)^p,$$

where positive constant C_1 depends only on the sequence $\{\lambda_\mu\}_{\mu=1}^\infty$. Now, we split the second sum into blocks of length 2^j and taking into consideration the fact that $a_\nu \downarrow$, and $\{\gamma_\nu\}_{\nu=1}^\infty$ is a lacunary monotone sequences, we have

$$I \le C_1 \sum_{i=N+1}^{M} 2^i \lambda_{2^i} \left(\sum_{j=i}^{M} a_{2^j} \sum_{\nu=2^j+1}^{2^{j+1}} \gamma_\nu \right)^p \le C_2 \sum_{i=N+1}^{M} 2^i \lambda_{2^i} \left(\sum_{j=i}^{M} a_{2^j} 2^j \gamma_{2^j} \right)^p.$$

By applying Theorem 1.1 and then changing the order of summation, we obtain

$$I \le C_2 \sum_{i=N+1}^{M} 2^i \lambda_{2^i} \sum_{j=i}^{M} a_{2^j}^p 2^{jp} \gamma_{2^j}^p \le C_2 \sum_{j=N+1}^{M} a_{2^j}^p 2^{jp} \gamma_{2^j}^p \sum_{i=N+1}^{j} 2^i \lambda_{2^i}.$$

Further, the fact that $\{\lambda_\mu\}_{\mu=1}^\infty$ is a quasi geometrically increasing sequence yields

$$I \le C_3 \sum_{j=N+1}^{M} a_{2^j}^p 2^{jp} \gamma_{2^j}^p 2^j \lambda_{2^j}.$$

Since $a_\nu \downarrow$, taking into consideration that $\{\gamma_\nu\}_{\nu=1}^\infty$ and $\{\lambda_\mu\}_{\mu=1}^\infty$ are quasi lacunary monotone sequences, we get

$$a_{2^j}^p 2^{j(p+1)} \gamma_{2^j}^p \lambda_{2^j} \le C_4 \sum_{\mu=2^{j-1}+1}^{2^j} a_\mu^p \gamma_\mu^p \mu^p \lambda_\mu.$$

Therefore,

$$I \le C_5 \sum_{j=N+1}^{M} \sum_{\mu=2^{j-1}+1}^{2^j} \lambda_\mu (\mu a_\mu \gamma_\mu)^p = C_5 \sum_{\mu=2^N+1}^{2^M} \lambda_\mu (\mu a_\mu \gamma_\mu)^p.$$

where positive constant C_5 does not depend on N and M. Since $2^M \le m$ and $2^N + 1 > m$, we obtain

$$I \le C_5 \sum_{\mu=n}^{m} \lambda_\mu (\mu a_\mu \gamma_\mu)^p,$$

which proves inequality (2.4).

Inequality (2.3) can be proved in an analogous way, but without a quasi geometrically increasing assumption for the sequence $\{2^\mu \lambda_{2^\mu}\}_{\mu=1}^\infty$.

5. Inequalities for non-negative decreasing sequences

For $\alpha, \lambda \in \mathbb{R}$ put

$$\lambda_\mu = \mu^{\alpha-1} \quad (\mu = 1, 2, \dots),$$
$$\gamma_\nu = \nu^\lambda \quad (\nu = 1, 2, \dots).$$

Obviously, the obtained sequences $\{\mu^{\alpha-1}\}_{\mu=1}^\infty$ and $\{\nu^\lambda\}_{\nu=1}^\infty$ are both quasi lacunary monotone sequences.

If, in addition, $\alpha > 0$, then $\{2^\mu 2^{\mu(\alpha-1)}\}_{\mu=1}^\infty$ is a quasi geometrically increasing sequence.

By applying Theorems 2.1 and 2.2 for such sequences $\{\lambda_\mu\}_{\mu=1}^\infty$ and $\{\gamma_\nu\}_{\nu=1}^\infty$, we deduce the following l_p-type inequalities for non-negative decreasing sequences, which are converse to inequalities (1.1), (1.2), (1.3) and (1.4).

Theorem 5.1. *Let a sequence $\{a_\nu\}_{\nu=1}^{\infty}$ be such that $a_\nu \downarrow$, and $\alpha > 0$, $\lambda \in \mathbb{R}$, m and n positive integers such that $m \geq 16n$. If $p \geq 1$, then the following inequalities hold*

$$\sum_{\mu=n}^{m} \mu^{\alpha-1} \left(\sum_{\nu=\mu}^{m} a_\nu \nu^\lambda \right)^p \geq C \sum_{\mu=8n}^{m} \mu^{\alpha-1} (a_\mu \mu^{\lambda+1})^p,$$

$$\sum_{\mu=n}^{m} \mu^{-\alpha-1} \left(\sum_{\nu=n}^{\mu} a_\nu \nu^\lambda \right)^p \geq C \sum_{\mu=4n}^{m} \mu^{-\alpha-1} (a_\mu \mu^{\lambda+1})^p.$$

Hereafter C denotes positive constant depending only on α, λ and p, and not depending on m, n and the sequence $\{a_\nu\}_{\nu=1}^{\infty}$.

Theorem 5.2. *Let a sequence $\{a_\nu\}_{\nu=1}^{\infty}$ be such that $a_\nu \downarrow$, and $\alpha > 0$, $\lambda \in \mathbb{R}$, m and n positive integers such that $m \geq 4n$. If $0 < p \leq 1$, then the following inequalities hold*

$$\sum_{\mu=4n}^{m} \mu^{\alpha-1} \left(\sum_{\nu=\mu}^{m} a_\nu \nu^\lambda \right)^p \leq C \sum_{\mu=n}^{m} \mu^{\alpha-1} (a_\mu \mu^{\lambda+1})^p,$$

$$\sum_{\mu=4n}^{m} \mu^{-\alpha-1} \left(\sum_{\nu=4n}^{\mu} a_\nu \nu^\lambda \right)^p \leq C \sum_{\mu=n}^{m} \mu^{-\alpha-1} (a_\mu \mu^{\lambda+1})^p.$$

Note that Theorems 5.1 and 5.2 given above imply several inequalities proved earlier [3, 4, 11].

Namely the following Corollaries are simple consequences of these theorems and the inequalities (1.1), (1.2), (1.3) and (1.4).

Corollary 5.1. *Let a sequence $\{a_\nu\}_{\nu=1}^{\infty}$ be such that $a_\nu \downarrow$, $\alpha > 0$, $\lambda \in \mathbb{R}$, and n a positive integer. If $p > 0$, then the following inequalities hold*

$$\sum_{\mu=1}^{n} \mu^{\alpha-1} \left(\sum_{\nu=\mu}^{n} a_\nu \nu^\lambda \right)^p \geq C \sum_{\mu=1}^{n} \mu^{\alpha-1} (a_\mu \mu^{\lambda+1})^p,$$

$$\sum_{\mu=1}^{n} \mu^{-\alpha-1} \left(\sum_{\nu=1}^{\mu} a_\nu \nu^\lambda \right)^p \geq C \sum_{\mu=1}^{n} \mu^{-\alpha-1} (a_\mu \mu^{\lambda+1})^p.$$

Corollary 5.2. *Let a sequence $\{a_\nu\}_{\nu=1}^{\infty}$ be such that $a_\nu \downarrow$, $\alpha > 0$, $\lambda \in \mathbb{R}$, and n a positive integer. If $p \geq 1$, then the following asymptotic equivalences hold*

$$\sum_{\mu=1}^{n} \mu^{\alpha-1} \left(\sum_{\nu=\mu}^{n} a_\nu \nu^\lambda \right)^p \asymp \sum_{\mu=1}^{n} \mu^{\alpha-1} (a_\mu \mu^{\lambda+1})^p,$$

$$\sum_{\mu=1}^{n} \mu^{-\alpha-1} \left(\sum_{\nu=1}^{\mu} a_\nu \nu^\lambda \right)^p \asymp \sum_{\mu=1}^{n} \mu^{-\alpha-1} (a_\mu \mu^{\lambda+1})^p.$$

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